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NEW NORM INEQUALITIES FOR COMMUTATORS OF HILBERT SPACE OPERATORS

Abstract. New norm inequalities for commutators of Hilbert space operators are given. Among other inequalities, it is shown that if A, $B \in \mathbb{B}(\mathbb{H})$ and there exists a real number z_0, such that $\|A - z_0 I\| = D_A$, then

$$\|AB \pm BA^*\| \leqslant 2D_A\|B\|,$$

where $D_A = \inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|$. In particular, under some conditions, we prove that

$$\|AB\| \leqslant D_A\|B\|,$$

which is an improvement of submultiplicative norm inequality. Also, we prove several numerical radius inequalities for products of two Hilbert space operators.

Key words: *bounded linear operator, Hilbert space, norm inequality, numerical radius*

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1. Introduction and preliminaries.

Let $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. A self-adjoint operator A is said to be positive if $\langle Ax, x\rangle \geqslant 0$ holds for all $x \in \mathbb{H}$. The numerical radius of $A \in \mathbb{B}(\mathbb{H})$ is defined by

$$\omega(A) = \sup\{|\langle Ax, x\rangle| \colon \|x\| = 1 \}.$$

It is well known that $\omega(\cdot)$ is a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for all $A \in \mathbb{B}(\mathbb{H})$,

$$\frac{\|A\|}{2} \leqslant \omega(A) \leqslant \|A\|. \tag{1}$$

The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. Several numerical radius inequalities improving the inequalities in (1) have been recently given in [2], [4], [5], [7], [8], [11], [12], [13], [14], [15] and [16]. If A and B are operators in $\mathbb{B}(\mathbb{H})$, we write the direct sum $A \oplus B$ for the 2×2 operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $H \oplus H$. Thus

$$\omega(A \oplus B) = \max(\omega(A), \omega(B)).$$

Also,

$$\|A \oplus B\| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| = \max(\|A\|,\|B\|). \tag{2}$$

The question about the best constant k, such that the inequality

$$w\,(AB) \leqslant k\|A\|\omega(B) \tag{3}$$

holds for all operators $A, B \in \mathbb{B}(\mathbb{H})$, is still open.

Concerning the inequality (3), it is shown in [1] that if $A, B \in \mathbb{B}(\mathbb{H})$, then

$$\omega(AB) \leqslant \omega(A)(D_B + \|B\|). \tag{4}$$

Also, if $A > 0$, then

$$\omega(AB) \leqslant \|A\|\Big(\frac{\omega(B) + \|B\|}{2}\Big) \tag{5}$$

and

$$\omega(AB) \leqslant \frac{3}{2}\|A\|\,\omega(B).$$

The commutator of two bounded linear operators A and B is the operator $AB - BA$. In [6], Dragomir proved that if $A, B \in \mathbb{B}(\mathbb{H})$ and $A - B$ is positive operator, then

$$\|AB - BA\| \leqslant \min(\|A\|, \|B\|)\|A - B\| \tag{6}$$

and, if A (or B) is positive, then

$$\|AB - BA\| \leqslant \|A\|\|A - B\|\ (\text{or } \|\mathrm{B}\|\|\mathrm{A} - \mathrm{B}\|).$$

Kittaneh in [9] proved that if $A, B, C \in \mathbb{B}(\mathbb{H})$, such that A or B is positive, then

$$\|AB - BA\| \leqslant \|A\|\|B\| \tag{7}$$

and, also, if A and C are positive operators, then

$$\|AB - BC\| \leqslant \max(\|A\|, \|C\|)\|B\|. \tag{8}$$

In Section 2, we establish norm inequalities for commutators of Hilbert space operators. Applications of these inequalities can be considered as improving some of the inequalities expressed in [6]; for example, we obtain refinements of the inequalities (6), (7), and (8). Finally, we obtain refinements of the inequality (5).

2. Main results. Let $D_A = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$ (the distance of A from the scalar operators), and let R_A denote the radius of the smallest disk in the complex plane containing $\sigma(A)$ (the spectrum of A). It is not hard to check that there exist a $\lambda_0 \in \mathbb{C}$, such that $D_A = \|A - \lambda_0 I\|$. It is known (see, e.g., [17]) that $D_A = R_A$ for any normal operator A.

In order to derive our main results, we need the following lemma, which can be found in [10]:

Lemma 1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. If $z_0 \in \mathbb{C}$, such that $\|A - z_0 I\| = D_A$, then*

$$\|Re(\alpha_0 AB)\| \leqslant \frac{\|B + B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + D_A\omega(B)$$

where $\alpha_0 = \frac{\bar{z}_0}{|z_0|}$.

Theorem 1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. If $z_0 \in R$, such that $\|A - z_0 I\| = D_A$, then*

$$\|AB \pm BA^*\| \leqslant 2D_A\|B\|.$$

Proof. By Theorem 1,

$$\|\operatorname{Re}(\alpha AB)\| \leqslant \frac{\|B + B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + D_A\omega(B). \tag{9}$$

From the assumption, $\|A - z_0 I\| = D_A$, it follows that $|\alpha| = 1$ and, by (9),

$$\|\operatorname{Re}(AB)\| \leqslant \frac{\|B + B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + D_A\omega(B). \tag{10}$$

Choose $A_1 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 & B \\ -B^* & 0 \end{bmatrix}$ in (10) to give

$$\| \operatorname{Re}(A_1 B_1)\| \leqslant D_{A_1}\omega(B_1).$$

Therefore,

$$\left\| \operatorname{Re}\left(\begin{bmatrix} 0 & AB \\ -AB^* & 0 \end{bmatrix} \right) \right\| \leqslant D_A \|B\|.$$

Consequently,

$$\left\| \begin{bmatrix} 0 & AB - BA^* \\ B^*A^* - AB^* & 0 \end{bmatrix} \right\| \leqslant 2D_A\|B\|$$

and, finally,

$$\|AB - BA^*\| \leqslant 2D_A\|B\|$$

by (2). Replacing A by iA gives the related inequality

$$\|AB + BA^*\| \leqslant 2D_A\|B\|.$$

This completes the proof. □

Corollary 1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. If A is a self-adjoint operator, then*

$$\|AB \pm BA\| \leqslant \left(\max_{\lambda \in \sigma(A)} \lambda - \min_{\lambda \in \sigma(A)} \lambda \right) \|B\|.$$

Proof. Let $z_0 = (\max_{\lambda \in \sigma(A)} \lambda + \min_{\lambda \in \sigma(A)} \lambda)/2$. Since $\|A - z_0 I\| = D_A$, from the Theorem 1 we have

$$\|AB \pm BA\| \leqslant 2D_A \; \|B\|. \tag{11}$$

On the other hand,

$$D_A = R_A = \frac{\max_{\lambda \in \sigma(A)} \lambda - \min_{\lambda \in \sigma(A)} \lambda}{2},$$

and, so,

$$\|AB \pm BA\| \leqslant \left(\max_{\lambda \in \sigma(A)} \lambda - \min_{\lambda \in \sigma(A)} \lambda \right) \|B\|, \qquad \text{(by (11))}$$

which is exactly the desired result. □

Remark 1. *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *and* $A > 0$. *Since* $\min\limits_{\lambda\in\sigma(A)} \lambda > 0$, *from Corollary 1 we get*

$$\begin{aligned}\|AB - BA\| &\leqslant \left(\max_{\lambda\in\sigma(A)} \lambda - \min_{\lambda\in\sigma(A)} \lambda\right) \|B\| = \\ &= \left(\|A\| - \min_{\lambda\in\sigma(A)} \lambda\right) \|B\| < \\ &< \|A\|\|B\|,\end{aligned}$$

which is a considerable improvement of inequality (7).

The following Corollary is a considerable improvement of the inequality (8):

Corollary 2. *Let* $A, B, C \in \mathbb{B}(\mathbb{H})$. *If* A *and* C *are positive operators, then*

$$\|AB - BC\| \leqslant \left(\max(\|A\|, \|C\|) - \min_{\lambda\in\{\sigma(A)\cup\sigma(C)\}} \lambda\right) \|B\|.$$

Proof. Let $A_1 = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$. Then A_1 is positive and $A_1B_1 - B_1A_1 = \begin{bmatrix} 0 & AB - BC \\ 0 & 0 \end{bmatrix}$. By Remark 1 and (2),

$$\begin{aligned}\|AB - BC\| &= \|A_1B_1 - B_1A_1\| \leqslant \\ &\leqslant (\|A_1\| - \min_{\lambda\in\sigma(A_1)} \lambda)\|B_1\| = \\ &= \left(\max(\|A\|, \|C\|) - \min_{\lambda\in\{\sigma(A)\cup\sigma(C)\}} \lambda\right) \|B\|.\end{aligned}$$

Consequently,

$$\|AB - BC\| \leqslant \left(\max(\|A\|, \|C\|) - \min_{\lambda\in\{\sigma(A)\cup\sigma(C)\}} \lambda\right) \|B\|.$$

This completes the proof. □

Theorem 2. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *If* $A - B$ *are self-adjoint operators, then*

$$\|AB - BA\| \leqslant \left(\max_{\lambda\in\sigma(A-B)} \lambda - \min_{\lambda\in\sigma(A-B)} \lambda\right) \min(\|B\|, \|A\|).$$

Proof. By Corollary 1,

$$\begin{aligned}\|AB - BA\| &= \|(A-B)B - B(A-B)\| \leqslant \\ &\leqslant \left(\max_{\lambda\in\sigma(A-B)} \lambda - \min_{\lambda\in\sigma(A-B)} \lambda\right)\|B\|.\end{aligned}$$

Consequently,

$$\|AB - BA\| \leqslant \left(\max_{\lambda\in\sigma(A-B)} \lambda - \min_{\lambda\in\sigma(A-B)} \lambda\right)\|B\| \tag{12}$$

and, similarly,

$$\begin{aligned}\|AB - BA\| &= \|(A-B)A - A(A-B)\| \leqslant \\ &\leqslant \left(\max_{\lambda\in\sigma(A-B)} \lambda - \min_{\lambda\in\sigma(A-B)} \lambda\right)\|A\|.\end{aligned}$$

Therefore,

$$\|AB - BA\| \leqslant \left(\max_{\lambda\in\sigma(A-B)} \lambda - \min_{\lambda\in\sigma(A-B)} \lambda\right)\|A\|. \tag{13}$$

Utilising (12) and (13), we deduce the desired result. □

Remark 2. *Let $A, B \in \mathbb{B}(\mathbb{H})$ and $A - B$ be a positive operator. Since $\min_{\lambda\in\sigma(A-B)} \lambda > 0$, from Theorem 2 we get*

$$\begin{aligned}\|AB - BA\| &\leqslant \left(\max_{\lambda\in\sigma(A-B)} \lambda - \min_{\lambda\in\sigma(A-B)} \lambda\right)\min(\|B\|, \|A\|) = \\ &= \left(\|A-B\| - \min_{\lambda\in\sigma(A-B)} \lambda\right)\min(\|B\|, \|A\|) < \\ &< \|A-B\| \min(\|B\|, \|A\|),\end{aligned}$$

which is a considerable improvement of inequality (6).

The following result for the self-commutator holds:

Corollary 3. *Let $A \in \mathbb{B}(\mathbb{H})$. If φ and ψ are the Cartesian decomposition of an operator A, i.e., $A = \varphi + i\psi$, then*

$$\|AA^* - A^*A\| \leqslant 2\left(\|\varphi - \psi\| - \min_{\lambda\in\sigma(\varphi-\psi)} \lambda\right)\min(\|\psi\|, \|\varphi\|).$$

Proof. Clearly,

$$\frac{1}{2i}(AA^* - A^*A) = \varphi\psi - \psi\varphi$$

and, applying Theorem 2,

$$\begin{aligned}\frac{1}{2}\|AA^* - A^*A\| &= \|\varphi\psi - \psi\varphi\| \leqslant \\ &\leqslant \left(\max_{\lambda\in\sigma(\varphi-\psi)}\lambda - \min_{\lambda\in\sigma(\varphi-\psi)}\lambda\right)\min(\|\psi\|, \|\varphi\|) \leqslant \\ &\leqslant \left(\|\varphi - \psi\| - \min_{\lambda\in\sigma(\varphi-\psi)}\lambda\right)\min(\|\psi\|, \|\varphi\|).\end{aligned}$$

Consequently,

$$\|AA^* - A^*A\| \leqslant 2\left(\|\varphi - \psi\| - \min_{\lambda\in\sigma(\varphi-\psi)}\lambda\right)\min(\|\psi\|, \|\varphi\|),$$

which is exactly the desired result. □

At the end of this section, we introduce some numerical radius inequalities for products of two operators. In order to derive our results, we need the following lemma, which can be found in [3]:

Lemma 2. *Let a, b and $e \in \mathbb{H}$. Then*

$$|\langle a, e\rangle\langle e, b\rangle| \leqslant \frac{\|e\|^2}{2}(\|a\|\|b\| + |\langle a, b\rangle|).$$

The following result holds:

Theorem 3. *Let $A, B, C \in \mathbb{B}(\mathbb{H})$ and $0 \leqslant \alpha \leqslant 1$. If A is a positive operator, then*

$$\omega(BAC) \leqslant \|A\|\Big(\frac{\|(1-\alpha)|C|^2 + \alpha|B|^2\|^{\frac{1}{2}}\|C\|^\alpha\|B\|^{1-\alpha} + \omega(BC)}{2}\Big).$$

Proof. Putting $a = Cx, e = ACx$, and $b = B^*x$, where $x \in \mathbb{H}$ and $\|x\| = 1$, in Lemma 2, gives:

$$|\langle Cx, ACx\rangle\langle ACx, B^*x\rangle| \leqslant \frac{\|ACx\|^2}{2}(\|Cx\|\|B^*x\| + |\langle Cx, B^*x\rangle|).$$

Hence,

$$\begin{aligned}
|\langle ACx, B^*x\rangle| &\leqslant \frac{\|ACx\|^2}{2\langle Cx, ACx\rangle}(\|Cx\|\|B^*x\| + |\langle Cx, B^*x\rangle|) \leqslant \\
&\leqslant \frac{\|A^{\frac{1}{2}}A^{\frac{1}{2}}Cx\|^2}{2\|A^{\frac{1}{2}}Cx\|^2}(\|Cx\|\|B^*x\| + |\langle Cx, B^*x\rangle|) \leqslant \\
&\leqslant \frac{\|A^{\frac{1}{2}}\|^2}{2}(\|Cx\|\|B^*x\| + |\langle Cx, B^*x\rangle|) = \\
&= \frac{\|A\|}{2}(\|Cx\|\|B^*x\| + |\langle Cx, B^*x\rangle|) = \\
&= \frac{\|A\|}{2}(\|Cx\|\|B^*x\| + |\langle BCx, x\rangle|) \leqslant \\
&\leqslant \frac{\|A\|}{2}(\|Cx\|\|B^*x\| + \sup_{\|x\|=1}|\langle BCx, x\rangle|) = \\
&= \frac{\|A\|}{2}(\|Cx\|\|B^*x\| + \omega(BC)) = \\
&= \frac{\|A\|}{2}(\langle Cx, Cx\rangle^{\frac{1}{2}}\langle B^*x, B^*x\rangle^{\frac{1}{2}} + \omega(BC)) = \\
&= \frac{\|A\|}{2}(\langle |C|^2, x\rangle^{\frac{1}{2}}\langle |B|^2x, x\rangle^{\frac{1}{2}} + \omega(BC)).
\end{aligned}$$

Therefore,

$$|\langle ACx, B^*x\rangle| \leqslant \frac{\|A\|}{2}(\langle |C|^2, x\rangle^{\frac{1}{2}}\langle |B|^2x, x\rangle^{\frac{1}{2}} + \omega(BC)). \tag{14}$$

Now, let $\alpha \in R$ and $0 \leqslant \alpha \leqslant 1$. By (14),

$$\begin{aligned}
|\langle BACx, x\rangle| &\leqslant \frac{\|A\|}{2}(\langle |C|^2, x\rangle^{\frac{1}{2}}\langle |B|^2x, x\rangle^{\frac{1}{2}} + \omega(BC)) = \\
&= \frac{\|A\|}{2}(\langle |C|^2x, x\rangle^{\frac{1-\alpha}{2}}\langle |B|^2x, x\rangle^{\frac{\alpha}{2}}\langle |C|^2x, x\rangle^{\frac{\alpha}{2}}\langle |B|^2x, x\rangle^{\frac{1-\alpha}{2}} + \omega(BC)) \leqslant \\
&\leqslant \frac{\|A\|}{2}(\langle |C|^2x, x\rangle^{\frac{1-\alpha}{2}}\langle |B|^2x, x\rangle^{\frac{\alpha}{2}}\|C\|^{\alpha}\|B\|^{1-\alpha} + \omega(BC)) = \\
&= \frac{\|A\|}{2}((\langle |C|^2x, x\rangle^{1-\alpha}\langle |B|^2x, x\rangle^{\alpha})^{\frac{1}{2}}\|C\|^{\alpha}\|B\|^{1-\alpha} + \omega(BC)) \leqslant \\
&\leqslant \frac{\|A\|}{2}(((1-\alpha)\langle |C|^2x, x\rangle + \alpha\langle |B|^2x, x\rangle)^{\frac{1}{2}}\|C\|^{\alpha}\|B\|^{1-\alpha} + \omega(BC)) \leqslant \\
&\leqslant \frac{\|A\|}{2}(\|(1-\alpha)|C|^2 + \alpha|B|^2\|^{\frac{1}{2}}\|C\|^{\alpha}\|B\|^{1-\alpha} + \omega(BC)).
\end{aligned}$$

Therefore,

$$|\langle BACx, x\rangle| \leqslant \|A\|(\frac{\|(1-\alpha)|C|^2 + \alpha|B|^2\|^{\frac{1}{2}}\|C\|^{\alpha}\|B\|^{1-\alpha} + \omega(BC)}{2}).$$

Taking the supremum over $x \in \mathbb{H}$, $\|x\| = 1$ gives

$$\omega(BAC) \leqslant \|A\|(\frac{\|(1-\alpha)|C|^2 + \alpha|B|^2\|^{\frac{1}{2}}\|C\|^{\alpha}\|B\|^{1-\alpha} + \omega(BC)}{2}),$$

which is exactly the desired result. □

Corollary 4. *If $B, C \in \mathbb{B}(\mathbb{H})$ and $0 \leqslant \alpha \leqslant 1$, then*

$$\omega(BC) \leqslant \|(1-\alpha)|C|^2 + \alpha|B|^2\|^{\frac{1}{2}} \ \|C\|^{\alpha}\|B\|^{1-\alpha}.$$

Proof. If we replace A by I in Theorem 3, we deduce the desired result. □

Remark 3. *Let $A, B \in \mathbb{B}(\mathbb{H})$ and A be positive operators. By Theorem 3, for $\alpha = 0$ and replacing C by I gives:*

$$\omega(BA) \leqslant \|A\|\frac{(\|B\| + \omega(B))}{2}.$$

This shows that Theorem 3 is a refinement of the inequality (5).

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