DOI: 10.15393/j3.art.2024.16550

UDC 517.954

B. Kh. Turmetov

ON SOLVABILITY OF SOME BOUNDARY-VALUE PROBLEMS FOR THE NON-LOCAL POISSON EQUATION WITH FRACTIONAL-ORDER BOUNDARY OPERATORS

Abstract. In this paper, a non-local analogue of the Laplace operator is introduced using involution-type mappings. For the corresponding non-local analogue of the Poisson equation in the unit ball, two types of boundary-value problems are considered. In the studied problems, the boundary conditions involve fractional-order operators with derivatives of the Hadamard type. The first problem generalizes the well-known Dirichlet, Neumann, and Robin problems for fractional-order boundary operators. The second problem is a generalization of periodic and antiperiodic boundary-value problems for circular domains. Theorems on the existence and uniqueness of solutions to the studied problems are proved. Exact conditions for solvability of the studied problems are found, and integral representations of the solutions are obtained.

Key words: *non-local equation, fractional derivative, Hadamard operator, periodic problem, Dirichlet problem, Neumann problem*

2020 Mathematical Subject Classification: *35J05,35J25*

1. Introduction. This paper is devoted to the study of correct formulations of boundary-value problems for equations with transformed arguments. In the equations, the considered transformation of arguments is carried out using involution-type mappings. A mapping S is called an involution if $S^2 = E$, where E is the identity mapping. For example, such a mapping is the Dunkl transformation [5]. Some applications of Dunkl-type mappings are considered in [7], [8].

Note that one of the first published papers for equations with involutive transformations is the work of T. Carleman [4], where equations with shifts of arguments of the type $\alpha = \alpha(t)$, $\alpha^2(t) = t$ were studied. Some issues of the application of equations with shifts of the Carleman type are considered in [6].

Further, let us consider statement of the problems studied in this work. Let $\Omega = \{x \in R^n : |x| < 1\}$ be a unit ball, $n \geqslant 2$, $\partial\Omega$ be a unit sphere, $u(x)$ be a smooth function in the domain Ω, $r = |x|$, $\theta = \frac{x}{r}$, $\delta = r\frac{d}{dr}$ be the Dirac operator, where $r\frac{d}{dr} = \sum\limits_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$.

Let us consider the modified Hadamard integro-differential operators ([11], p. 116)

$$J_\mu^\alpha [u](x) = \begin{cases} u(x), & \alpha = 0 \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^1 \left(\ln \frac{1}{\tau}\right)^{\alpha-1} \tau^{\mu-1} u(\tau x)\, d\tau, & \alpha > 0,\ \mu \geqslant 0, \end{cases}$$

$$D_\mu^\alpha [u](x) = r^{-\mu} J^{m-\alpha} [\delta^m [\tau^\mu \cdot u]](x), \quad m-1 < \alpha \leqslant m,\ m \geqslant 1.$$

For any $x \in R^n$ consider the mappings $S_i x = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$, $1 \leqslant i \leqslant n$. For the index i, we will use not only the usual notation, but also its representation in the binary number system $i = (i_n \ldots i_2 i_1)_2 \equiv$ $\equiv i_n \cdot 2^{n-1} + \cdots + i_2 \cdot 2^1 + i_1 \cdot 2^0$. Using this notation, we can consider mappings of the type $S_n^{i_n} \ldots S_2^{i_2} S_1^{i_1} x$, where $i_k = 0$ or $i_k = 1$. The total number of such mappings is 2^n.

Let $a_i, i = 0, 1, \ldots, 2^n - 1$ be some real numbers, Δ be a Laplace operator. Let us introduce the operator

$$L_n u(x) = \sum_{i=0}^{2^n-1} a_i (-\Delta) u \left(S_n^{i_n} \ldots S_2^{i_2} S_1^{i_1} x\right),$$

which we will call a non-local Laplace operator.

Then, for any point $x = (x_1, x_2, \ldots, x_n) \in \Omega$, we match the «opposite» point $x^* = (\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_n x_n) \in \Omega$, where $\sigma_1 = -1$, and σ_j, $j = 2, \ldots, n$ take one of the values ± 1. Let us denote

$$\partial\Omega_+ = \{x \in \partial\Omega \colon x_1 \geqslant 0\}, \quad \partial\Omega_- = \{x \in \partial\Omega \colon x_1 \leqslant 0\},$$

$$I = \{x \in \partial\Omega \colon x_1 = 0\}.$$

Note that the point x^* can be represented as $x^* = S_n^{j_n} \dots S_2^{j_2} S_1^1 x$. Moreover, if $x \in \partial\Omega_+$, then $x^* \in \partial\Omega_-$.

In the domain Ω, we consider the equation

$$L_n u(x) = f(x), x \in \Omega. \tag{1}$$

In [20], for equation (1) the main boundary-value problems with the Dirichlet and Neumann conditions were investigated. Spectral issues for the operator L_n were studied in [21]. The present work is a continuation of these studies, and for equation (1) we will consider the following problems.

Problem 1. Let $0 \leqslant \mu, 0 \leqslant \alpha \leqslant 1$. Find a function $u(x)$ from the class $C^2(\Omega) \cap C(\bar{\Omega})$, for which $D_\mu^\alpha[u](x) \in C(\bar{\Omega})$, satisfying equation (1) and the condition

$$D_\mu^\alpha[u](x) = g(x), x \in \partial\Omega. \tag{2}$$

Problem 2. Let $0 \leqslant \beta < \alpha \leqslant 1$. Find a function $u(x)$ from the class $C^2(\Omega) \cap C(\bar{\Omega})$, for which $D_0^\alpha[u](x) \in C(\bar{\Omega})$, satisfying equation (1) and the conditions

$$D_0^\beta[u](x) - (-1)^k D_0^\beta[u](x^*) = g_0(x), x \in \partial\Omega_+, \tag{3}$$

$$D_0^\alpha[u](x) + (-1)^k D_0^\alpha[u](x^*) = g_1(x), x \in \partial\Omega_+, \tag{4}$$

where k takes one of the values $k = \pm 1$.

As in the case $x \in \partial\Omega_-$ there is an inclusion $x^* \in \partial\Omega_+$; then, from condition (3) it follows that

$$D_0^\beta[u](x^*) - (-1)^k D_0^\beta[u](x) = g_0(x^*), x \in \partial\Omega_-.$$

In this case, if $x \in I \Leftrightarrow x = (0, x_2, \dots, x_n) \in \partial\Omega_+$, then for the point $x^* \in \partial\Omega_+$ corresponding to it we get: $x^* = (0, \sigma_2 x_2, \dots, \sigma_n x_n) \in \partial\Omega_- \Leftrightarrow x^* \in I$. Therefore, for points $x \in I$ it is necessary to fulfill the conditions of agreement:

$$g_0(x) = D_0^\beta[u](x) - (-1)^k D_0^\beta[u](x^*)\Big|_{x \in I} =$$

$$= -(-1)^k \left[D_0^\beta[u](x^*) - (-1)^k D_0^\beta[u](x) \right]\Big|_{x^* \in I} = -(-1)^k g_0(x^*).$$

Let

$$\partial^m u(x) = \frac{\partial^m}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}, m = (m_1, \dots, m_n).$$

Further, we will find solutions to Problem 2 from the class $C^{\lambda+2}\left(\bar{\Omega}\right)$, $0<\lambda<1$. Then a necessary condition for the existence of a solution to Problem 2 from this class is the fulfillment of the following matching conditions:

$$\partial^m g_0(0,x_2,\ldots,x_n)=(-1)^k\partial^m g_0(0,\sigma_2 x_2,\ldots,\sigma_n x_n), x\in I, m=0,1,2, \quad (5)$$

$$\partial^m g_1(0,x_2,\ldots,x_n)=-(-1)^k\partial^m g_1(0,\sigma_2 x_2,\ldots,\sigma_n x_n), x\in I, m=0,1,2. \quad (6)$$

In what follows, we assume that conditions (5), (6) are satisfied. As $J_\mu^0 u(x)=u(x)$, then in the case $\alpha=1$ the operator D_μ^1 coincides with the operator $r\dfrac{\partial}{\partial r}+\mu$. In this case, Problem 1 for $\mu>0$ coincides with the Robin problem, and in the case $\mu=0$ it coincides with the Neumann problem.

Note that boundary-value problems for an elliptic equation with fractional-order boundary operators were studied in [1], [10], [12], [13], [18], [24]. In these works, operators with Hadamard, Riemann-Liouville, Caputo derivatives and some of their modifications were considered as boundary operators.

Boundary-value problems with periodic and antiperiodic conditions for the Poisson equation in circular domains were first studied in [14], [15], and for the non-local analogue of the Poisson equation in the case $n=2$ they were considered in [23]. Later, some generalizations of these problems with conditions of the Dirichlet, Neumann, and Robin type, as well as the Samarskii-Ionkin type, were studied in [16], [17], [19], [25].

Also note that the boundary conditions in the considered problems are specified as a relationship between the values of the unknown function at different points of the boundary. Problems of this type are usually called non-local problems of the Bitsadze-Samarskii type [2], [3].

2. Properties of integro-differentiation operators. In this section, we present some known properties of operators J_μ^α and D_μ^α in the class of smooth functions. The statements below clarify the conditions for reversibility and action of these operators in the Hölder class. Note that studies in this direction were conducted in [9], where the properties of fractional differential operators associated with the derivative q were studied in the class of harmonic functions. The following statements were proved in [18]:

Lemma 1. *Let $\alpha>0$, $\mu\geqslant 0$, $0<\lambda<1$ and $u(x)\in C^{\lambda+p}\left(\bar{\Omega}\right), p\geqslant 0$. Then*

1) *if* $\mu > 0$, *then* $J_\mu^\alpha [u] (x) \in C^{\lambda+p} \left(\bar{\Omega}\right)$;
2) *if* $\mu = 0$ *and the condition* $u (0) = 0$ *is satisfied, the function* $J_0^\alpha [u] (x)$ *also belongs to the class* $C^{\lambda+p} \left(\bar{\Omega}\right)$ *and the equality* $J_0^\alpha [u] (0) = 0$ *is satisfied.*

Lemma 2. *Let* $\mu \geqslant 0$, $p - 1 < \alpha \leqslant p$, $p = 1, 2, \ldots,$ $0 < \lambda < 1$, *and* $u (x) \in C^{\lambda+q} \left(\bar{\Omega}\right)$, $q \geqslant p$. *Then the function* $D_\mu^\alpha [u] (x)$ *belongs to the class* $C^{\lambda+q-p} \left(\bar{\Omega}\right)$ *and the equality* $D_0^\alpha [u] (0) = 0$ *is satisfied.*

Lemma 3. *Let* $\mu \geqslant 0$, $p - 1 < \alpha \leqslant p$, $p = 1, 2, \ldots,$ $0 < \lambda < 1$ *and* $u (x) \in C^{\lambda+q} \left(\bar{\Omega}\right)$, $q \geqslant p$. *Then for any* $x \in \bar{\Omega}$ *the equality*

$$J_\mu^\alpha \left[D_\mu^\alpha [u]\right] (x) = \begin{cases} u (x) , & \mu > 0, \\ u (x) - u (0) , & \mu = 0. \end{cases}$$

is valid.

Lemma 4. *Let* $\mu \geqslant 0$, $p - 1 < \alpha \leqslant p$, $p = 1, 2, \ldots,$ $0 < \lambda < 1$, *and* $u (x) \in C^{\lambda+q} \left(\bar{\Omega}\right)$, $q \geqslant p$. *Then for any* $x \in \bar{\Omega}$ *if* $\mu > 0$ *the equality*

$$D_\mu^\alpha \left[J_\mu^\alpha [u]\right] (x) = u (x) \tag{7}$$

is valid; in the case $\mu = 0$, *equality* (7) *is also valid under the additional condition* $u (0) = 0$.

Lemma 5. *Let* $\mu \geqslant 0$, $p - 1 < \alpha \leqslant p$, $p = 1, 2, \ldots,$ $f (x)$ *be a smooth function in the domain* $\bar{\Omega}$ *and* $-\Delta u (x) = f (x)$, $x \in \Omega$. *Then the equality*

$$- \Delta D_\mu^\alpha [u] (x) = F (x) , \ x \in \Omega, \tag{8}$$

is valid, where

$$F (x) = D_{\mu+2}^\alpha [f] (x) . \tag{9}$$

Lemma 6. *If* $\mu = 0, 0 < \alpha \leqslant 1$, *then for the function* $F (x)$ *from equality* (9) *there is a representation*

$$F (x) = \left(r \frac{d}{dr} + 2\right) f_{1-\alpha} (x) ,$$

where $f_{1-\alpha} (x) = J_2^{1-\alpha} [f] (x)$.

3. Existence and uniqueness of a solution to Problem 1. Let us introduce the notation

$$\varepsilon_k = \sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i, \ k = 0, 1, \ldots, 2^n - 1,$$

where $k \otimes i = k_n i_n + \cdots + k_1 i_1$ is a "scalar" product of numbers $(k)_2$ and $(i)_2$, $(i)_2 = (i_n \ldots i_1)_2$ is the notation of the index i in the binary number system.

Note that in case $n = 2$ the numbers $\varepsilon_k = \sum\limits_{i=0}^{3} (-1)^{k\otimes i} a_i$, $k = 0, 1, 2, 3,$ are written as:

$$\begin{cases} \varepsilon_0 \equiv \varepsilon_{(00)_2} = a_0 + a_1 + a_2 + a_3, \varepsilon_1 \equiv \varepsilon_{(01)_2} = a_0 - a_1 + a_2 - a_3, \\ \varepsilon_2 \equiv \varepsilon_{(10)_2} = a_0 + a_1 - a_2 - a_3, \varepsilon_3 \equiv \varepsilon_{(11)_2} = a_0 - a_1 - a_2 + a_3. \end{cases}$$

As we have already noted, in the case of the classical Dirichlet ($\alpha = 0$) and Neumann ($\alpha = 1$, $\mu = 0$) boundary conditions, Problem 1 was investigated in [20]. The following statement was proved:

Theorem 1. *Let $\mu = 0$, the coefficients of the operator L_n be such that the conditions $\varepsilon_k \neq 0$, $k = 0, 1, \ldots, 2^n - 1$ are satisfied, and $f(x) \in C^\lambda\left(\bar{\Omega}\right)$, $g(x) \in C^{\lambda+2}\left(\partial\Omega\right)$, $0 < \lambda < 1$. Then*

1) *if $\alpha = 0$, then a solution to Problem 1 exists and is unique;*
2) *if $\alpha = 1$, then for the solvability of Problem 1 it is necessary and sufficient that the condition*

$$\int\limits_{\Omega} f(y)dy + \left(\sum_{i=0}^{2^n-1} a_i\right) \int\limits_{\Omega} g(y)ds_y = 0 \tag{10}$$

is satisfied.

If a solution to the problem exists, then it is unique up to a constant term and belongs to the class $C^{\lambda+2}\left(\bar{\Omega}\right)$.

Example. Let $x^* = (-x_1, -x_2, \ldots, -x_n)$, $H_m(x)$ be a homogeneous harmonic polynomial of degree m and $u(x) = (1 - |x|^2) H_m(x)$. It is obvious that $u(x)|_{\partial\Omega} = 0$. Moreover, $\Delta u(x) = 2\,(2m + n)\, H_m(x)$. Hence,

$$a_0 \Delta u(x) + a_1 \Delta u(x^*) = -2\,(2m + n)\left[a_0 H_m(x) + a_0 H_m(x^*)\right].$$

If the polynomial $H_m(x)$ has the property $H_m(x) = H_m(x^*)$ and $a_0 + a_1 = 0$, then we see that the function $u(x) = (1 - |x|^2)\, H_m(x)$ is a solution to the following homogeneous problem

$$a_0 \Delta u(x) + a_1 \Delta u(x^*) = 0,\ x \in \Omega;\ \ u(x)|_{\partial\Omega} = 0.$$

It follows from this example that if the coefficients of the operator L_n are such that the condition $\varepsilon_k = 0$, $k = 0, 1, \ldots, 2^n - 1$, is satisfied, then the homogeneous Problem 1 can have infinitely many solutions.

In the general case, the following statement is valid:

Theorem 2. *Let $0 < \alpha < 1$, $\mu \geqslant 0$, the coefficients of the operator L_n be such that the conditions $\varepsilon_k \neq 0$, $k = 0, 1, \ldots, 2^n - 1$, are satisfied, and $f(x) \in C^{\lambda+1}\left(\bar{\Omega}\right)$, $g(x) \in C^{\lambda+2}(\partial\Omega)$, $0 < \lambda < 1$. Then the following statements hold:*

1) *if $\mu > 0$, then a solution to Problem 1 exists and is unique;*
2) *if $\mu = 0$, then for the solvability of Problem 1 it is necessary and sufficient that the condition*

$$\int_{\Omega} f_{1-\alpha}(y)dy + \left(\sum_{i=0}^{2^n-1} a_i\right) \int_{\Omega} g(y)ds_y = 0 \tag{11}$$

is satisfied. If a solution to the problem exists, it is unique up to a constant term;

3) *if a solution to the problem exists, it is represented in the form*

$$u\left(x\right) = J_{\mu}^{\alpha}\left[v\right]\left(x\right), \tag{12}$$

where the function $v(x)$ is a solution to the problem

$$L_n v(x) = F(x),\ x \in \Omega, \tag{13}$$

$$v\left(x\right)|_{\partial\Omega} = g(x), \tag{14}$$

where $F(x) = D_{\mu+2}^{\alpha}[f](x)$. In case $\mu = 0$, the function $v(x)$ satisfies the additional condition $v(0) = 0$;

4) *if a solution to the problem exists, then $u(x) \in C^{\lambda+2}\left(\bar{\Omega}\right)$.*

Proof. Let $\mu > 0$ and the function $u\left(x\right)$ be a solution to Problem 1. Let us consider the function $v\left(x\right) = D_{\mu}^{\alpha} u\left(x\right)$. If we apply the operator Δ to this function, then, by virtue of equality (8), we obtain $\Delta v(x) = D_{\mu+2}^{\alpha}[\Delta u](x)$. Let S be an orthogonal matrix and $I_S u(x) = u(Sx)$. Then the operators I_S and Δ, as well as I_S and D_{μ}^{α}, commute. Therefore, for all $i = 0, \ldots, 2^n - 1$ we get $\Delta v(S_n^{i_n} \ldots S_1^{i_1} x) = D_2^{\alpha}[\Delta u](S_n^{i_n} \ldots S_1^{i_1} x)$ and, hence,

$$L_n v(x) = \sum_{i=0}^{2^n-1} a_i \Delta D^\alpha u(S_n^{i_n} \dots S_1^{i_1} x) =$$
$$= D_{\mu+2}^\alpha \left[\sum_{i=0}^{2^n-1} a_i \Delta^2 u(S_n^{i_n} \dots S_1^{i_1} x) \right] = D_{\mu+2}^\alpha [f](x), \ x \in \Omega.$$

Moreover, from the boundary condition (2) it follows that

$$v\,(x)|_{\partial\Omega} = D_\mu^\alpha [u](x)\big|_{\partial\Omega} = g(x).$$

Thus, if $u\,(x)$ is a solution to Problem 1, then for the function $v\,(x) = D_\mu^\alpha[u](x)$ we obtain the Dirichlet problem (13), (14) with the function $F(x) = D_{\mu+2}^\alpha[f](x)$. If $f\,(x) \in C^{\lambda+1}\left(\bar\Omega\right)$, then, due to Lemma 2, we get $D_{\mu+2}^\alpha\,[f]\,(x) \in C^\lambda\left(\bar\Omega\right)$. Then, by Theorem 1, for the functions $F(x) = D_{\mu+2}^\alpha[f](x) \in C^\lambda\left(\bar\Omega\right)$ and $g\,(x) \in C^{\lambda+2}\,(\partial\Omega)$ a solution to problem (13), (14) exists, is unique, and $v(x) \in C^{\lambda+2}\left(\bar\Omega\right)$. If we apply the operator J_μ^α, to the equality $v\,(x) = D_\mu^\alpha\,[u]\,(x)$ on both sides, then, by virtue of the assertion of Lemma 3, we obtain $u\,(x) = J_\mu^\alpha\,[v]\,(x)$, i. e., the solution to the problem is represented in the form (12). The inverse assertion is also valid, i.e., if the function $v\,(x)$ is a solution to problem (13), (14), then the function $u\,(x) = J_\mu^\alpha\,[v]\,(x)$ satisfies all the conditions of Problem 1. Indeed, as $v(x) \in C^{\lambda+2}\left(\bar\Omega\right)$, by the assertion of Lemma 1 the function $u\,(x) = J_\mu^\alpha\,[v]\,(x)$ also belongs to the class $C^{\lambda+2}\left(\bar\Omega\right)$. Further, if we apply the operator L_n to the function $u\,(x) = J_\mu^\alpha\,[v]\,(x)$, we get

$$L_n u\,(x) = L_n\left[J_\mu^\alpha\,[v]\right](x) = J_{\mu+2}^\alpha\,[L_n v]\,(x) =$$
$$= J_{\mu+2}^\alpha\left[D_{\mu+2}^\alpha\,[f]\right](x) = f(x), \ x \in \Omega.$$

Therefore, function (12) satisfies equation (1). In addition, from equality (7) it follows that

$$D_\mu^\alpha[u]\,(x)\big|_{\partial\Omega} = D_\mu^\alpha\left[J_\mu^\alpha\,[v]\right](x)\big|_{\partial\Omega} = v\,(x)|_{\partial\Omega} = g(x),$$

i. e., the boundary condition is also satisfied.

Further, we will consider the case $\mu = 0$. In this case, for the function $v\,(x) = D_0^\alpha u\,(x)$ we also obtain problem (13), (14) with the function $F(x) = D_2^\alpha[f](x)$. In addition, by virtue of the assertion of Lemma 2, the function $v\,(x) = D_0^\alpha u\,(x)$ must satisfy the additional condition $v(0) = 0$. From Lemma 6 it also follows that the function $F(x) = D_2^\alpha\,[f]\,(x)$ can be represented as $F(x) = (\delta + 2) f_{1-\alpha}(x)$. Further, in [20] it is proved (see problem (5.4)) that for the equality $v(0) = 0$ to be satisfied, it is necessary and sufficient that

$$\int\limits_{\Omega} f_{1-\alpha}(y)dy + \left(\sum_{i=0}^{2^n-1} a_i\right) \int\limits_{\partial\Omega} g(y)dy = 0.$$

Thus, the necessity of fulfilling condition (11) for the existence of a solution to Problem 1 is proved. The other part of the theorem is proved in the same way as in the case $\mu > 0$. The theorem is proved. □

Remark 1. *If $\alpha = 1$ and $\mu = 0$, then $f_0(x) = J_2^0[f](x) \equiv f(x)$ and then the solvability condition (11) coincides with condition (10).*

4. Existence and uniqueness of a solution to Problem 2. First, let us study the uniqueness of the solution to Problem 2. The following assertion is valid:

Theorem 3. *Let coefficients of the operator L_n be such that the conditions $\varepsilon_p \neq 0$, $p = 0, 1, \ldots, 2^n - 1$, are satisfied and a solution to Problem 2 exists. Then*

1) if $k = 1$ and $\beta = 0$, then the solution is unique;

2) in other cases, the solution is unique up to a constant term.

Proof. Let the function $u(x)$ be a solution to the homogeneous Problem 2. In [20] it is proved (see Lemma 2) that under the condition $\varepsilon_k \neq 0$, $k = 0, 1, \ldots, 2^n - 1$, the function $u(x)$ satisfying the equation $L_n u(x) = 0$, $x \in \Omega$, is harmonic in Ω. Hence, $u(x)$ satisfies the conditions of the following problem:

$$\Delta u(x) = 0, \ x \in \Omega, \tag{15}$$

$$D_0^\beta u(x) - (-1)^k D_0^\beta u(x^*) = 0, \ x \in \partial\Omega_+, \tag{16}$$

$$D_0^\alpha u(x) + (-1)^k D_0^\alpha u(x^*) = 0, \ x \in \partial\Omega_+. \tag{17}$$

Let $k = 1$ and $v(x) = u(x) - u(x^*)$. Note that for any $x \in \Omega$ there is the equality $v(x) = u(x) - u(x^*) = -[u(x^*) - u(x)] = -v(x^*)$.

Hence, if $x \in \partial\Omega_+$, then

$$D_0^\alpha v(x)|_{\partial\Omega_+} = D_0^\alpha u(x) - D_0^\alpha u(x^*)|_{\partial\Omega_+} = 0,$$

and if $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$ and

$$D_0^\alpha v(x)|_{x\in\partial\Omega_-} = -[D_0^\alpha u(x^*) - D_0^\alpha u(x)]|_{x^*\in\partial\Omega_+} = 0.$$

Therefore, the function $v(x)$ satisfies the conditions of the problem

$$\Delta v(x) = 0,\ x \in \Omega,\ D_0^\alpha v(x)|_{\partial\Omega} = 0. \tag{18}$$

Then, by the assertion of Theorem 2 (case $a_0 = 1$, $a_j = 0$, $j = 2, 3, \ldots, 2^n - 1$) we get $v(x) \equiv Const$. As $v(x) = -v(x^*)$, we see that $v(x) \equiv 0$, $x \in \bar{\Omega}$. Hence, $u(x) \equiv u(x^*)$, $x \in \bar{\Omega}$. From this equality it follows that $D_0^\beta u(x) \equiv D_0^\beta u(x^*)$, $x \in \partial\Omega$. On the other hand, from condition (16) we have $D_0^\beta u(x) = -D_0^\beta u(x^*)$, $x \in \partial\Omega_+$. So, we obtain the equality $D_0^\beta u(x) = 0$, $x \in \partial\Omega$. Thus, the function $u\,(x)$ satisfies the conditions of the problem

$$\Delta u(x) = 0,\ x \in \Omega,\ D_0^\beta u(x)\Big|_{\partial\Omega} = 0. \tag{19}$$

If $\beta = 0$, then problem (19) coincides with the Dirichlet problem and, therefore, $u(x) \equiv 0$, $x \in \bar{\Omega}$. If $\beta > 0$, then by the assertion of Theorem 2 we obtain $u(x) \equiv C$. If $k = 2$, then for the function $v(x) = u(x) - u(x^*)$ we obtain the problem

$$\Delta v(x) = 0,\ x \in \Omega,\ D_0^\beta v(x)\Big|_{\partial\Omega} = 0. \tag{20}$$

In this case, for all $0 \leqslant \beta < \alpha$ we also get $v(x) \equiv 0$, $x \in \bar{\Omega}$. Then $u(x) \equiv u(x^*)$, $x \in \bar{\Omega}$ and, hence, $D_0^\alpha u(x) \equiv D_0^\alpha u(x^*)$, $x \in \bar{\Omega}$. On the other hand, from condition (17) it follows that $D_0^\alpha u(x) = -D_0^\alpha u(x^*)$, $x \in \partial\Omega_+$, which is possible only in the case $D_0^\alpha u(x) = 0$, $x \in \partial\Omega$. Thus, the function $u\,(x)$ satisfies the conditions of the problem

$$\Delta u(x) = 0,\ x \in \Omega,\ D_0^\alpha u(x)|_{\partial\Omega} = 0. \tag{21}$$

Then, by the assertion of Theorem 2 we obtain $u(x) \equiv C$. The theorem is proved. □

Let us consider the existence of a solution to Problem 2. In the case $k = 1$, the following assertion is valid:

Theorem 4. *Let $k = 1$, and the coefficients of the operator L_n be such that the conditions $\varepsilon_p \neq 0$, $p = 0, 1, \ldots, 2^n - 1$, are satisfied and $g_0(x), g_1(x) \in C^{\lambda+2}\,(\partial\Omega_+)$, $f(x) \in C^\lambda\left(\bar{\Omega}\right), 0 < \lambda < 1$. Then*

1) if $\beta = 0$, then a solution to problem 2 exists and is unique;

2) *if $\beta > 0$, then for the solvability of Problem 2 it is necessary and sufficient that the condition*

$$\int\limits_{\Omega} f_{1-\alpha}(y)dy + \left(\sum_{i=0}^{2^n-1} a_i\right) \int\limits_{\partial\Omega_+} g_0(y)ds_y = 0 \qquad (22)$$

be satisfied.

If a solution to the problem exists, it is unique up to a constant term and belongs to the class $C^{\lambda+2}\left(\bar{\Omega}\right)$.

Proof. Let $k = 1$ and $u(x)$ be the solution to Problem 2. Let us represent this function as $u(x) = v(x) + w(x)$, where

$$v(x) = \frac{1}{2}\left[u(x) - u(x^*)\right], \quad w(x) = \frac{1}{2}\left[u(x) + u(x^*)\right].$$

Let us find the problems that the functions $v(x)$ and $w(x)$ satisfy. By assumption, $u(x)$ satisfies equation (1), i. e.,

$$\sum_{i=0}^{2^n-1} a_i \Delta u\left(S_n^{i_n} \dots S_2^{i_2} S_1^1 x\right) = f(x), \;\; x \in \Omega.$$

As the operators $I_{S_n^{j_n} \dots S_2^{j_2} S_1^1}$ and Δ commute, at the point $x^* = S_n^{j_n} \dots S_2^{j_2} S_1^1 x$ we get

$$\sum_{i=0}^{2^n-1} a_i \Delta u\left(S_n^{i_n} \dots S_2^{i_2} S_1^{i_1} x^*\right) = \sum_{i=0}^{2^n-1} a_i \Delta I_{S_n^{j_n} \dots S_2^{j_2} S_1^1} u\left(S_n^{i_n} \dots S_2^{i_2} S_1^{i_1} x\right) =$$

$$= I_{S_n^{j_n} \dots S_2^{j_2} S_1^1} \sum_{i=0}^{2^n-1} a_i \Delta u\left(S_n^{i_n} \dots S_2^{i_2} S_1^{i_1} x\right) = f(x^*).$$

Hence, for the functions $v(x)$ and $w(x)$ we get:

$$L_n v(x) = \frac{1}{2}\left[L_n u(x) - L_n u(x^*)\right] = \frac{1}{2}\left[f(x) - f(x^*)\right] \equiv f^-(x), \;\; x \in \Omega,$$

$$L_n w(x) = \frac{1}{2}\left[L_n u(x) + L_n u(x^*)\right] = \frac{1}{2}\left[f(x) + f(x^*)\right] \equiv f^+(x), \;\; x \in \Omega.$$

It is clear, that if $f(x) \in C^{\lambda+k}\left(\bar{\Omega}\right)$, $0 < \lambda < 1$, $k = 0, 1, \dots$, then the functions $f^{\pm}(x)$ also belong to the class $C^{\lambda+k}\left(\bar{\Omega}\right)$.

If $x \in \partial\Omega_+$, then, by virtue of condition (3), we have

$$D_0^\alpha v(x)|_{\partial\Omega_+} = \frac{1}{2}\left[D_0^\alpha u(x) - D_0^\alpha u(x^*)\right]\Big|_{\partial\Omega_+} = \frac{1}{2}g_1(x),$$

and if $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$, and thus:

$$D_0^\alpha v(x)|_{\partial\Omega_-} = \frac{1}{2}\left[D_0^\alpha u(x) - D_0^\alpha u(x^*)\right]|_{\partial\Omega_-} =$$

$$= -\frac{1}{2}\left[D_0^\alpha u(x^*) - D_0^\alpha u(x)\right]|_{x^*\in\partial\Omega_+} = -\frac{1}{2}g_1(x^*).$$

Similarly, taking into account condition (2) for the function $w(x)$, we obtain

$$D_0^\beta w(x)\Big|_{\partial\Omega_+} = \frac{1}{2}\left[D_0^\beta u(x) + D_0^\beta u(x^*)\right]\Big|_{\partial\Omega_+} = \frac{1}{2}g_0(x),$$

$$D_0^\beta w(x)\Big|_{x\in\partial\Omega_-} = \frac{1}{2}\left[D_0^\beta u(x^*) + D_0^\beta u(x)\right]\Big|_{x^*\in\partial\Omega_+} = \frac{1}{2}g_0(x^*).$$

Let us introduce the functions

$$2\tilde{g}_0(x) = \begin{cases} g_0(x), & x \in \partial\Omega_+, \\ g_0(x^*), & x \in \partial\Omega_-, \end{cases} \quad 2\tilde{g}_1(x) = \begin{cases} g_1(x), & x \in \partial\Omega_+, \\ -g_1(x^*), & x \in \partial\Omega_-. \end{cases}$$

Then, for $v(x)$ and $w(x)$, we get the following problems:

$$L_{\,n} v(x) = f^-(x), \;\; x \in \Omega; \;\; D_0^\alpha v(x)|_{\partial\Omega} = \tilde{g}_1(x), \tag{23}$$

$$L_{\,x} w(x) = f^+(x), \;\; x \in \Omega; \;\; D_0^\beta w(x)\Big|_{\partial\Omega} = \tilde{g}_0(x). \tag{24}$$

If $g_1(x) \in C^{\lambda+2}(\partial\Omega_+)$ and the matching condition (6) is satisfied, then $\tilde{g}_1(x) \in C^{\lambda+2}(\partial\Omega)$. Then, according to Theorem 2, for the solvability of problem (23) it is necessary and sufficient that the condition

$$\int\limits_\Omega f_{1-\alpha}^-(y)dy + \left(\sum_{i=0}^{2^n-1} a_i\right) \int\limits_{\partial\Omega} \tilde{g}_1(y)ds_y = 0. \tag{25}$$

be satisfied, where $f_{1-\alpha}^-(y) = \dfrac{1}{2}\left[f_{1-\alpha}(y) - f_{1-\alpha}(y^*)\right]$. If this condition is satisfied, then the solution to the problem exists, and it is unique up to a constant term. Let us study the integrals in equality (25). In [22] it is proved that if S is an orthogonal matrix, then the equalities

$$\int\limits_{\Omega} f(Sy)dy = \int\limits_{\Omega} f(y)dy, \quad \int\limits_{\partial\Omega_-} g(Sy)ds_y = \int\limits_{\partial\Omega_+} g(y)ds_y. \tag{26}$$

are valid.

From these equalities for the functions $f_{1-\alpha}^{-}(y)$ and $\tilde{g}_1(y)$, we get

$$\int\limits_{\Omega} f_{1-\alpha}^{-}(y)dy = \frac{1}{2}\int\limits_{\Omega} \left[f_{1-\alpha}(y) - f_{1-\alpha}(y^*)\right] dy = 0, \quad \int\limits_{\partial\Omega} \tilde{g}_1(y)ds_y = 0.$$

Thus, the solvability condition (25) is satisfied and, therefore, a solution to problem (23) exists and is unique up to a constant term. As the function $v(x)$ has the property $v(x^*) = -v(x)$, this is possible only in the case $C \equiv 0$.

Now we can consider problem (24). If $\beta = 0$, then this problem coincides with the Dirichlet problem and, therefore, according to Theorem 1, the problem is unconditionally solvable. In the case $\beta > 0$, the solvability condition is written as

$$\int\limits_{\Omega} f_{1-\alpha}^{+}(y)dy + \left(\sum_{i=0}^{2^n-1} a_i\right) \int\limits_{\partial\Omega} \tilde{g}_0(y)ds_y = 0. \tag{27}$$

where

$$f_{1-\alpha}^{+}(y) = \frac{1}{2}\left[f_{1-\alpha}(y) + f_{1-\alpha}(y^*)\right].$$

If this condition is satisfied, the solution to the problem exists and is unique up to a constant term. Further, from equalities (26), for the integrals from (27) we have:

$$\int\limits_{\Omega} f_{1-\alpha}^{+}(y)dy = \frac{1}{2}\int\limits_{\Omega} \left[f_{1-\alpha}(y) + f_{1-\alpha}(y^*)\right] dy = \int\limits_{\Omega} f_{1-\alpha}(y)dy,$$

$$\int\limits_{\partial\Omega} \tilde{g}_0(y)ds_y = \frac{1}{2}\left[\int\limits_{\partial\Omega_+} g_0(y)ds_y + \int\limits_{\partial\Omega_-} g_0(y^*)ds_y\right] = \int\limits_{\partial\Omega_+} g_0(y)ds_y.$$

Then the solvability condition (27) can be rewritten as (22). Thus we have found the conditions under which solutions to problems (21) and (24) exist. The function $u(x) = v(x) + w(x)$ constructed from the solutions to these problems satisfies all the conditions of Problem 2. The theorem is proved. $\square$

The following assertion is proved in a similar way.

Theorem 5. *Let $k = 2$, the coefficients of the operator L_n be such that the following conditions $\varepsilon_p \neq 0$, $p = 0, 1, \ldots, 2^n - 1$, are satisfied, and $g_0(x), g_1(x) \in C^{\lambda+2}(\partial\Omega_+)$, $f(x) \in C^{\lambda}(\bar{\Omega})$, $0 < \lambda < 1$. Then, for the solvability of Problem 2, it is necessary and sufficient that the condition*

$$\int\limits_{\Omega} f_{1-\alpha}(y)dy + \left(\sum_{i=0}^{2^n-1} a_i\right) \int\limits_{\partial\Omega_+} g_1(y)ds_y = 0$$

be satisfied.

If a solution to the problem exists, then it is unique up to a constant term and belongs to the class $C^{\lambda+2}(\bar{\Omega})$.

Acknowledgment. This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488086).

References

[1] Ashurov R., Fayziev Y. *On some boundary value problems forequations with boundary operatorsof fractional order.* IJAM. 2021, vol. 34, no. 2, pp. 283 – 295. DOI: http://dx.doi.org/10.12732/ijam.v34i2.6

[2] Ashurov R. R., Kadirkulov B. J., Turmetov B. Kh. *On the inverse problem of the Bitsadze-Samarskii type for a fractional parabolic equation.* Probl. Anal. Issues Anal., 2023, no. 3, pp. 20 – 40. DOI: https://doi.org/10.15393/j3.art.2023.14370

[3] Bitsadze A. V., Samarskiy A. A. *On some simplest generalizations of linear elliptic boundary value problems.* Dokl. Akad. Nauk SSSR. 1969, vol. 185, no. 4, pp. 739 – 740. (in Russian)

[4] Carleman T. *Sur la theorie des equations integrales et ses applications.* Verh. Int. Math. Kongress. Zurich. Bd I, 1932, pp. 138– 151.

[5] Dunkl C. F. *Integral kernels reflection group invariance.* Canad. J. Math., 1991, vol. 43, pp. 1213 – 1227. DOI: http://doi.org/10.4153/CJM-1991-069-8

[6] Garif'yanov F. N., Strezhneva E. V. *On a Carleman problem in the case of a doubly periodic group.* Probl. Anal. Issues Anal., 2022, no. 3, pp. 45 – 55. DOI: https://doi.org/10.15393/j3.art.2023.14570

[7] Habbachi Y. *Second structure relation for the Dunkl-classical orthogonal polynomials.* Probl. Anal. Issues Anal., 2023, no. 3, pp. 86 – 104. DOI: https://doi.org/10.15393/j3.art.2023.13333

[8] Habbachi Y., Bouras B. *A note for the Dunkl-classical polynomials.* Probl. Anal. Issues Anal., 2022, no. 2, pp. 29 – 41. DOI: https://doi.org/10.15393/j3.art.2022.11310

[9] Hadi S. H., Darus M. *A class of harmonic (p, q)-starlike functions involving a generalized (p, q)-Bernardi integral operator.* Probl. Anal. Issues Anal., 2023, no. 2, pp. 17 – 36. DOI: https://doi.org/10.15393/j3.art.2023.12850

[10] Kadirkulov B. J., Kirane M. *On solvability of a boundary value problem for the Poisson equation with a nonlocal boundary operator.* Acta Math. Sci., 2015, vol 35, no. 5, pp. 970. – 980. DOI: https://doi.org/10.1016/S0252-9602(15)30031-X

[11] Kilbas A. A., Srivastava H. M., Trujillo J. J. *Theory and applications of fractional differential equations.* North-Holland Mathematics Studies, 2006.

[12] Kirane M., Torebek B. T. *On a nonlocal problem for the Laplace equation in the unit ball with fractional boundary conditions.* Math. Methods Appl. Sci., 2016, vol. 39, no. 6, pp. 1121 – 1128. DOI: https://doi.org/10.1002/mma.3554

[13] Krasnoschok M., Vasylyeva N. *On a nonclassical fractional boundary-value problem for the Laplace operator.* J. of Differ. Equat., 2014, vol. 257, no. 6, pp. 1814 – 1839. DOI: https://doi.org/10.1016/j.jde.2014.05.022

[14] Sadybekov M. A., Turmetov B. Kh. *On an analog of periodic boundary value problems for the Poisson equation in the disk.* Differ. Equ., 2014, vol. 50, no. 7, pp. 268 – 273. DOI: https://doi.org/10.1134/S0012266114020153

[15] Sadybekov M. A., Turmetov B. Kh. *On analogues of periodic boundary value problems for the Laplace operator in a ball.* Eurasian Math. J., 2012, vol. 3, no. 1, pp. 143 – 146.

[16] Sadybekov M. A., Dukenbayeva A. A. *Direct and inverse problems for the Poisson equation with equality of flows on a part of the boundary.* Complex Var. Elliptic Equ., 2019, vol. 64, no. 5, pp. 777 – 791. DOI: https://doi.org/10.1080/17476933.2018.1517340

[17] Sadybekov M. A., Dukenbayeva A. A. *On boundary value problem of the Samarskii-Ionkin type for the Laplace operator in a ball.* Complex Var. Elliptic Equ.,2022, vol. 67, no. 2, pp. 369 – 383.
DOI: https://doi.org/10.1080/17476933.2020.1828377

[18] Turmetov B. Kh. *On the solvability of some boundary value problems for the inhomogeneous polyharmonic equation with boundary operators of the Hadamard type.* Differ. Equ., 2017, vol. 53, no. 3, pp. 333 – 344.
DOI: https://doi.org/10.1134/S0012266117030053

[19] Turmetov B. Kh. *Generalization of the Robin Problem for the Laplace Equation.* Differ. Equ., 2019, vol. 55, no. 9, pp. 1134 – 1142.
DOI: https://doi.org/10.1134/S0012266119090027

[20] Turmetov B. Kh., Karachik V. V. *On solvability of the Dirichlet and Neumann boundary value problems for the Poisson equation with multiple involution.* Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 2021, vol. 31, no. 4, pp. 651 – 667. (in Russian)
DOI: https://doi.org/10.35634/vm210409

[21] Turmetov B. Kh., Karachik V. V. *On Eigenfunctions and Eigenvalues of a Nonlocal Laplace Operator with Multiple Involution* . Symmetry, 2021, vol. 13, no. 1781, pp. 1 – 20.
DOI: https://doi.org/10.3390/sym13101781

[22] Turmetov B. Kh., Karachik V. V. *On Solvability of Some Boundary Value Problems for a Biharmonic Equation with Periodic Conditions.* Filomat, 2018, vol. 32, no. 3, pp. 947 – 953.
DOI: https://doi.org/10.2298/FIL1803947T

[23] Turmetov B., Koshanova M., Muratbekova M. *On periodic problems for the nonlocal Poisson equation in the circle.* IJAM. 2023, vol. 36, no. 5, pp. 735 – 746. DOI: http://dx.doi.org/10.12732/ijam.v36i5.11

[24] Umarov S. R. *On some boundary value problems for elliptic equations with a boundary operator of fractional order.* Dokl. Math., 1994, vol. 48, no. 3, pp. 655 – 658.

[25] Yessirkegenov N. *Spectral properties of the generalized Samarskii Ionkin type problems.* Filomat, 2018, vol. 32, no. 3, pp. 1019 – 1024.
DOI: https://doi.org/10.2298/FIL1803019Y

Received August 22, 2024.
In revised form, October 13, 2024.
Accepted October 13, 2024.
Published online November 10, 2024.

B. Kh. Turmetov
Khoja Akhmet Yassawi International Kazakh-Turkish University
29 B. Sattarhanov ave., Turkistan 161200, Kazakhstan
Alfraganus University
2a Yukori Karakamish Str., Tashkent, 100190, Uzbekistan
E-mail: batirkhan.turmetov@ayu.edu.kz