DOI: 10.15393/j3.art.2025.16610

UDC 517.53

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REFINEMENT OF ERDÖS-LAX INEQUALITY FOR N-OPERATOR

Abstract. Let $\mathcal{P}_n$ be the space of all polynomials of degree less than or equal to n. In this paper, we establish a refinement of Erdös-Lax inequality in which the classical derivative (as an operator on $\mathcal{P}_n$) is replaced by a B_n operator. The result obtained includes some interesting inequalities as special cases.

Key words: *inequalities, N-operator, polynomials, zeros*

2020 Mathematical Subject Classification: *30A06, 30A64, 30E10*

1. Introduction. Let $\mathbb{C}$ denote the set of all complex numbers. For a subset $H \subseteq \mathbb{C}$, we denote by $\mathcal{P}_n(H) \subseteq \mathcal{P}_n$ the set of all those polynomials in $\mathcal{P}_n$ whose zeros lie in H. Further, let $\Omega^+ = \{z \in \mathbb{C}\colon |z| > 1\}$, $\Omega^- = \{z \in \mathbb{C}\colon |z| < 1\}$ and $\partial\Omega = \{z \in \mathbb{C}\colon |z| = 1\}$. For a complex function $f(z)$, the Hardy space q-norm is given by

$$\|f(z)\|_q := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}, \qquad 0 < q < \infty.$$

It is a well known fact that $\lim\limits_{q\to\infty} \|f(z)\|_q = \max\limits_{z\in\partial\Omega} |f(z)|$ and for this reason we write

$$\|f(z)\|_\infty := \max_{z\in\partial\Omega} |f(z)|.$$

The basic result on the extremal problems of Markov and Bernstein type [26] was related with some investigations by the well-known Russian chemist Mendeleev [14]. Mendeleev's problem, after some reductions, was that if $P(x)$ is an arbitrary quadratic polynomial and $|P(x)| \leqslant 1$ on $[-1, 1]$, how large can $|P'(x)|$ be on $[-1, 1]$? Mendeleev himself found that

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$|P'(x)| \leqslant 4$ on $[-1, 1]$, which is the best possible result with the extremal polynomial $P(x) = 1 - 2x^2$.

Later, in 1889, A. A. Markov [13] managed to solve the original problem of Mendeleev. In fact, Markov established that *if $P \in \mathcal{P}_n$ is a polynomial of degree n, such that $|P(x)| \leqslant M$ for $x \in [a, b]$*, then

$$|P'(x)| \leqslant Mn^2.$$

Here the equality is attained only if

$$P(x) = \pm MT_n\left(\frac{2x - a - b}{b - a}\right),$$

where $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials.

An analogue of Markov's theorem for the unit disk in the complex plane instead of the interval $[-1, 1]$ was formulated by Bernstein [5]. In terms of the supremum norm, the Bernstein inequality [5] states that if $P(z)$ is a polynomial of degree n, then

$$\|P'(z)\|_\infty \leqslant n\|P(z)\|_\infty. \tag{1}$$

There are many results on the aforementioned inequalities due to Markov and Bernstein, and there are generalizations in various metrics and restricted classes of polynomials. Several monographs and papers have been published in this area (see, for example, [11], [19], [20], [25]).

For any $P \in \mathcal{P}_n$ of degree n, Zygmund [28] extended inequality (1) to the integral mean and proved for $q \geqslant 1$ that

$$\|P'(z)\|_q \leqslant n\|P(z)\|_q. \tag{2}$$

De Bruijin and Springer [8] proved the inequality (2) for $q = 0$, and for the remaining values $0 < q < 1$, the inequality (2) was established by Arestov [2]. Furthermore, we have the following inequality for $R > 1$:

$$\|P(Rz)\|_q \leqslant R^n\|P(z)\|_q, \qquad q > 0. \tag{3}$$

Inequality (3) is a simple consequence of a result due to Hardy [10].

While restricting the zeros of $P(z)$, De-Bruijin [7] refined inequality (2) and established the fact that for every $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$ and $q \geqslant 1$, the following inequality holds:

$$\|P'(z)\|_q \leqslant \frac{n}{\|1 + z\|_q}\|P(z)\|_q. \tag{4}$$

The corresponding case of inequality (3) was verified by Boas and Rahman [6] by proving the fact that

$$\|P(Rz)\|_q \leqslant \frac{\|R^n z+1\|_q}{\|1+z\|_q}\|P(z)\|_q, \qquad \text{for } R>1 \text{ and } q\geqslant 1. \tag{5}$$

Inequalities (4) and (5) have also been proved for $0\leqslant q<1$ by Rahman and Schmeisser [21].

An operator $T\colon \mathcal{P}_n \to \mathcal{P}_n$ is said to be a B_n-operator if for every polynomial $P\in\mathcal{P}_n$ that has all zeros in $|z-z_0|\leqslant 1$ for some complex number z_0, the image $T[P]$ has all its zeros in $|z-z_0|\leqslant 1$.

Now, for a polynomial $P\in\mathcal{P}_n$ of degree n, if we choose the complex numbers λ_0, λ_1, and λ_2, such that

$$g(z)=\lambda_0+\binom{n}{1}\lambda_1 z+\binom{n}{2}\lambda_2 z^2,$$

has all zeros in the half-plane $|z|\leqslant \left|z-\frac{n}{2}\right|$, then it is an established fact (see [15], corollary (18, 3)) that the polynomial

$$B[P(z)]=\lambda_0 P(z)+\lambda_1\left(\frac{nz}{2}\right)\frac{P'(z)}{1!}+\lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!},$$

has all zeros in $\Omega^-\cup\partial\Omega$ whenever P has all zeros in $\Omega^-\cup\partial\Omega$. This shows that B is a B_n operator. This operator was first investigated by Rahman [18]. He observed that Bernstein-type inequalities remain preserved if the classical derivative is replaced by the operator B. In fact, in [18] it is proved that if $P\in\mathcal{P}_n$ is of degree n, then for $R\geqslant 1$

$$\|B[P(Rz)]\|_\infty \leqslant R^n|\Lambda|\|P(z)\|_\infty, \tag{6}$$

where

$$\Lambda=\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}. \tag{7}$$

Clearly, inequality (6) yields Bernstein inequality (1) as a special case by just choosing $\lambda_0=\lambda_2=0$ and $R=1$.

Recently Rather and Shah [24] proved a more general result than (6) by extending the sup-norm $\|\cdot\|_\infty$ to the Hardy space q-norm. In fact, they proved the following results:

Theorem 1. *If $P \in \mathcal{P}_n$, then for $R > 1$ and $0 \leqslant q < \infty$, the following inequality holds:*

$$\|B[P(Rz)]\|_q \leqslant R^n|\Lambda|\|P(z)\|_q, \tag{8}$$

where Λ is defined in (7). The result is sharp as shown by $P(z) = \alpha z^n$, $\alpha \neq 0$.

Theorem 2. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for every $R > 1$ and $0 \leqslant q < \infty$, we have*

$$\|B[P(Rz)]\|_q \leqslant \frac{\|R^n\Lambda z + \lambda_0\|_q}{\|1+z\|_q}\|P(z)\|_q, \tag{9}$$

where Λ is defined in (7). The result is sharp and the extremal polynomial is $P(z) = az^n + b$, $|a| = |b| \neq 0$.

By introducing the minimum value of $|P(z)|$ on $|z| = 1$, S. L. Wali [27] proved the following refinement of Theorem 2:

Theorem 3. *Let $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$. Then, for every complex number β with $|\beta| \leqslant 1$, $R > 1$ and $0 \leqslant q < \infty$,*

$$\Big\|B[P(Rz)] + \Big(\frac{R^n|\Lambda| - |\lambda_0|}{2}\Big)m|\beta|\Big\|_q \leqslant \frac{\|R^n\Lambda z + \lambda_0\|_q}{\|1+z\|_q}\|P(z)\|_q, \tag{10}$$

where $m = \min_{z\in\partial\Omega}|P(z)|$ and Λ is defined as in (7). The result is sharp and equality holds if $P(z) = z^n + 1$.

It can be seen in [23] that the operator B has been recently extended to an operator $N\colon \mathcal{P}_n \to \mathcal{P}_n$ by Rather et. al, who defined it by involving first s derivatives of the underlying polynomial through the expression

$$N[P(z)] := \sum_{\nu=0}^{s} \lambda_\nu \Big(\frac{nz}{2}\Big)^\nu \frac{P^{(\nu)}(z)}{\nu!}, \quad \forall\ P \in \mathcal{P}_n, \quad \text{and} \quad z \in \mathbb{C}, \tag{11}$$

where the numbers λ_ν for $\nu = 0, 1, \ldots, s$ are chosen such that the polynomial $\phi(z) = \sum_{\nu=0}^{s} \binom{s}{\nu}\lambda_\nu z^\nu$ has all zeros in the half-plane $\mathrm{Re}(z) \leqslant \frac{n}{4}$. To confirm that the operator N is a B_n operator, N. A. Rather et. al. [23] first proved the following result:

Theorem 4. *If all zeros of a polynomial $P(z)$ of degree n lie in $|z| \leqslant k$ and if all zeros of the polynomial*

$$\phi(z) = \lambda_0 + \binom{n}{1}\lambda_1 z + \cdots + \binom{n}{s}\lambda_s z^s, \qquad s \leqslant n,$$

lie in $|z| \leqslant \mu|z-\sigma|$, $\mu > 0$, then the polynomial

$$h(z) = \lambda_0 P(z) + \lambda_1 \frac{(\sigma z)}{1!} P'(z) + \cdots + \lambda_s \frac{(\sigma z)^s}{s!} P^{(s)}(z)$$

has all its zeros in $|z| \leqslant k\max(1,\mu)$.

Taking $\mu = 1$, $\sigma = \frac{n}{2}$ and $k = 1$ in Theorem 4, we observe that the map N defined in (11) is a B_n-operator.

It is pertinent to mention that the operator N reduces to operator B by simply choosing $\lambda_\nu = 0$ for $\nu = 3, 4, \ldots, s$ in (11). Like the operator B, operator N also preserves different types of inequalities involving a complex polynomial. For instance, the following results obtained by A. Mir [16] show that the Bernstein inequality in [4] and Erdös-Lax inequality in [12] do not alter if the classical derivative is replaced by N-operator.

Theorem 5. *If $P \in \mathcal{P}_n$ has degree n, then*

$$\|N[P(Rz)]\|_\infty \leqslant R^n |N[z^n]|\ \|P\|_\infty, \qquad for\ z \in \partial\Omega \text{ and } R \geqslant 1. \tag{12}$$

Equality holds in (12) if $P(z) = az^n$, $a \neq 0$.

Theorem 6. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for $R \geqslant 1$:*

$$\|N[P(Rz)]\|_\infty \leqslant \frac{1}{2}\{R^n N[z^n] + |\lambda_0|\}\|P(z)\|_\infty \quad for\ z \in \Omega^+ \cup \partial\Omega. \tag{13}$$

Equality in (13) holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$.

The following theorem, which generalizes Theorem 2 to the Hardy-space q-norm, has been recently established by A. Mir et. al [17].

Theorem 7. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for any complex number α with $|\alpha| \leqslant 1$, $0 \leqslant p \leqslant \infty$, and $R \geqslant 1$, the following holds:*

$$\|N[P(Rz)] - \alpha N[P(z)]\|_q \leqslant \frac{\|(R^n - \alpha)\Lambda_s z + (1-\alpha)\lambda_0\|_q}{\|1+z\|_q}\|P(z)\|_q. \tag{14}$$

The result is the best possible and equality holds in (14) for $P(z) = az^n + b$, $|a| = |b| = 1$.

Definition 1. *(**Admissible C_γ-operator**) If $P(z) = \sum\limits_{\nu=o}^{n} a_\nu z^\nu \in \mathcal{P}_n$, then for any $n+1$ dimensional complex vector $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{C}^{n+1}$ Arestov [2] defined an operator C_γ acting on $P(z)$ as*

$$C_\gamma P(z) = \sum_{\nu=0}^{n} \gamma_\nu a_\nu z^\nu.$$

The operator C_γ is said to be admissible if it preserves one of the following properties:

(i) $P \in \mathcal{P}_n(\Omega^- \cup \partial\Omega)$,

(ii) $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$.

In this paper, we establish a refinement of Theorem 7, which simultaneously provides the extension of inequality (13) in Hardy space q-norm. In fact, we prove

Theorem 8. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for α, $\beta \in \mathbb{C}$ with $|\alpha| \leqslant 1$ and $|\beta| \leqslant 1$, $R > r \geqslant 1$ and $0 \leqslant q < \infty$,*

$$\left\| |N[P(Rz)] - \alpha N[P(rz)]| + \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_s| - |1-\alpha||\lambda_0|)}{2} \right\|_q \leqslant$$
$$\leqslant \frac{\|(R^n - \alpha r^n)\Lambda_s z + (1-\alpha)\lambda_0\|_q}{\|1+z\|_q} \|P(z)\|_q, \quad (15)$$

where $m = \min\limits_{z\in\partial\Omega} |P(z)|$ and $\Lambda_s = \sum\limits_{\nu=0}^{s} \lambda_\nu \left(\frac{n}{2}\right)^\nu \binom{n}{\nu}$ with $\lambda_\nu, 0 \leqslant \nu \leqslant s$, such that the polynomial $\phi(z) = \sum\limits_{\nu=0}^{s} \binom{n}{\nu} \lambda_\nu z^\nu$, $s \leqslant n$, has all zeros in the region $\mathrm{Re}(z) \leqslant \frac{n}{4}$. *The result is sharp and equality holds for $P(z) = cz^n + d$ with $|c| = |d| = 1$.*

Assume that the polynomial $\phi(z)$ has zeros as $w_1, w_2, \ldots, w_s$; then

$$|w_1 w_2 \cdots w_s| = |w_1||w_2| \cdots |w_s| \leqslant |w_1 - n/2||w_2 - n/2| \cdots |w_s - n/2| =$$
$$= \left|\sigma_s - \left(\frac{n}{2}\right)\sigma_{s-1} + \cdots + (-1)^s \left(\frac{n}{2}\right)^s \sigma_0\right|, \quad (16)$$

where σ_i are elementary symmetric polynomials in the zeros $w_1, w_2, \ldots, w_s$ for each $i = 1, 2, \ldots, s$ and $\sigma_0 = 1$. Using Viéte formulae (see Rahman and Schmeisser [22], p.6) in (16), we get $|\lambda_0| \leqslant |\Lambda_s|$. Again $R > r \geqslant 1$ and $|\alpha| \leqslant 1$; therefore, we have due to Lemma 8 (proved in lemma section):

$$|R^n - \alpha r^n||\Lambda_s| = r^n \left|\frac{R^n}{r^n} - \alpha\right||\Lambda_s| \geqslant r^n|1-\alpha||\Lambda_s| \geqslant |1-\alpha||\lambda_0|.$$

This gives

$$\|N[P(Rz)] - \alpha N[P(rz)]\|_q \leqslant \Bigg\||N[P(Rz)] - \alpha N[P(rz)]| + \\ + \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_s| - |1-\alpha||\lambda_0|)}{2}\Bigg\|_q,$$

which shows, after taking $(r = 1)$, that Theorem 8 is a refinement of Theorem 7. Further, if we take $\alpha = 0$ in Theorem 8, we get the following result:

Corollary 1. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $R \geqslant 1$ and $0 \leqslant q < \infty$:*

$$\left\||N[P(Rz)]| + \frac{m|\beta|(|R^n|\Lambda_s| - |\lambda_0|)}{2}\right\|_q \leqslant \frac{\|R^n\Lambda_s z + \lambda_0\|_q}{\|1+z\|_q}\|P(z)\|_q, \quad (17)$$

where $m = \min_{z\in\partial\Omega}|P(z)|$ and $\Lambda_s = \sum_{\nu=0}^{s} \lambda_\nu \left(\frac{n}{2}\right)^\nu \binom{n}{\nu}$ with $\lambda_\nu, 0 \leqslant \nu \leqslant s$, such that the polynomial $\phi(z) = \sum_{\nu=0}^{s}\binom{n}{\nu}\lambda_\nu z^\nu$, $s \leqslant n$, has all zeros in the region $\mathrm{Re}(z) \leqslant \frac{n}{4}$. The result is sharp and equality holds for $P(z) = cz^n + d$ with $|c| = |d| = 1$.

Taking $s = 0$ in the above Corollary 1, we get the following result.

Corollary 2. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $R \geqslant 1$ and $0 \leqslant q < \infty$,*

$$\left\||P(Rz)| + \frac{m|\beta|(|R^n - 1)}{2}\right\|_q \leqslant \frac{\|R^n z + 1\|_q}{\|1+z\|_q}\|P(z)\|_q, \quad (18)$$

where $m = \min_{z\in\partial\Omega}|P(z)|$. The result is sharp and equality holds for $P(z) = cz^n + d$ with $|c| = |d| = 1$.

If we let $q \to \infty$ in (18), we get a generalization of the result due to Ankeny and Rivilin [1].

Further if we choose $\lambda_\nu = 0$ for all $\nu = 3, 4, \cdots, s$ in Theorem 8, we obtain the following result:

Corollary 3. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leqslant 1$ and $|\beta| \leqslant 1$, $R > r \geqslant 1$ and $0 \leqslant q < \infty$,*

$$\left\| |B[P(Rz)] - \alpha B[P(rz)]| + \frac{m|\beta|(|R^n - \alpha r^n||\Lambda| - |1-\alpha||\lambda_0|)}{2} \right\|_q \leqslant$$
$$\leqslant \frac{\|(R^n - \alpha r^n)\Lambda z + \lambda_0(1-\alpha)\|_q}{\|1+z\|_q}\|P(z)\|_q,$$

where Λ is defined as in (7). The result is the best possible and equality holds for $P(z) = az^n + b$ with $|a| = |b| = 1$.

Remark 1. *For $\alpha = 0$, Corollary 3 reduces to Theorem 3.*

Remark 2. *If we make $q \to \infty$ in (15) and choose argument of β suitably, we obtain the following inequality due to A. Mir [16]:*

$$\|N[P(Rz)] - \alpha N[P(rz)]\|_\infty \leqslant \frac{1}{2}\Big[\big(|R^n - \alpha r^n||\Lambda_m| + |1-\alpha||\lambda_0|\big)\|P(z)\|_\infty -$$
$$- \big(|R^n - \alpha r^n|\Lambda_s| - |1-\alpha||\lambda_0|\big)m\Big],$$

with equality for $P(z) = az^n + b$ with $|a| = |b| = 1$.

2. Lemmas. We need the following lemmas to prove our main theorem.

Lemma 1. *If $P \in \mathcal{P}_n(\Omega^- \cup \partial\Omega)$, then $N[P] \in \mathcal{P}_n(\Omega^- \cup \partial\Omega)$.*

This lemma follows by taking $k = \mu = 1$ and $\sigma = \frac{n}{2}$ in Theorem 4. The next lemma is due to Govil et al. [9]:

Lemma 2. *If $P \in \mathcal{P}_n(\Omega^- \cup \partial\Omega)$, then for $R > r \geqslant 1$ we have*

$$|P(Rz)| \geqslant \left(\frac{R+1}{r+1}\right)^n |P(rz)| \quad \textit{for } z \in \partial\Omega.$$

The following lemma is due to A. Mir [16]:

Lemma 3. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for every complex number α with $|\alpha| \leqslant 1$ and $R > r \geqslant 1$, we have, for $z \in \partial\Omega$:*

$$|N[P(Rz)] - \alpha N[P(rz)]| \leqslant |N[P^*(Rz)] - \alpha N[P^*(rz)]|,$$

where $P^*(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$. *The result is sharp and equality holds if* $P(z) = z^n + 1$.

The next lemma is due to Arestov ([2], Theorem 2).

Lemma 4. *Let* $\Phi(x) = \psi(\log x)$, *where* ψ *is a convex non-decreasing function on* $\mathbb{R}$. *Then for all* $P \in \mathcal{P}_n$ *and each admissible operator* C_γ*:*

$$\int_0^{2\pi} \Phi(|C_\gamma P(e^{i\theta})|)d\theta \leqslant \int_0^{2\pi} \Phi(c(\gamma)|P(e^{i\theta})|)d\theta,$$

where $c(\gamma) = \max(|\gamma_0|, |\gamma_n|)$.

In particular, Lemma 4 applies with $\Phi(x)\colon x \to x^q$ for every $q \in (0,\infty)$ and with $\Phi(x)\colon x \to \log x$, so, we have for $0 \leqslant q < \infty$:

$$\left\{\int_0^{2\pi} |C_\gamma P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leqslant c(\gamma)\left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{19}$$

Lemma 5. *If* $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \leqslant 1$, $q > 0$, $R > r \geqslant 1$, *and real* ϕ, $0 \leqslant \phi < 2\pi$*:*

$$\int_0^{2\pi} \Big||N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]|e^{i\phi} +$$

$$+ |N[P^*(Re^{i\theta})]^* - \bar\alpha N[P^*(re^{i\theta})]^*|\Big|^q d\theta \leqslant$$

$$\leqslant |(R^n - \alpha r^n)\Lambda_s e^{i\phi} + \bar\lambda_0(1-\bar\alpha)|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta,$$

where Λ_s *is defined as in Theorem 8,* $P^*(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$, *and* $N[P^*(z)]^* := (N[P^*(z)])^*$. *The result is sharp and equality holds if* $P(z) = z^n + 1$.

Proof. Since $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, therefore, by Lemma 3, we have for every complex number α with $|\alpha| \leqslant 1$ and $R > r \geqslant 1$:

$$|N[P(Rz)] - \alpha N[P(rz)]| \leqslant |N[P^*(Rz)] - \alpha N[P^*(rz)]|,\ z \in \partial\Omega. \tag{20}$$

Now,

$$P^*(Rz)-\alpha P^*(rz)=R^nz^n\overline{P\Big(\frac{1}{R\bar{z}}\Big)}-\alpha r^nz^n\overline{P\Big(\frac{1}{r\bar{z}}\Big)}.$$

Therefore,

$$\begin{aligned}
N[P^*(Rz)-\alpha P^*(rz)] &= N[P^*(Rz)]-\alpha N[P^*(rz)] = \\
&= \lambda_0\left\{R^nz^n\overline{P\left(1/R\bar{z}\right)}-\alpha r^nz^n\overline{P\left(1/r\bar{z}\right)}\right\}+ \\
&+\lambda_1\left(\frac{nz}{2}\right)\Big[\Big(nR^nz^{n-1}\overline{P\left(1/R\bar{z}\right)}-R^{n-1}z^{n-2}\overline{P'\left(1/R\bar{z}\right)}\Big)- \\
&-\alpha\Big(nr^nz^{n-1}\overline{P\left(1/r\bar{z}\right)}-r^{n-1}z^{n-2}\overline{P'\left(1/r\bar{z}\right)}\Big)\Big]+ \\
&+\cdots+\frac{\lambda_s}{s!}\left(\frac{nz}{2}\right)^s\Big[\Big\{\binom{s}{0}n(n-1)(n-2)\cdots \\
&\cdots(n-s+1)R^nz^{n-s}\overline{P\left(1/R\bar{z}\right)}-\binom{s}{1}(n-1)(n-2)\cdots \\
&\cdots(n-s+1)R^{n-1}z^{n-s-1}\overline{P'\left(1/R\bar{z}\right)}+\cdots+ \\
&+(-1)^s\binom{s}{s}R^{n-s}z^{n-2s}\overline{P^{(s)}\left(1/R\bar{z}\right)}\Big\}- \\
&-\alpha\Big\{\binom{s}{0}n(n-1)(n-2)\cdots \\
&\cdots(n-s+1)r^nz^{n-s}\overline{P\left(1/r\bar{z}\right)}-\binom{s}{1}(n-1)(n-2)\cdots \\
&\cdots(n-s+1)r^{n-1}z^{n-s-1}\overline{P'\left(1/r\bar{z}\right)}+ \\
&+\cdots+(-1)^s\binom{s}{s}r^{n-s}z^{n-2s}\overline{P^{(s)}\left(1/r\bar{z}\right)}\Big\}\Big]= \\
&=\sum_{\nu=0}^{s}\frac{\lambda_\nu}{\nu!}\left(\frac{nz}{2}\right)^\nu\sum_{k=0}^{\nu}(-1)^k\frac{(n-k)!}{(n-\nu)!}\binom{\nu}{k}z^{n-\nu-k}\Big[R^{n-k}\overline{P^{(k)}\left(1/R\bar{z}\right)}- \\
&\qquad -\alpha r^{n-k}\overline{P^{(k)}\left(1/r\bar{z}\right)}\Big].
\end{aligned}$$

This gives

$$\begin{aligned}
N[P^*(Rz)]^*-\bar{\alpha}N[P^*(rz)]^* &= \Big(N[P^*(Rz)]-\alpha N[P^*(rz)]\Big)^* = \\
&=\sum_{\nu=0}^{s}\frac{\bar{\lambda_\nu}}{\nu!}\left(\frac{n}{2}\right)^\nu z^{n-\nu}\sum_{k=0}^{\nu}(-1)^k\frac{(n-k)!}{(n-\nu)!}\binom{\nu}{k}\frac{1}{z^{n-\nu-k}}\Big[R^{n-k}P^{(k)}\left(z/R\right)-
\end{aligned}$$

$$-\bar{\alpha}r^{n-k}P^{(k)}\left(z/r\right)\Big].$$

Further, for $z \in \partial\Omega$, we have

$$|N[P^*(Rz)] - \alpha N[P^*(rz)]| = |N[P^*(Rz)]^* - \alpha N[P^*(rz)]^*|;$$

using this fact in (20), we get for $z \in \partial\Omega$ and $R > r \geqslant 1$, that

$$|N[P(Rz) - \alpha N[P(rz)]| \leqslant |N[P^*(Rz)]^* - \alpha N[P^*(rz)]^*|.$$

Since $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$ implies $P^* \in \mathcal{P}_n(\Omega^-)$, therefore, by Lemma 2, we have for $R > r \geqslant 1$ and $z \in \partial\Omega$:

$$|P^*(Rz)| \geqslant \left(\frac{R+1}{r+1}\right)^n |P^*(rz)| > |P^*(rz)|.$$

This gives, by the application of Rouche's Theorem, that for every real or complex number α with $|\alpha| \leqslant 1$, the polynomial $P^*(Rz) - \alpha P^*(rz)$ has all zeros in Ω^-. Therefore, using linearity of the operator N, it follows by Lemma 1 that the polynomial $N[P^*(Rz)]-\alpha N[P^*(rz)]$ has all zeros in Ω^- and, hence, $(N[P^*(Rz)]-\alpha N[P^*(rz)])^* = N[P^*(Rz)]^* - \alpha N[P^*(rz)]^* \neq 0$ for $|z| < 1$. Therefore, by the maximum modulus principle, we have for $z \in \Omega^-$:

$$|N[P(Rz)] - \alpha N[P(rz)]| < |N[P^*(Rz)]^* - \alpha N[P^*(rz)]^*|. \tag{21}$$

Inequality (21) implies for $P(z) = a_n z^n + \cdots + a_0$ that operator C_γ taking $P(z)$ to

$$\begin{aligned}C_\gamma P(z) &= (N[P(Rz)] - \alpha N[P(rz)])e^{i\phi} + (N[P^*(Rz)]^* - \alpha N[P^*(rz)]^*) = \\ &= \Big\{(R^n - \alpha r^n)\Big(\sum_{k=0}^{s} \lambda_k \left(\frac{n}{2}\right)^k \binom{s}{k}\Big)e^{i\phi} + (1-\bar{\alpha})\bar{\lambda_0}\Big\}a_n z^n + \\ &+ \cdots + \left\{(R^n - \bar{\alpha} r)\left(\sum_{k=0}^{s} \bar{\lambda_k} \left(\frac{n}{2}\right)^k \binom{s}{k}\right) + e^{i\phi}(1-\alpha)\lambda_0\right\} a_0,\end{aligned}$$

is an admissible operator. Therefore, applying inequality (19) of Lemma 4 with $c(\gamma) = (R^n - \alpha r^n)\Lambda_s e^{i\phi} + (1-\bar{\alpha})\bar{\lambda_0}$, we get

$$\int_0^{2\pi} \left|(N[P(Rz)] - \alpha N[P(rz)])e^{i\phi} + N[P^*(Rz)]^* - \alpha N[P^*(rz)]^*\right|^q d\theta \leqslant$$

$$\leqslant \left|(R^n - \alpha r^n)\Lambda_s e^{i\phi} + (1-\bar{\alpha})\bar{\lambda_0}\right|^q \int\limits_0^{2\pi} |P(e^{i\theta})|^q d\theta.$$

Equivalently, since ϕ is arbitrary,

$$\int\limits_0^{2\pi} \left| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| e^{i\phi} + |N[P^*(Re^{i\theta})]^* - \alpha N[P^*(re^{i\theta})]^*| \right|^q d\theta \leqslant$$

$$\leqslant |(R^n - \alpha r^n)\Lambda_s e^{i\phi} + \bar{\lambda_0}(1-\bar{\alpha})|^q \int\limits_0^{2\pi} |P(e^{i\theta})|^q d\theta,$$

which is the desired inequality. □

Lemma 6. *If $P \in \mathcal{P}_n(\Omega^+ \cup \partial\Omega)$, then for every $q > 0$, $R > r \geqslant 1$, $|\alpha| \leqslant 1$ and real ϕ, $0 \leqslant \phi < 2\pi$, we have:*

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} \Big| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| +$$

$$+ e^{i\phi}|N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]| \Big|^q d\theta d\phi \leqslant$$

$$\leqslant \int\limits_0^{2\pi} |(R^n - \alpha r^n)\Lambda_s e^{i\phi} + \lambda_0(1-\alpha)|^q d\phi \int\limits_0^{2\pi} |P(e^{i\theta})|^q d\theta,$$

where Λ_s is defined as in Theorem 8.

Proof. Since for $0 \leqslant \theta < 2\pi$

$$|N[P^*(Re^{i\theta})]^* - \alpha N[P^*(re^{i\theta})]^*| = |N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]|,$$

therefore, for $q > 0$, $R > r \geqslant 1$ and $0 \leqslant \theta < 2\pi$ fixed, we have

$$\int\limits_0^{2\pi} \Big| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| +$$

$$+ e^{i\phi}|N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]| \Big|^q d\phi =$$

$$= \int_0^{2\pi} \Big| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| e^{i\phi}$$

$$+ |N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]| \Big|^q d\phi =$$

$$= \int_0^{2\pi} \Big| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| e^{i\phi} +$$

$$+ |N[P^*(Re^{i\theta})]^* - \alpha N[P^*(re^{i\theta})]^*| \Big|^q d\phi. \quad (22)$$

Integrating both sides of (22) with respect to θ from 0 to 2π and applying Lemma 5, we get

$$\int_0^{2\pi}\int_0^{2\pi} \Big| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| +$$

$$+ e^{i\phi} |N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]| \Big|^q d\phi d\theta =$$

$$= \int_0^{2\pi}\int_0^{2\pi} \Big| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| e^{i\phi} +$$

$$+ |N[P^*(Re^{i\theta})]^* - \alpha N[P^*(re^{i\theta})]^*| \Big|^q d\phi d\theta =$$

$$= \int_0^{2\pi} \Big[\int_0^{2\pi} \Big| |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| e^{i\phi} +$$

$$+ |N[P^*(Re^{i\theta})]^* - \alpha N[P^*(re^{i\theta})]^*| \Big|^q d\theta \Big] d\phi \leqslant$$

$$\leqslant \int_0^{2\pi} |(R^n - \alpha r^n)\Lambda_s e^{i\phi} + \bar{\lambda_0}(1-\bar{\alpha})|^q d\phi \int_0^{2\pi} |P(e^{i\theta})|^q d\theta =$$

$$= \int_0^{2\pi} |(R^n - \alpha r^n)\Lambda_s e^{i\phi} + \lambda_0(1-\alpha)|^q d\phi \int_0^{2\pi} |P(e^{i\theta})|^q d\theta.$$

This completes the proof of Lemma 6. □

Note that none of the inequalities in lemmas 2, 4, and 6 are known to be sharp.

The next lemma is due to Aziz and Shah [3].

Lemma 7. *Let L, M, and N be non-negative real numbers, such that $M + N \leqslant L$; then for every real number ϕ:*

$$|(L - N) + e^{i\phi}(M + N)| \leqslant |L + e^{i\phi}M|.$$

Lemma 8. *Let a, b be real numbers and α be real or complex, such that $|\alpha| \leqslant 1 \leqslant a \leqslant b$; then*

$$|a - \alpha| \leqslant |b - \alpha|.$$

Proof. If α is real, then the result is trivial. So let $\alpha = x + iy$, where $x, y \in R$; then $x \leqslant |\alpha| \leqslant 1 \leqslant a \leqslant b$. This gives

$$a - x \leqslant b - x \implies (a - x)^2 \leqslant (b - x)^2 \implies |a - \alpha| \leqslant |b - \alpha|.$$

This proves the lemma. □

3. Proof of Theorem 8. If $P(z)$ has a zero on $|z| = 1$, then $m = \min_{z\in\partial\Omega} |P(z)| = 0$ and the result follows from Theorem 7. Suppose that $P(z)$ has no zero on $|z| = 1$, so that $m > 0$, and we have

$$m|\beta z^n| < |P(z)|,$$

on $|z| = 1$, for every real or complex number β with $|\beta| \leqslant 1$. Therefore, by Rouche's Theorem, the polynomial $h(z) = P(z) + \beta m z^n$ is in $\mathcal{P}_n(\Omega^+)$. Applying Lemma 3 to $h(z)$, we get for $z \in \partial\Omega$:

$$|N[h(Rz)] - \alpha N[h(rz)]| \leqslant |N[h^*(Rz)] - \alpha N[h^*(rz)]|,$$

where $h^*(z) = z^n\overline{h(\frac{1}{\overline{z}})}$. This gives

$$|N[P(Rz)] - \alpha N[P(rz)] + m\beta(R^n - \alpha r^n)N[z^n]| \leqslant \\ \leqslant |N[P^*(Rz)] - \alpha N[P^*(rz)] + \overline{\beta} m\lambda_0(1 - \alpha)|, \quad (23)$$

for $z \in \partial\Omega$. Choosing the argument of β suitably in the left-hand side of (23) and noting that N is a linear operator, we get for $z \in \partial\Omega$:

$$|N[P(Rz)] - \alpha N[P(rz)]| + m|\beta||R^n - \alpha r^n||\Lambda_m| \leqslant \\ \leqslant |N[P^*(Rz)] - \alpha N[P^*(rz)]| + m|\beta||1 - \alpha||\lambda_0|.$$

Equivalently, we have for $z \in \partial\Omega$:

$$|N[P(Rz)] - \alpha N[P(rz)]| + \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_m| - |1-\alpha||\lambda_0|)}{2} \leqslant \\ \leqslant |N[P^*(Rz)] - \alpha N[P^*(rz)]| - \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_m| - |1-\alpha||\lambda_0|)}{2}. \tag{24}$$

Now define non-negative numbers $L = |N[P^*(Rz)] - \alpha N[P^*(rz)]|$ and $M = |N[P(Rz)] - \alpha N[P(rz)]|$. At the end of the statement of Theorem 8, we have proved that the number $N = \left\{\frac{|R^n - \alpha r^n||\Lambda_m| - |1-\alpha||\lambda_0|}{2}\right\} m|\beta|$ is non-negative. Hence, from (24) we have $M + N \leqslant L - N \leqslant L$. Therefore, invoking Lemma 7, we get for every real ϕ and $z \in \partial\Omega$ that

$$\Bigg||N[P^*(Rz)] - \alpha N[P^*(rz)]| - \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_s| - |1-\alpha||\lambda_0|)}{2} + \\ + e^{i\phi}\left(|N[P(Rz)] - \alpha N[P(rz)]| + \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_s| - |1-\alpha||\lambda_0|)}{2}\right)\Bigg| \leqslant \\ \leqslant |N[P^*(Rz)] - \alpha N[P^*(rz)]| + e^{i\phi}|N[P(Rz)] - \alpha N[P(rz)]|.$$

The inequality above gives for each $q > 0$ and $0 \leqslant \theta < 2\pi$:

$$\int_0^{2\pi} |U(\theta) + e^{i\phi}V(\theta)|^q d\theta \leqslant \int_0^{2\pi} |N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]| + \\ + e^{i\phi}|N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]|^q d\theta, \tag{25}$$

where

$$U(\theta) = |N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]| - \\ - \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_s| - |1-\alpha||\lambda_0|)}{2},$$

$$V(\theta) = |N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]| + \\ + \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_s| - |1-\alpha||\lambda_0|)}{2}.$$

Integrating both sides of inequality (25) with respect to ϕ from 0 to 2π, we get

$$\int_0^{2\pi}\int_0^{2\pi} |U(\theta) + e^{i\phi}V(\theta)|^q d\theta d\phi \leqslant \int_0^{2\pi}\int_0^{2\pi} \Big||N[P^*(Re^{i\theta})] - \alpha N[P^*(re^{i\theta})]|+ \\ + e^{i\phi}|N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]|\Big|^q d\theta d\phi.$$

Applying Lemma 6, we obtain for every real ϕ and $0 \leqslant \theta < 2\pi$:

$$\int_0^{2\pi}\int_0^{2\pi} |U(\theta) + e^{i\phi}V(\theta)|^q d\theta d\phi \leqslant \\ \leqslant \int_0^{2\pi} |(R^n - \alpha r^n)\Lambda_s e^{i\phi} + \lambda_0(1-\alpha)|^q d\phi \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \quad (26)$$

Since for every real ϕ and $t \geqslant 1$, we have $|t + e^{i\phi}| \geqslant |1 + e^{i\phi}|$, which gives for $q > 0$ that

$$\int_0^{2\pi} |t + e^{i\phi}|^q d\phi \geqslant \int_0^{2\pi} |1 + e^{i\phi}|^q d\phi. \quad (27)$$

Now, if $V(\theta) \neq 0$, then from (24) we have $t = \frac{|U(\theta)|}{|V(\theta)|} \geqslant 1$. This gives, with the help of (27), that

$$\int_0^{2\pi} |U(\theta) + e^{i\phi}V(\theta)|^q d\phi = |V(\theta)|^q \int_0^{2\pi} \Big|\frac{U(\theta)}{V(\theta)} + e^{i\phi}\Big|^q d\phi = \\ = |V(\theta)|^q \int_0^{2\pi} \Big|\Big|\frac{U(\theta)}{V(\theta)}\Big| + e^{i\phi}\Big|^q d\phi \geqslant |V(\theta)|^q \int_0^{2\pi} |1 + e^{i\phi}|^q d\phi. \quad (28)$$

Clearly, inequality (28) is already true when $V(\theta) = 0$. Hence, for every θ with $0 \leqslant \theta < 2\pi$, we have:

$$\int_0^{2\pi} |U(\theta) + e^{i\phi}V(\theta)|^q d\phi \geqslant \Big||N[P(Re^{i\theta})] - \alpha N[P(re^{i\theta})]|+ \\ + \frac{m|\beta|(|R^n - \alpha r^n||\Lambda_s| - |1-\alpha||\lambda_0|)}{2}\Big|^q \int_0^{2\pi} |1 + e^{i\phi}|^q d\phi. \quad (29)$$

Now, integrating both sides of inequality (29) with respect to θ and using inequality (26), we get

$$\int_0^{2\pi}\Big||N[P(Re^{i\theta})]-\alpha N[P(re^{i\theta})]|+ \\ +\frac{m|\beta|(|R^n-\alpha r^n||\Lambda_s|-|1-\alpha||\lambda_0|)}{2}\Big|^q d\theta\int_0^{2\pi}|1+e^{i\phi}|^q d\phi \leqslant \\ \leqslant \int_0^{2\pi}|(R^n-\alpha r^n)\Lambda_s e^{i\phi}+\lambda_0(1-\alpha)|^q d\phi\int_0^{2\pi}|P(e^{i\theta})|^q d\theta,$$

for every β with $|\beta|<1, R>r\geqslant 1$, $q>0$, real α, and $0\leqslant\theta<2\pi$. This, in particular, gives

$$\Big\||N[P(Rz)]-\alpha N[P(rz)]|+\frac{m|\beta|(|R^n-\alpha r^n||\Lambda_s|-|1-\alpha||\lambda_0|)}{2}\Big\|_q \leqslant \\ \leqslant \frac{\|(R^n-\alpha r^n)\Lambda_s z+\lambda_0(1-\alpha)\|_q}{\|1+z\|_q}\|P(z)\|_q.$$

This completes the proof.

Acknowledgment: The author is highly indebted to the referees for their valuable suggestions.

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Received August 15, 2024.
In revised form, December 13, 2024.
Accepted December 20, 2024.
Published online January 7, 2025.

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