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A CLASS OF BI-BAZILEVIČ AND BI-PSEUDO-STARLIKE FUNCTIONS INVOLVING TREMBLAY FRACTIONAL DERIVATIVE OPERATOR

Abstract. In this paper, we introduce a new class of bi-univalent functions, which exhibit connections to Bazilevič and Pseudo-Starlike functions. By employing the Tremblay fractional derivative operator, we establish this comprehensive class. For functions within this class, we derive coefficient inequalities and the Fekete-Szegö inequalities. Furthermore, we discuss various significant observations related to the presented results. Additionally, we provide some non-trivial examples within this class, supplemented by illustrative figures to enhance understanding.

Key words: *pseudo-starlike function, Bazilevič function, bi-univalent functions, fractional derivative, Fekete-Szegö inequality*

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1. Introduction. Let $\mathcal{A}$ indicate the family of functions f that are analytic in the open unite disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Correspondingly, let $\mathcal{S}$ represent the subclass of functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$, normalized such that $f(0) = 0$ and $f'(0) = 1$, and can be expressed as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

An analytic function f is said to be subordinate to another analytic function g in $\mathbb{U}$, denoted $f \prec g$, if there exists an analytic function u with $u(0) = 0$ and $|u(z)| < 1$ $(z \in \mathbb{U})$ satisfying $f(z) = g(u(z))$. According to the Schwarz Lemma, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq$ $\subseteq g(\mathbb{U})$ $(z \in \mathbb{U})$ whenever g is univalent (for details, refer to [7]).

In the geometric function theory, important subclasses of $\mathcal{S}$ include starlike and convex functions. Let $f \in \mathcal{A}$ in the form (1) be called starlike and convex if

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z),$$

respectively, where $p(z) = \dfrac{1+z}{1-z}$.

Babalola [3] introduced and studied the class of Pseudo-Starlike functions of type α $(\alpha > 0)$ and order ρ $(0 \leqslant \rho < 1)$. Geometrically, these functions satisfy the condition:

$$Re\left(\frac{z\big(f'(z)\big)^{\alpha}}{f(z)}\right) > \rho.$$

He demonstrates that all Pseudo-Starlike functions are Bazilevič of order $\rho^{\frac{1}{\alpha}}$ and type $1 - \frac{1}{\alpha}$. When $\alpha = 1$, these functions are called Starlike functions of order ρ.

The Koebe One-Quarter Theorem [7] guarantees that for every $f \in \mathcal{S}$, $f(\mathbb{U})$ forms a disk of radius $\frac{1}{4}$. Therefore, every $f \in \mathcal{S}$ possesses an inverse f^{-1}, such that $f^{-1}\big(f(z)\big) = z$ and $f\big(f^{-1}(w)\big) = w$ $(z \in \mathbb{U},\ r(f) \leqslant \frac{1}{4},\ |w| < r(f))$.

The class of bi-univalent functions is denoted by Σ, meaning both f and f^{-1} belong to $\mathcal{S}$. For $f \in \Sigma$, computation reveals that its inverse has the expression:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \dots. \tag{2}$$

Additionally, Σ is not empty, since the functions $-\log(1-z)$, $\dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$, and $\dfrac{z}{1-z}$ belong to Σ. On the other hand, the functions $\dfrac{z}{1-z^2}$, $z - \dfrac{z^2}{2}$, and the Koebe function are not members of Σ. In recent years, Srivastava et al. [21] have revitalized the study of bi-univalent functions, followed by several other studies (for example, [2, 4, 5, 16, 18, 19]).

2. Fractional Calculus. Fractional calculus is a fundamental branch of mathematical analysis that revolves around the differentiation operator $D = \dfrac{d}{dz}$, expanding its scope to encompass real and complex numbers. This field has played a crucial role in the development of geometric function theory. Researchers have extensively investigated its wide-ranging applications, as detailed in [8], [10], [23], [24]. Srivastava et al. [12], [20] have made substantial contributions by extending the definitions of fractional operators as follows:

Definition 1. [20] *For a function f, the fractional integral of order γ is defined by*

$$D_z^{-\gamma} f(z) = \frac{1}{\Gamma(\gamma)} \int_0^z (z-\xi)^{\gamma-1} f(\xi)\, d\xi \quad (\gamma > 0), \tag{3}$$

where $f(z)$ is an analytic function in a simply connected region of $\mathbb{C}$ and the multiplicity of $(z-\xi)^{\gamma-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 2. [20] *For an analytic function f, the fractional derivatives of order γ is defined by*

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z (z-\xi)^{-\gamma} f(\xi)\, d\xi \quad (0 \leqslant \gamma < 1),$$

and the multiplicity of $(z-\xi)^{-\gamma}$ is removed, as in Definition 1. In general,

$$D_z^{n+\gamma} f(z) = \frac{d^n}{dz^n} D_z^{\gamma} f(z) \quad (0 \leqslant \gamma < 1, n \in \mathbb{N}_0).$$

Definition 3. [12] *The Tremblay fractional derivative operator $\mathcal{T}_z^{\mu,\gamma}$ of a function $f \in \mathcal{A}$ is defined as*

$$\mathcal{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z) \quad (\gamma > 0;\ \mu \leqslant 1;\ 0 \leqslant \mu-\gamma < 1). \tag{4}$$

As consequences of Definitions 1, 2, and 3, we note that

(1) $D_z^{-\gamma} z^n = \dfrac{n!}{\Gamma(n+\gamma+1)} z^{n+\gamma}$ $(\gamma > 0;\ n \in \mathbb{N})$;

(2) $D_z^{\gamma} z^n = \dfrac{n!}{\Gamma(n-\gamma+1)} z^{n-\gamma}$ $(0 \leqslant \gamma < 1;\ n \in \mathbb{N})$;

(3) $\mathcal{T}_z^{\mu,\gamma} f(z) = \dfrac{\mu}{\gamma} z + \displaystyle\sum_{n=2}^{\infty} \frac{\Gamma(\gamma)\Gamma(n+\mu)}{\Gamma(\mu)\Gamma(n+\gamma)} a_n z^n$ $(f \in \mathcal{A};\ \gamma > 0;\ \mu \leqslant 1;\ 0 \leqslant \mu-\gamma < 1)$;

(4) $\mathcal{T}_z^{1,1} f(z) = f(z)$, where $f \in \mathcal{A}$.

3. Motivation. The exploration of bi-univalent functions is driven by their remarkable connections to various classes of functions, such as Bazilevič and Pseudo-Starlike functions. These functions play a vital role in complex analysis, particularly in the study of geometric properties and coefficient inequalities. The introduction of the class $\mathcal{BPS}_\Sigma(\varepsilon, \lambda, \theta, \gamma, \mu; h)$ (see Definition 4) represents a significant step forward in our understanding of bi-univalent functions.

We aim at establishing new results and insights within this framework by employing the Tremblay fractional derivative operator. The development of coefficient inequalities and Fekete-Szegö inequalities enhances our theoretical understanding and opens up new possibilities for applications in mathematical analysis. Additionally, providing non-trivial examples further demonstrates the practical implications of this research. This motivates our inquiry into the intricate relationships among these functions and their properties, thereby contributing to the advancement of the field.

4. Objectives and Methodology. This paper aims at introducing a new class of bi-univalent functions that exhibit significant connections to Bazilevic and Pseudo-Starlike functions. The methodology employed in this study involves the use of the Tremblay fractional derivative operator to establish and explore this class comprehensively. Coefficient inequalities and Fekete-Szegö inequalities are derived for functions within this class. Additionally, the Mathematica$^{\text{TM}}$ software (version 13.2) was used to generate graphical representations that visually demonstrate the differential subordination relationships within these functions, thereby enhancing the understanding of the presented results.

5. Main results. Unless indicated otherwise, we will suppose throughout the rest of this section that $\gamma > 0$, $\mu \leqslant 1$, $0 \leqslant \mu - \gamma < 1$, $0 \leqslant \varepsilon \leqslant 1$, $\theta \geqslant 1$, $\lambda \geqslant 0$, and h is analytic in $\mathbb{U}$ with $h(0) = 1$.

Definition 4. *A function $f \in \Sigma$ is a member of $\mathcal{BPS}_\Sigma(\varepsilon, \lambda, \theta, \gamma, \mu; h)$ if the following subordination holds:*

$$\varepsilon \left(\frac{\gamma z^{1-\lambda} \big(\mathcal{T}_z^{\mu,\gamma} f(z) \big)'}{\mu \big(f(z) \big)^{1-\lambda}} \right) + (1 - \varepsilon) \left(\frac{\gamma^\theta z \Big(\big(\mathcal{T}_z^{\mu,\gamma} f(z) \big)' \Big)^\theta}{\mu^\theta f(z)} \right) \prec h(z), \qquad (5)$$

and for $g := f^{-1}$

$$\varepsilon\left(\frac{\gamma w^{1-\lambda}\left(\mathcal{T}_w^{\mu,\gamma}g(w)\right)'}{\mu\big(g(w)\big)^{1-\lambda}}\right)+(1-\varepsilon)\left(\frac{\gamma^\theta w\Big(\left(\mathcal{T}_w^{\mu,\gamma}g(w)\right)'\Big)^\theta}{\mu^\theta g(w)}\right)\prec h(w), \qquad (6)$$

where $z,\ w \in \mathbb{U}$.

Remark 1. *The class $\mathcal{BPS}_\Sigma(\varepsilon,\lambda,\theta,\gamma,\mu;h)$ of bi-univalent functions constitutes a generalization of several well-known classes that have been widely studied in previous works. For reference, some of these classes are listed as follows:*

(1) *Let $\varepsilon = 1$, $\lambda = 1$, and h have positive real part in $\mathbb{U}$ with $h'(0) > 0$; then $\mathcal{BPS}_\Sigma(1,1,\theta,\gamma,\mu;h) := \mathcal{A}_\Sigma(1,\gamma,\mu;h)$ was studied by Srivastava et al. [22].*

(2) *Let $\varepsilon = 0, \mu = \gamma = 1$, and $h(z) = \Pi(x,z) + 1 - a$ be the generating function of the Horadom polynomials $h_n(x)$ (see [11]) given by*

$$\Pi(x,z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1},$$

where $h_1(x) = a$, $h_2(x) = bx$, $h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x)$, $n\in\mathbb{N}/\{1,2\}$ and $a,b,p,q\in\mathbb{R}$. Then $\mathcal{BPS}_\Sigma(0,\lambda,\theta,1,1;\Pi(x,z){+}1{-}a) :=$ $:= \mathcal{G}_\Sigma(\theta,x)$ was studied by Abirami et al. [1].

(3) *Let $\varepsilon = \mu = \gamma = 1$ and $h(z) = \Pi(x,z){+}1{-}a$; then $\mathcal{BPS}_\Sigma(1,\lambda,\theta,1,1;$ $\Pi(x,z) + 1 - a) := \mathcal{R}_\Sigma^{1,\lambda}(x)$ was studied by Abirami et al. [1].*

(4) *If we choose $a = 1$, $b = p = 2$, and $q = -1$ in (2) and (3) above, then $\mathcal{BPS}_\Sigma(0,\lambda,\theta,1,1;\Pi(x,z){+}1{-}a)$ and $\mathcal{BPS}_\Sigma(1,\lambda,\theta,1,1;\Pi(x,z){+}1{-}a)$ were studied by Magesh and Bulut [14] and by Bulut et al. [5], respectively.*

(5) *If $\varepsilon = \mu = \gamma = \lambda + 1 = 1$ or $\mu = \gamma = \theta = \varepsilon + 1 = 1$, then our class $\mathcal{BPS}_\Sigma(\varepsilon,\lambda,\theta,\gamma,\mu;h)$ was studied by Srivastava et al. [18] associated to $h(z) = \Pi(x,z) + 1 - a$.*

(6) *If $\varepsilon = \mu = \gamma = 1$, then $\mathcal{BPS}_\Sigma\left(1,\lambda,\theta,1,1;\dfrac{1}{1-2tz+z^2}\right), \dfrac{1}{2} < t \leqslant 1$ was studied by Bulut et al. [4].*

(7) *If $\varepsilon = \mu = \gamma = \lambda + 1 = 1$ or $\mu = \gamma = \theta = \varepsilon + 1 = 1$, then $\mathcal{BPS}_\Sigma(\varepsilon,\lambda,\theta,\gamma,\mu;h)$ was studied by Janowski [13] associated to $h(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leqslant B < A \leqslant 1)$.*

(8) *If* $\varepsilon = \mu = \gamma = \lambda + 1 = 1$ *or* $\mu = \gamma = \theta = \varepsilon + 1 = 1$, *then* $\mathcal{BPS}_\Sigma(\varepsilon, \lambda, \theta, \gamma, \mu; h)$ *was studied by Raina and Sokol* [15] *associated to* $h(z) = z + \sqrt{1 + z^2}$.

(9) *If* $\varepsilon = \mu = \gamma = \lambda + 1 = 1$ *or* $\mu = \gamma = \theta = \varepsilon + 1 = 1$, *then* $\mathcal{BPS}_\Sigma(\varepsilon, \lambda, \theta, \gamma, \mu; h)$ *was studied by Sharma et al.* [17] *associated to* $h(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$.

(10) *If* $\varepsilon = \mu = \gamma = \lambda + 1 = 1$ *or* $\mu = \gamma = \theta = \varepsilon + 1 = 1$, *then* $\mathcal{BPS}_\Sigma(\varepsilon, \lambda, \theta, \gamma, \mu; h)$ *was studied by Cho et al.* [6] *associated to* $h(z) = 1 + \sin(z)$.

(11) *If* $\varepsilon = \mu = \gamma = 1$, *then* $\mathcal{BPS}_\Sigma(1, \lambda, \theta, 1, 1; \sqrt{1 + z})$ *and* $\mathcal{BPS}_\Sigma\Big(1, \lambda, \theta, 1, 1; \dfrac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}\Big)$, $\tau = \dfrac{1 - \sqrt{5}}{2}$ *were studied by Srivastava et al.* [19].

Below we will provide non-trivial examples of $\mathcal{BPS}_\Sigma(\varepsilon, \lambda, \theta, \gamma, \mu; h)$, supported by appropriate explanations or figures.

Example 1.

(1) If $f(z) = \dfrac{z}{1 + z}$, then $g(w) = f^{-1}(w) = \dfrac{w}{1 - w}$. Hence, $f \in \Sigma$. After executing some calculations and using Definition 3, we obtain

$$\frac{\gamma\big(\mathcal{T}_z^{\mu,\gamma} f(z)\big)'}{\mu} = {}_2F_1(\mu + 1, 2; \gamma + 1; -z),$$

and

$$\frac{\gamma\big(\mathcal{T}_w^{\mu,\gamma} g(w)\big)'}{\mu} = {}_2F_1(\mu + 1, 2; \gamma + 1; w).$$

Thus, $f \in \mathcal{BPS}_\Sigma(1, 1, \theta, \gamma, \mu; {}_2F_1(\mu + 1, 2; \gamma + 1; z))$, where ${}_2F_1$ is a hypergeometric function.

(2) If $f(z) = \dfrac{z}{2 - z}$, then $g(w) = \dfrac{2w}{1 + w}$, and, hence, $f \in \Sigma$. Consequently,

$$\frac{zf'(z)}{f(z)} = \frac{2}{2 - z}, \qquad \text{and} \qquad \frac{wg'(w)}{g(w)} = \frac{1}{1 + w}.$$

Therefore, $f \in \mathcal{BPS}_\Sigma\Big(0, \lambda, 1, 1, 1; \dfrac{1}{1 - z}\Big)$.

(3) The function $(\sin^{-1}\sqrt{1-z})^2$ belongs to $\mathcal{BPS}_\Sigma\Big(0,\lambda,2,1,1;$ $\dfrac{1}{1-2tz+z^2}\Big)$, $\dfrac{1}{2}<t\leqslant 1$ (see [14]).

(4) If $f_\delta(z)=\dfrac{z}{1-\delta z}$, $|\delta|\leqslant 1$, then $g_\delta(w)=f_\delta^{-1}(w)=\dfrac{w}{1+\delta w}$. It is clear that $f\in\Sigma$. The function

$$h(z)=1+z-\frac{z^3}{3}\quad(z\in\mathbb{U})$$

has positive real part and is symmetric with respect to the real axis. After some computations, we have

$$F(z):=\frac{z\big(f'(z)\big)^2}{f(z)}=\frac{1}{(1-\delta z)^3},$$

and

$$G(w):=\frac{w\big(g'(w)\big)^2}{g(w)}=\frac{1}{(1+\delta w)^3}.$$

Figure 1 demonstrates that $F(\mathbb{U})=G(\mathbb{U})\subseteq h(\mathbb{U})(|\delta|\leqslant 0.15)$, so we conclude that $f_\delta\in\mathcal{BPS}_\Sigma\big(0,\lambda,2,1,1;1+z-\frac{z^3}{3}\big)(|\delta|\leqslant 0.15)$. Therefore, $|a_2|=0.15$ and $|a_3|=0.0225$, and so this applies to Corollary 1, as the right-hand sides of inequalities (30) and (31) are equal to 0.397 and 0.311, respectively.

(5) Consider the following formulas:

$$F(z):=\frac{1}{2(1-\delta z)}\Big(\frac{1}{(1-\delta z)^3}+1\Big)$$

and

$$G(w):=\frac{1}{2(1+\delta w)}\Big(\frac{1}{(1+\delta w)^3}+1\Big).$$

Figure 2 demonstrates that $F(\mathbb{U})=G(\mathbb{U})\subseteq h^q(\mathbb{U})(q\to 1^-,|\delta|\leqslant 0.19)$, so we conclude that $f_\delta\in\mathcal{BPS}_\Sigma(\frac{1}{2},3,1,1,1;h^q(z))$ $(q\to 1^-,|\delta|\leqslant 0.19)$, where

$$h^q(z)=\frac{(1+q)z}{2+(1-q)z}+\sqrt[3]{1+\Big(\frac{(1+q)z}{2+(1-q)z}\Big)^3}\quad(z\in\mathbb{U},\ q\in(0,1)).$$

Therefore, $|a_2|=0.19$ and $|a_3|=0.036$, and so this applies to Theorem 1, as the right-hand sides of inequalities (7) and (8) are equal to 0.426 and 0.285, respectively.

(6) Using the same argument as in (5), we have $f_\delta \in \mathcal{BPS}_\Sigma(\frac{1}{2}, 4, 1, 1, 1; h^{\frac{3}{4}}(z))$ $(|\delta| \leqslant 0.17)$ (see Figure 3).

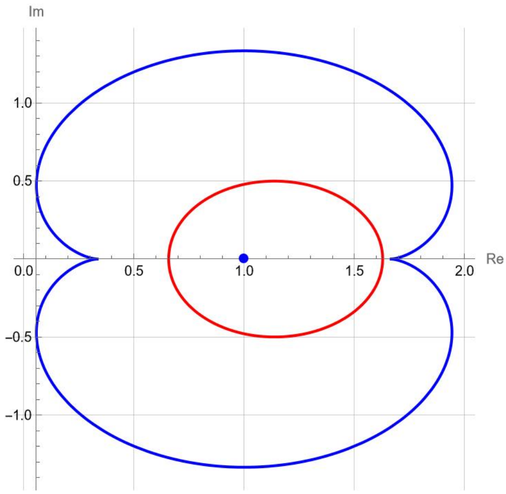

Figure 1: An image of $F(\mathbb{U})$ (red) and $h(\mathbb{U})$ (blue).

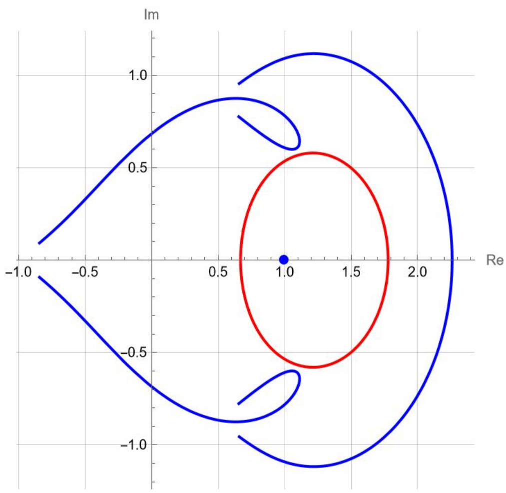

Figure 2: An image of $F(\mathbb{U})$ (red) and $h^q(\mathbb{U})$ (blue), $(q \to 1^-)$.

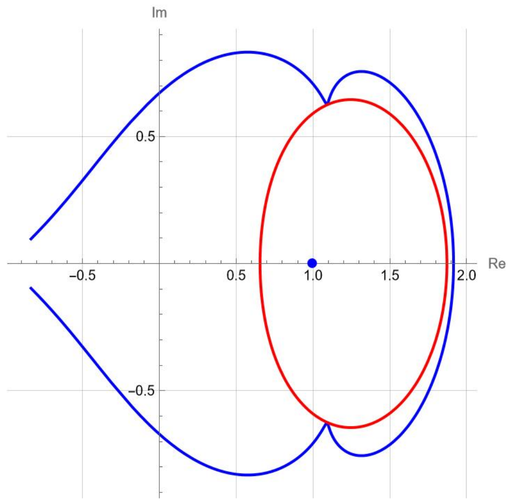

Figure 3: An image of $F(\mathbb{U})$ (red) and $h^{\frac{3}{4}}(\mathbb{U})$ (blue).

Theorem 1. *Consider f to be in the class $\mathcal{BPS}_{\Sigma}(\varepsilon, \lambda, \theta, \gamma, \mu; h)$. Then*

$$|a_2| \leqslant \frac{|h_1|\sqrt{|h_1|}}{\sqrt{|h_1^2[\chi_1(\varepsilon, \lambda, \theta, \psi_3) + \chi_2(\varepsilon, \lambda, \theta, \psi_2)] - h_2\chi_3(\varepsilon, \lambda, \theta, \psi_2)|}}, \tag{7}$$

and

$$|a_3| \leqslant \frac{|h_1|}{|\chi_1(\varepsilon, \lambda, \theta, \psi_3)|} + \frac{|h_1^2|}{|\chi_3(\varepsilon, \lambda, \theta, \psi_2)|}, \tag{8}$$

where

$$\psi_n = \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)} \quad (n \in \mathbb{N}), \tag{9}$$

$$\chi_1(\varepsilon, \lambda, \theta, \psi_3) = \varepsilon(3\psi_3 - (1-\lambda)) + (1-\varepsilon)(3\theta\psi_3 - 1), \tag{10}$$

$$\chi_2(\varepsilon, \lambda, \theta, \psi_2) = \varepsilon(1-\lambda)\left(\frac{1}{2}(2-\lambda) - 2\psi_2\right) + (1-\varepsilon)(1-2\theta(\psi_2-\theta+1)), \tag{11}$$

$$\chi_3(\varepsilon, \lambda, \theta, \psi_2) = [\varepsilon(2\psi_2 - (1-\lambda)) + (1-\varepsilon)(2\theta\psi_2 - 1)]^2. \tag{12}$$

Proof. Suppose that $f \in \mathcal{BPS}_{\Sigma}(\varepsilon, \lambda, \theta, \gamma, \mu; h)$ and $g = f^{-1}$; then there exist two analytic functions $u, v : \mathbb{U} \longrightarrow \mathbb{U}$ given by

$$u(z) = u_1 + u_2 z^2 + u_3 z^3 + \ldots, \quad z \in \mathbb{U}, \tag{13}$$

$$v(w) = v_1 + v_2 w^2 + v_3 w^3 + \ldots, \quad w \in \mathbb{U}, \tag{14}$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, and $|v(w)| < 1$ $(z, w \in \mathbb{U})$, such that

$$\varepsilon\left(\frac{\gamma z^{1-\lambda}\left(\mathcal{T}_z^{\mu,\gamma}f(z)\right)'}{\mu\left(f(z)\right)^{1-\lambda}}\right)+(1-\varepsilon)\left(\frac{\gamma^\theta z\left(\left(\mathcal{T}_z^{\mu,\gamma}f(z)\right)'\right)^\theta}{\mu^\theta f(z)}\right)= \\ = 1+h_1u(z)+h_2u^2(z)+\ldots, \quad (15)$$

and

$$\varepsilon\left(\frac{\gamma w^{1-\lambda}\left(\mathcal{T}_w^{\mu,\gamma}g(w)\right)'}{\mu\left(g(w)\right)^{1-\lambda}}\right)+(1-\varepsilon)\left(\frac{\gamma^\theta w\left(\left(\mathcal{T}_w^{\mu,\gamma}g(w)\right)'\right)^\theta}{\mu^\theta g(w)}\right)= \\ = 1+h_1v(w)+h_2v^2(w)+\ldots. \quad (16)$$

Combining (13) and (14) as well as (15) and (16), yields

$$\varepsilon\left(\frac{\gamma z^{1-\lambda}\left(\mathcal{T}_z^{\mu,\gamma}f(z)\right)'}{\mu\left(f(z)\right)^{1-\lambda}}\right)+(1-\varepsilon)\left(\frac{\gamma^\theta z\left(\left(\mathcal{T}_z^{\mu,\gamma}f(z)\right)'\right)^\theta}{\mu^\theta f(z)}\right)= \\ = 1+h_1u_1z+\left(h_1u_2+h_2u_1^2\right)z^2+\ldots, \quad (17)$$

and

$$\varepsilon\left(\frac{\gamma w^{1-\lambda}\left(\mathcal{T}_w^{\mu,\gamma}g(w)\right)'}{\mu\left(g(w)\right)^{1-\lambda}}\right)+(1-\varepsilon)\left(\frac{\gamma^\theta w\left(\left(\mathcal{T}_w^{\mu,\gamma}g(w)\right)'\right)^\theta}{\mu^\theta g(w)}\right)= \\ = 1+h_1v_1w+\left(h_1v_2+h_2v_1^2\right)w^2+\ldots. \quad (18)$$

It is well known that if $|u(z)|<1$ and $|v(w)|<1$ $(z,w\in\mathbb{U})$, then we have

$$|u_n|\leqslant 1, \text{ and } |v_n|\leqslant 1 \ (n\in\mathbb{N}). \quad (19)$$

Comparing the corresponding coefficients in light of (17) and (18), after simplifying, we obtain

$$[\varepsilon(2\psi_2-(1-\lambda))+(1-\varepsilon)(2\theta\psi_2-1)]a_2=h_1u_1, \quad (20)$$

$$[\varepsilon(3\psi_3-(1-\lambda))+(1-\varepsilon)(3\theta\psi_3-1)]a_3+\Big[\varepsilon(1-\lambda)\Big(\frac{1}{2}(2-\lambda)-2\psi_2\Big)+ \\ +(1-\varepsilon)(1-2\theta(\psi_2-\theta+1))\Big]a_2^2=h_1u_2+h_2u_1^2, \quad (21)$$

$$-\left[\varepsilon(2\psi_2-(1-\lambda))+(1-\varepsilon)(2\theta\psi_2-1)\right]a_2=h_1v_1, \tag{22}$$

$$[\varepsilon(3\psi_3-(1-\lambda))+(1-\varepsilon)(3\theta\psi_3-1)](2a_2^2-a_3)+\Big[\varepsilon(1-\lambda)\Big(\frac{1}{2}(2-\lambda)-2\psi_2\Big)+ + (1-\varepsilon)(1-2\theta(\psi_2-\theta+1))\Big]a_2^2=h_1v_2+h_2v_1^2. \tag{23}$$

From (20) and (22), we obtain

$$u_1=-v_1, \tag{24}$$

and

$$2[\varepsilon(2\psi_2-(1-\lambda))+(1-\varepsilon)(2\theta\psi_2-1)]^2a_2^2=h_1^2(u_1^2+v_1^2). \tag{25}$$

Now, summing (21) and (23), we obtain

$$2\Big\{[\varepsilon(3\psi_3-(1-\lambda))+(1-\varepsilon)(3\theta\psi_3-1)]+\Big[\varepsilon(1-\lambda)\Big(\frac{1}{2}(2-\lambda)-2\psi_2\Big)+ + (1-\varepsilon)(1-2\theta(\psi_2-\theta+1))\Big]\Big\}a_2^2=h_1(u_2+v_2)+h_2(u_1^2+v_1^2). \tag{26}$$

Putting (25) into the right-hand side of (26) and simplifying, we have

$$a_2^2=\frac{h_1^3(u_2+v_2)}{2[h_1^2[\chi_1(\varepsilon,\lambda,\theta,\psi_3)+\chi_2(\varepsilon,\lambda,\theta,\psi_2)]-h_2\chi_3(\varepsilon,\lambda,\theta,\psi_2)]}, \tag{27}$$

where χ_1, χ_2, and χ_3 are as defined in parametric equations (10), (11), and (12), respectively.

Taking the absolute value of both sides of (27) and substituting (19) into the result, followed by some computations, we obtain

$$|a_2|\leqslant\frac{|h_1|\sqrt{|h_1|}}{\sqrt{|h_1^2[\chi_1(\varepsilon,\lambda,\theta,\psi_3)+\chi_2(\varepsilon,\lambda,\theta,\psi_2)]-h_2\chi_3(\varepsilon,\lambda,\theta,\psi_2)|}}.$$

To determine the bound on $|a_3|$, we subtract (23) from (21), obtaining

$$2[\varepsilon(3\psi_3-(1-\lambda))+(1-\varepsilon)(3\theta\psi_3-1)](a_3-a_2^2)=h_1(u_2-v_2). \tag{28}$$

Substituting (10), (12), (24), and (25) into (28), we have

$$a_3=\frac{h_1(u_2-v_2)}{2\chi_1(\varepsilon,\lambda,\theta,\psi_3)}+\frac{h_1^2(u_1^2+v_1^2)}{2\chi_3(\varepsilon,\lambda,\theta,\psi_2)}. \tag{29}$$

Taking the absolute value of both sides of (29) and substituting (19) into the result, we obtain

$$|a_3| \leqslant \frac{|h_1|}{|\chi_1(\varepsilon,\lambda,\theta,\psi_3)|} + \frac{|h_1^2|}{|\chi_3(\varepsilon,\lambda,\theta,\psi_2)|}.$$

The proof is completed. □

As a consequence of Theorem 1, these two corollaries result from setting ε to 0 and 1:

Corollary 1. *Consider f to be in the class $\mathcal{BPS}_\Sigma(0,\lambda,\theta,\gamma,\mu;h)$. Then*

$$|a_2| \leqslant \frac{|h_1|\sqrt{|h_1|}}{\sqrt{|h_1^2[3\theta\psi_3 - 2\theta(\psi_2-\theta+1)] - h_2(2\theta\psi_2-1)|}}, \tag{30}$$

and

$$|a_3| \leqslant \frac{|h_1|}{|3\theta\psi_3 - 1|} + \frac{|h_1^2|}{(2\theta\psi_2-1)^2}. \tag{31}$$

Corollary 2. *Consider f to be in the class $\mathcal{BPS}_\Sigma(1,\lambda,\theta,\gamma,\mu;h)$. Then*

$$|a_2| \leqslant \frac{|h_1|\sqrt{|h_1|}}{\sqrt{\left|h_1^2\left[3\psi_3 - \frac{1}{2}(1-\lambda)(\lambda+4\psi_2)\right] - h_2(2\psi_2-(1-\lambda))\right|}}, \tag{32}$$

and

$$|a_3| \leqslant \frac{|h_1|}{|3\psi_3-(1-\lambda)|} + \frac{|h_1^2|}{(2\psi_2-(1-\lambda))^2}. \tag{33}$$

In their foundational, Fekete and Szegö [9] introduced the generalized functional $|a_3 - \beta a_2^2|$, where β represented a real number. In the result that follows, we establish the Fekete-Szegö functional for f belonging to $\mathcal{BPS}_\Sigma(\varepsilon,\lambda,\theta,\gamma,\mu;h)$.

Theorem 2. *Let a function f be in the class $\mathcal{BPS}_\Sigma(\varepsilon,\lambda,\theta,\gamma,\mu;h)$ and $\beta\in\mathbb{R}$. Then*

$$|a_3-\beta a_2^2| \leqslant \begin{cases} \dfrac{|h_1|}{|\chi_1|}; & |1-\beta| \leqslant \left|1+\dfrac{\chi_2}{\chi_1}-\dfrac{h_2\chi_3}{h_1^2\chi_1}\right|, \\ \dfrac{|h_1|^3|1-\beta|}{|h_1^2(\chi_1+\chi_2)-h_2\chi_3|}; & |1-\beta| \geqslant \left|1+\dfrac{\chi_2}{\chi_1}-\dfrac{h_2\chi_3}{h_1^2\chi_1}\right|, \end{cases}$$

where χ_1, χ_2, and χ_3 are given by (10), (11), and (12), respectively.

Proof. From (28), for $\beta \in \mathbb{R}$, we write

$$a_3 - \beta a_2^2 = \frac{h_1(u_2 - v_2)}{2[\varepsilon(3\psi_3 - (1-\lambda)) + (1-\varepsilon)(3\theta\psi_3 - 1)]} + (1-\beta)a_2^2. \quad (34)$$

By substituting (27) in (34), we obtain

$$a_3 - \beta a_2^2 = \frac{h_1(u_2 - v_2)}{2\chi_1} + (1-\beta)\frac{h_1^3(u_2+v_2)}{2[h_1^2(\chi_1+\chi_2) - h_2\chi_3]} =$$
$$= h_1\Big[\Big(\Xi(\beta,\varepsilon,\lambda,\theta,\psi_2,\psi_3) + \frac{1}{2\chi_1}\Big)u_2 + \Big(\Xi(\beta,\varepsilon,\lambda,\theta,\psi_2,\psi_3) - \frac{1}{2\chi_1}\Big)v_2\Big], \quad (35)$$

where

$$\Xi(\beta,\varepsilon,\lambda,\theta,\psi_2,\psi_3) = \frac{(1-\beta)h_1^2}{2[h_1^2(\chi_1+\chi_2) - h_2\chi_3]}.$$

Now, taking modulus of (35), we have

$$|a_3 - \beta a_2^2| \leqslant \begin{cases} \frac{|h_1|}{|\chi_1|}; & 0 \leqslant \Xi(\beta,\varepsilon,\lambda,\theta,\psi_2,\psi_3) \leqslant \dfrac{1}{2|\chi_1|}, \\ 2|h_1||\Xi(\beta,\varepsilon,\lambda,\theta,\psi_2,\psi_3)|; & \Xi(\beta,\varepsilon,\lambda,\theta,\psi_2,\psi_3) \geqslant \dfrac{1}{2|\chi_1|}, \end{cases}$$

which, after simple computation, we obtain

$$|a_3 - \beta a_2^2| \leqslant \begin{cases} \dfrac{|h_1|}{|\chi_1|}; & |1-\beta| \leqslant \left|1 + \dfrac{\chi_2}{\chi_1} - \dfrac{h_2\chi_3}{h_1^2\chi_1}\right|, \\ \dfrac{|h_1|^3|1-\beta|}{|h_1^2(\chi_1+\chi_2) - h_2\chi_3|}; & |1-\beta| \geqslant \left|1 + \dfrac{\chi_2}{\chi_1} - \dfrac{h_2\chi_3}{h_1^2\chi_1}\right|. \end{cases}$$

The proof is completed. □

Set the parameter values $\varepsilon = 0$, $\varepsilon = 1$, and $\beta = 1$ to derive the three corollaries from Theorem 2, as follows:

Corollary 3. *Let a function f be in the class $\mathcal{BPS}_\Sigma(0,\lambda,\theta,\gamma,\mu;h)$ and $\beta \in \mathbb{R}$. Then*

$$|a_3 - \beta a_2^2| \leqslant \begin{cases} \dfrac{|h_1|}{|3\theta\psi_3 - 1|}, \\ \text{if } |1-\beta| \leqslant \left|1 + \dfrac{h_1^2(1 - 2\theta(\psi_2 - \theta + 1)) - h_2(2\theta\psi_2 - 1)^2}{h_1^2(3\theta\psi_3 - 1)}\right|, \\ \dfrac{|h_1|^3|1-\beta|}{|h_1^2((3\theta\psi_3 - 2\theta(\psi_2 - \theta + 1)) - h_2(2\theta\psi_2 - 1))|}, \\ \text{if } |1-\beta| \geqslant \left|1 + \dfrac{h_1^2(1 - 2\theta(\psi_2 - \theta + 1)) - h_2(2\theta\psi_2 - 1)^2}{h_1^2(3\theta\psi_3 - 1)}\right|. \end{cases}$$

Corollary 4. *Let a function f be in the class $\mathcal{BPS}_\Sigma(1,\lambda,\theta,\gamma,\mu;h)$ and $\beta \in \mathbb{R}$. Then*

$$|a_3-\beta a_2^2| \leqslant \begin{cases} \dfrac{|h_1|}{|3\psi_3-(1-\lambda)|}; \\ \text{if } |1-\beta| \leqslant \left|1+\dfrac{h_1^2(1-\lambda)(\frac{1}{2}(2-\lambda)-2\psi_2)-h_2(2\psi_2-(1-\lambda))^2}{h_1^2(3\psi_3-(1-\lambda))}\right|, \\ \dfrac{|h_1|^3|1-\beta|}{|h_1^2\left(3\psi_3-\frac{1}{2}(1-\lambda)(\lambda+4\psi_2)\right)-h_2(2\psi_2-(1-\lambda))|}; \\ \text{if } |1-\beta| \geqslant \left|1+\dfrac{h_1^2(1-\lambda)(\frac{1}{2}(2-\lambda)-2\psi_2)-h_2(2\psi_2-(1-\lambda))^2}{h_1^2(3\psi_3-(1-\lambda))}\right|. \end{cases}$$

Corollary 5. *Let a function f be in the class $\mathcal{BPS}_\Sigma(\varepsilon,\lambda,\theta,\gamma,\mu;h)$ and $\beta \in \mathbb{R}$. Then*

$$|a_3-a_2^2| \leqslant \frac{|h_1|}{|\varepsilon(3\psi_3-(1-\lambda))+(1-\varepsilon)(3\theta\psi_3-1)|}.$$

6. Conclusion. In this study, we have investigated coefficient estimates related to recently defined subclass of bi-univalent function in open unit disk $\mathbb{U}$, as specified in Definition 4. Several examples and specific cases have been examined. We compute the Tylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, along with estimations for the Fekete-Szegö functional problem, for functions belonging to this class. In subsequent studies, exploring Hankel determinants of second, third, and fourth orders within the aforementioned subclasses holds promise for future research avenues.

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