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SOME BOHR-TYPE INEQUALITIES FOR SENSE-PRESERVING HARMONIC MAPPINGS

Abstract. In this paper, we investigate the Bohr-type radii for various forms of Bohr-type inequalities for the sense-preserving harmonic mapping of the form $f(z) = h(z) + \overline{g(z)}$.

Key words: *Bohr-type inequality, sense-preserving harmonic mapping, Taylor series coefficient*

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1. Introduction and Preliminaries. One of the inequalities that exist in the theory of majorant series $M_f(r) = \sum\limits_{n=0}^{\infty} |a_n| r^n$, is the classical Bohr inequality established by Harald Bohr [3] in 1914. The inequality of Bohr [3] is stated as follows:

Theorem A. *If $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon z < 1\}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then*

$$\sum_{n=0}^{\infty} |a_n||z|^n = \sum_{n=0}^{\infty} |a_n| r^n \leqslant 1 \qquad \textit{for} \quad r \leqslant \frac{1}{3}. \tag{1}$$

The number 1/3 cannot be improved.

Initially, Bohr [3] obtained this inequality for $r \leqslant \frac{1}{6}$ and was thereafter independently sharpened by Riesz, Schur, and Wiener for $r \leqslant \frac{1}{3}$. Thus, the constant 1/3 is now referred to in the literature as the classical Bohr radius.

Now, let $\mathcal{B}$ denote the class of analytic functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$, such that $|f(z)| < 1$. Several researchers have studied Bohr's inequality for $f(z) \in \mathcal{B}$ in various settings and the inequality has been extended to some special functions, such as harmonic mapping, univalent and convex functions, locally univalent harmonic mapping, etc.; for example, (see [7], [14]). Other extensions and improvements in this topic include [9], [10], [11], [12], [13], [15], [16]. The following concept of harmonic mappings in the complex plane was discussed by Duren in [6].

A complex-valued function $f(z) = u(x, y) + iv(x, y)$ is said to be *harmonic* if the real and imaginary parts u and v satisfy the Laplace equation $\triangle f = 0$. The complex-valued harmonic function $f(z)$ is called harmonic mapping of a domain $\mathbb{D} \subset \Omega$ if and only if it is univalent (one-to-one) in $\mathbb{D}$. Thus, by harmonic mapping, we mean a complex-valued univalent harmonic function. If $f(0) = h(0)$, then $f(z)$ can be written in the canonical form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n},$$

where $g(0) = 0$, $h(z)$ is called the analytic part and $g(z)$ is called the co-analytic part of f. The Jacobian of f is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2. \tag{2}$$

We say that f is *sense-preserving* if $J_f(z) > 0$. Thus, univalent and sense-preserving if and only if $J_f(z) > 0$. That is, if $|g'(z)| < |h'(z)|$. Kayumov et al. [14] established the Bohr inequality for sense-preserving harmonic mappings in some settings; we state several of their results in the following theorems:

Theorem B. *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ *is a sense- preserving harmonic mapping of the disk* $\mathbb{D}$*, where* h *is a bounded function in* $\mathbb{D}$*. Then*

$$|a_0| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leqslant 1 \quad \textit{for all} \quad r \leqslant \frac{1}{5}, \tag{3}$$

and the number $1/5$ *is sharp. Moreover, if* $a_0 = 0$ *or* $|a_0|$ *in* (3) *is replaced by* $|a_0|^2$*, then the constant* $1/5$ *could be replaced by* $1/3$*, which is also sharp.*

Theorem C. *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ *is a sense-preserving harmonic mapping of the disk* $\mathbb{D}$*, where* h *is a bounded* *function in* $\mathbb{D}$*. Then*

$$|a_0| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leqslant 1 \quad \textit{for} \quad r \leqslant 0.2942\ldots. \tag{4}$$

The number $0.2942\ldots$ *cannot be replaced by a number greater than* $R = 0.299825\ldots$*, where* R *is the positive root of the equation*

$$\frac{4R}{1-R} + 2\ln(1-R) = 1.$$

The main purpose of this paper is to obtain some sharp Bohr-type radii versions of Theorems B and C by replacing $|a_0|$ with the Taylor series coefficient $|h(z)|$, $|a_1|$ with $|h'(z)|$, $|a_2|$ with $\frac{|h''(z)|}{2!}$ and then $|a_n|$ with order $\frac{|h^{(n)}(z)|}{n!}$. For this purpose, we need the following well-known lemmas.

Lemma 1. [16] *If* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic on the unit disk* $\mathbb{D}$ *and* $|h(z)| \leqslant 1$ *for all* $z \in \mathbb{D}$*. Then*

$$|h(z)| \leqslant \frac{r + |a_0|}{1 + |a_0| r}, \quad \textit{where } r = |z| \textit{ and } |a_0| \in [0, 1). \tag{5}$$

Lemma 2. [16] *If* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic and* $|h(z)| < 1$ *all* $z \in \mathbb{D}$*, then for* $n = 1, 2, \ldots,$ *have*

$$|h^{(n)}(z)| \leqslant \frac{n!\,(1 - |h(z)|^2)}{(1 - |z|^2)^n}(1 + |z|)^{n-1}, \ |z| < 1. \tag{6}$$

Lemma 3. [11] *Suppose* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *with* $h(z) \in \mathcal{B}$*. Then*

$$\sum_{n=1}^{\infty} |a_n| r^n \leqslant r \frac{1 - |a_0|^2}{1 - |a_0| r}, \qquad \textit{for} \quad r \leqslant \frac{1}{3}. \tag{7}$$

Lemma 4. [11] *Let* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *with* $h(z) \in \mathcal{B}$, *then*

$$\sum_{n=1}^{\infty} |a_n|^2 r^n \leqslant \frac{(1-|a_0|^2)^2 r}{1-|a_0|^2 r}, \quad \text{for } r < 1. \tag{8}$$

Lemma 5. [11] *Let* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *with* $h \in \mathcal{B}$ *and* $|g'(z)| \leqslant |h'(z)|$. *Then*

$$\sum_{n=0}^{\infty} |b_n|^2 r^n \leqslant \sum_{n=0}^{\infty} |a_n|^2 r^n. \tag{9}$$

Lemma 6. [4] *If* $f(z) = h(z) + \overline{g(z)}$ *is a sense-preserving harmonic mapping with* $g'(0) = 0$, *then*

$$\sum_{n=2}^{\infty} n|b_n| r^n \leqslant \sum_{n=2}^{\infty} \left(\frac{n-1}{n}\right) |a_{n-1}| r^n,\ n \geqslant 2. \tag{10}$$

2. Main Results.

Theorem 1. *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ *is a harmonic mapping preserving the sense of the disk* $\mathbb{D}$ *where* $|h(z)| < 1$ *for* $z \in \mathbb{D}$. *Then*

$$M_{h,g}(r) = |h(z)| + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leqslant 1 \quad \text{for } r \leqslant R_1 = \frac{2\sqrt{3}-3}{3}, \tag{11}$$

where $r = |z|$ *and the constant* R_1 *cannot be improved. Moreover,*

$$|h(z)|^2 + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leqslant 1 \quad \text{for all } r \leqslant R_2 = \sqrt{5}-2, \tag{12}$$

and the constant R_2 *cannot be improved.*

Proof. Let $|a_0| = a$. Then, by the classical Cauchy-Schwarz inequality and Lemmas 4 and 5, we have

$$\sum_{n=1}^{\infty}|b_n|r^n \leqslant \sqrt{\sum_{n=1}^{\infty}|b_n|^2 r^n}\sqrt{\sum_{n=1}^{\infty} r^n} \leqslant \sqrt{\sum_{n=1}^{\infty}|a_n|^2 r^n}\sqrt{\sum_{n=1}^{\infty} r^n} \leqslant$$
$$\leqslant \sqrt{r\frac{(1-a^2)^2}{1-a^2 r}}\sqrt{\frac{r}{1-r}} = \frac{r(1-a^2)}{\sqrt{(1-r)(1-a^2 r)}}. \quad (13)$$

From (11) and applying (13), Lemma 1 and, Lemma 3, we have

$$M_{h,g}(r) = |h(z)| + \sum_{n=1}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n \leqslant$$
$$\leqslant \frac{a+r}{1+ar} + r\frac{1-a^2}{1-ar} + \frac{r(1-a^2)}{\sqrt{(1-r)(1-a^2 r)}} = P(a,r),$$

where

$$P(a,r) = \frac{a+r}{1+ar} + r\frac{1-a^2}{1-ar} + \frac{r(1-a^2)}{\sqrt{(1-r)(1-a^2 r)}}. \quad (14)$$

Easy computations show that for each fixed $r \in [0, \frac{2\sqrt{3}-3}{3}]$, $P(a,r)$ is a strictly increasing function of $a \in [0,1]$. Since $|a_0| = a < 1$, then for each fixed $r \in [0, \frac{2\sqrt{3}-3}{3}]$, $P(a,r) < P(1,r)$, that is,

$$P(a,r) < \frac{1+r}{1+r} + 0 + 0 = 1.$$

Therefore, for each fixed $r \in [0, \frac{2\sqrt{3}-3}{3}]$, $M_{h,g} \leqslant P(a,r) < 1$. We now need to show that for each fixed $r \in [0, \frac{2\sqrt{3}-3}{3}]$, $P(a,r)$ is a strictly increasing function of $a \in [0,1]$.

Now, differentiating $P(a,r)$ *w.r.t.* a, we obtain

$$\frac{\partial P(a,r)}{\partial a} = \frac{1-r^2}{(1+ar)^2} + r\frac{(r-2a+a^2 r)}{(1-ar)^2} + \frac{ar(a^2 r + r - 2)}{(1-a^2 r)\sqrt{(1-r)(1-a^2 r)}},$$

$$\frac{\partial^2 P(a,r)}{\partial a^2} = -\frac{2r(1-r^2)}{(1+ar)^3} - \frac{2r(1-r^2)}{(1-ar)^3} - \frac{r(2-r+a^2 r-2a^2 r^2)}{(1-a^2 r)^2\sqrt{(1-r)(1-a^2 r)}}.$$

It is easy to see (with simple computations) that $\frac{\partial^2 P(a,r)}{\partial a^2} \leqslant 0$ for $a \in [0,1)$ and $r \in (0,1)$. For $|a_0| = a < 1$, clearly, $\frac{\partial P(a,r)}{\partial a} > \frac{\partial P(1,r)}{\partial a}$.

Thus $\frac{\partial P(a,r)}{\partial a} > 0$ if $\frac{\partial P(1,r)}{\partial a} \geqslant 0$, which is equivalent to

$$\frac{1-r^2}{(1+r)^2} + r\frac{2r-2}{(1-r)^2} + \frac{r(2r-2)}{(1-r)^2} \geqslant 0,$$

and simplifying gives

$$3r^3 + 9r^2 + 5r - 1 = 3(1+r)\left(r + \frac{3+2\sqrt{3}}{3}\right)\left(r + \frac{3-2\sqrt{3}}{3}\right) \leqslant 0. \quad (15)$$

Thus, for $r \in (0,1)$, (15) holds only if $r \leqslant \frac{2\sqrt{3}-3}{3}$.

To complete the proof, we need to show the sharpness of the constant $R_1 = \frac{2\sqrt{3}-3}{3}$. To do this, choose $a \in [0,1)$ and consider the function $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = \frac{a-z}{1-az} = a - (1-a^2)\sum_{n=1}^{\infty} a^{n-1}z^n = a + \sum_{n=1}^{\infty} a_n z^n, z \in \mathbb{D}, \quad (16)$$

and $g(z) = \lambda h(z)$, where $|\lambda| = 1$. Here $a_n = -(1-a^2)a^{n-1}$ and $b_n = \lambda a_n$ for $n \geqslant 1$. For this function, we have

$$|h(-r)| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n =$$
$$= \frac{a+r}{1+ar} + (1-a^2)\sum_{n=1}^{\infty} a^{n-1}r^n + (1-a^2)\sum_{n=1}^{\infty} a^{n-1}r^n =$$
$$= \frac{a+r}{1+ar} + \frac{2(1-a^2)r}{1-ar},$$

and the last expression is greater than or equal to 1 if and only if

$$r \geqslant \frac{\sqrt{17a^2+22a+9} - 3 - 3a}{2a(1+2a)}.$$

Since $a < 1$, a could be chosen arbitrarily close to 1^-, thus, $r \geqslant \frac{2\sqrt{3}-3}{3}$. This shows that the constant $\frac{2\sqrt{3}-3}{3}$ cannot be improved. Hence, the proof of the first part of the theorem is complete. For the second part of Theorem 1, we proceed from (14) by squaring $(a+r)/(1+ar)$ and following the style of proof of the first part of the theorem to obtain the desired Bohr-type radius. Thus, the proof of Theorem 1 is complete. □

Theorem 2. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a sense-preserving harmonic mapping of the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$. Then*

$$M'_{h,g}(r) = |h(z)| + r|h'(z)| + \sum_{n=2}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leqslant 1, r = |z| \tag{17}$$

for $r \leqslant R_3 = 0.16709\ldots$, where the constant R_3 is the best possible. However

$$|h(z)|^2 + r|h'(z)| + \sum_{n=2}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leqslant 1 \tag{18}$$

for all $r \leqslant R_4 = 0.2555\ldots$. The constant R_4 is the best possible.

Proof. Let $z = re^{i\theta}$ and $n = 1$ in Lemma 2. We get

$$|h'(z)| \leqslant \frac{1 - |h(z)|^2}{1 - r^2}. \tag{19}$$

By Lemma 1 and (19), we have the following:

$$\begin{aligned}
|h(z)| + r|h'(z)| &\leqslant |h(z)| + \frac{r}{1-r^2}\left(1 - |h(z)|^2\right) = \\
&= \frac{r}{1-r^2}(1 + |h(z)|)(1 - |h(z)|) + |h(z)| \leqslant \\
&\leqslant \frac{r}{1-r^2}\Big(1 + a + r1 + ar\Big)(1 - |h(z)|) + |h(z)| \leqslant \\
&\leqslant \frac{2r}{1-r^2}(1 - |h(z)|) + |h(z)| = \\
&= \frac{2r}{1-r^2} + \left(1 - \frac{2r}{1-r^2}\right)|h(z)| \leqslant \\
&\leqslant \frac{a+r}{1+ar} + \frac{r}{1-r^2}\left(1 - \left(\frac{a+r}{1+ar}\right)^2\right),
\end{aligned} \tag{20}$$

where the last inequality holds for any $r \in [0, \sqrt{2}-1]$, since $\dfrac{2r}{1-r^2} \leqslant 1$ if $r \in [0, \sqrt{2}-1]$.

From (17), employing (20) and (13), we have

$$M'_{h,g}(r) = |h(z)| + r|h'(z)| + \sum_{n=2}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n \leqslant$$

$$\leqslant \frac{a+r}{1+ar} + \frac{r}{1-r^2}\left(1-\left(\frac{a+r}{1+ar}\right)^2\right) + \frac{a(1-a^2)r^2}{1-ar} + \frac{r(1-a^2)}{\sqrt{(1-r)(1-a^2r)}} =$$

$$= \frac{r+a}{1+ar} + \frac{r(1-a^2)}{(1+ar)^2} + \frac{a(1-a^2)r^2}{1-ar} + \frac{r(1-a^2)}{\sqrt{(1-r)(1-a^2r)}} =$$

$$= P(a,r), \qquad \text{for} \quad 0 \leqslant r \leqslant \sqrt{2}-1.$$

Differentiating $P(a,r)$ partially *w.r.t.* a, we obtain

$$\frac{\partial P(a,r)}{\partial a} = \frac{1-r^2}{(1+ar)^2} - \frac{2r(a+r)}{(1+ar)^3} + \frac{(1-3a^2+2a^3r)r^2}{(1-ar)^2} + \\ + \frac{ar(a^2r+r-2)}{(1-a^2r)\sqrt{(1-r)(1-a^2r)}}. \tag{21}$$

For $a \in [0,1)$ and $r \in (0,1)$, short computations show that $\frac{\partial P(a,r)}{\partial a} > 0$ i.e. $P(a,r)$ is an increasing function. Hence,

$$M'_{h,g} \leqslant P(a,r) < P(1,r) = \frac{r+1}{1+r} = 1.$$

Differentiating $P(a,r)$ again for all $a \in [0,1)$ and $r \in (0,1)$, we get $\frac{\partial^2 P(a,r)}{\partial a^2} \leqslant 0$. Thus $\frac{\partial P(a,r)}{\partial a} > \frac{\partial P(1,r)}{\partial a}$. Therefore, $\frac{\partial P(a,r)}{\partial a} > 0$ if $\frac{\partial P(1,r)}{\partial a} \geqslant 0$, and this is equivalent to

$$\frac{1-r}{1+r} - \frac{2r}{(1+r)^2} - \frac{2r^2}{1-r} - \frac{2r}{1-r} \geqslant 0. \tag{22}$$

Simplifying (22), we obtain $2r^4+5r^3+5r^2+5r-1 \leqslant 0$, which holds for $r \in (0,1)$ only if $r \leqslant R_3$, where R_3 is the minimum positive root of the equation $2r^4+5r^3+5r^2+5r-1=0$. To show that the number R_3 is sharp, consider $f(z) = h(z) + \overline{g(z)}$ as in (16). For the function, we have

$$|h(-r)| + r|h'(-r)| + \sum_{n=2}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n =$$

$$= \frac{a+r}{1+ar} + \frac{r(1-a^2)}{(1+ar)^2} + \frac{a(1-a^2)r^2}{1-ar} + \frac{(1-a^2)r}{1-ar}. \quad (23)$$

Expression (23) is greater than 1 if and only if

$$(1-a)(-1+(3+2a)r+(3+4a+2a^2)r^2+a(6+6a+a^2)r^3+ \\ +3a^2(1+a)r^4) > 0. \quad (24)$$

Now, let $Q(a,r) = -1+(3+2a)r+(2a+3a^2)r^2+(2a^2+3a^3)r^3+a^3(1+a)r^4$. Then $\frac{\partial Q}{\partial a} = 2r+(2+6a)r^2+(4a+9a^2)r^3+(3a^2+4a^3)r^4$. Easy computations for $r \in [0,1)$ reveal that $\frac{\partial Q}{\partial a} \geqslant 0$. Since $a < 1$, we have $Q(a,r) \leqslant Q(1,r)$, that is,

$$Q(a,r) \leqslant Q(1,r) = -1+5r+5r^2+5r^3+2r^4.$$

Hence, (23) is less than or equal to 1 for all $a \in [0,1)$ only when $r \leqslant R_3$, where R_3 is minimum positive real root of $2r^4+5r^3+5r^2+5r-1=0$. This proves the sharpness of R_3 and, thus, the proof of the first part of Theorem 2 is complete. The proof of the second part easily follows the same style of proof as in the first part. □

Theorem 3. *Let* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ *be a sense-preserving harmonic mapping of the disk* $\mathbb{D}$*, where* $h(z) \in \mathcal{B}$*. Then*

$$M''_{h,g}(r) = |h(z)| + r|h'(z)| + \frac{r^2}{2!}|h''(z)| + \sum_{n=3}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n \leqslant 1 \quad (25)$$

for $r \leqslant R_5 = 0.16817\ldots$*, where* R_5 *cannot be improved. Moreover,*

$$M_{h^2,g}(r) = |h(z)|^2 + r|h'(z)| + \frac{r^2}{2!}|h''(z)| + \sum_{n=3}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n \leqslant 1 \quad (26)$$

for $r \leqslant R_6 = 0.25782\ldots$*. The constant* R_6 *is the best possible.*

Proof. Let $|a_0| = a$. Since $h(z) \in \mathcal{B}$, then $|a_n| \leqslant 1-|a_0|^2$, $n \geqslant 1$. Hence,

$$\sum_{n=3}^{\infty}|a_n|r^n \leqslant (1-a^2)\sum_{n=3}^{\infty} r^n = \frac{(1-a^2)r^3}{1-r}. \quad (27)$$

Let $z = re^{i\theta}$ and $n = 2$ in Lemma 2. We get

$$\frac{|h''(z)|}{2!} \leqslant \frac{1-|h(z)|^2}{(1-r)(1-r^2)}.$$

From (19), we have

$$\begin{aligned}
&|h(z)|+r|h'(z)|+\frac{1}{2!}r^2|h''(z)|\leqslant\\
&\leqslant |h(z)|+\frac{r}{1-r^2}(1-|h(z)|^2)+\frac{r^2(1-|h(z)|^2)}{(1-r)(1-r^2)}=\\
&=\frac{r}{(1-r)(1-r^2)}(1+|h(z)|)(1-|h(z)|)+|h(z)|\leqslant\\
&\leqslant \frac{r}{(1-r)(1-r^2)}\left(1+\frac{a+r}{1+ar}\right)(1-|h(z)|)+|h(z)|\leqslant\\
&\leqslant \frac{2r}{(1-r)(1-r^2)}(1-|h(z)|)+|h(z)|=\\
&=\frac{2r}{(1-r)(1-r^2)}+\left(1-\frac{2r}{(1-r)(1-r^2)}\right)|h(z)|\leqslant\\
&\leqslant \frac{2r}{(1-r)(1-r^2)}+\left(1-\frac{2r}{(1-r)(1-r^2)}\right)\frac{a+r}{1+ar}=\\
&=\frac{a+r}{1+ar}+\frac{r}{(1-r)(1-r^2)}\left(1-\left(\frac{a+r}{1+ar}\right)^2\right). \qquad (28)
\end{aligned}$$

Since $\frac{2r}{(1-r)(1-r^2)}\leqslant 1$ if $r_5\in(0.3,0.4)$, then the last inequality holds for any $r_5\in(0.3,0.4)$, where r_5 is the unique root of $1-3r-r^2+r^3=0$. Now, from (25) applying (28), (27), and (13), we therefore have

$$\begin{aligned}
&M''_{h,g}(r)=|h(z)|+r|h'(z)|+\frac{1}{2!}r^2|h''(z)|+\sum_{n=3}^{\infty}|a_n|r^n+\sum_{n=1}^{\infty}|b_n|r^n\leqslant\\
&\leqslant \frac{a+r}{1+ar}+\frac{r}{(1-r)(1-r^2)}\left(1-\left(\frac{a+r}{1+ar}\right)^2\right)+\frac{(1-a^2)r^3}{1-r}+\\
&\qquad+\frac{r(1-a^2)}{\sqrt{(1-r)(1-a^2r)}}=\\
&=\frac{r+a}{1+ar}+\frac{(1-a^2)r}{(1-r)(1+ar)^2}+\frac{(1-a^2)r^3}{1-r}+\frac{(1-a^2)r}{\sqrt{(1-r)(1-a^2r)}}.
\end{aligned}$$

That is, $M''_{h,g}\leqslant P(a,r)$, where

$$P(a,r)=\frac{r+a}{1+ar}+\frac{(1-a^2)r}{(1-r)(1+ar)^2}+\frac{(1-a^2)r^3}{1-r}+\frac{(1-a^2)r}{\sqrt{(1-r)(1-a^2r)}}. \qquad (29)$$

Then, differentiating $P(a,r)$ *w.r.t.* a, we obtain

$$\frac{\partial P(a,r)}{\partial a} = \frac{1-r^2}{(1+ar)^2} - \frac{2r(a+r)}{(1-r)(1+ar)^3} - \frac{2ar^3}{1-r} + \\ + \frac{ar(a^2r+r-2)}{(1-a^2r)\sqrt{(1-r)(1-a^2r)}}$$

With some computations for $a \in [0,1)$ and $r \in (0,1)$, it is evident that $\dfrac{\partial P(a,r)}{\partial a} > 0$ and $\dfrac{\partial^2 P(a,r)}{\partial a^2} \leqslant 0$. Thus, for $|a_0| = a < 1$, $P(a,r) < P(1,r)$, and $\dfrac{\partial P(a,r)}{\partial a} > \dfrac{\partial P(1,r)}{\partial a}$, respectively. Therefore,

$$M''_{h,g} \leqslant P(a,r) < \frac{r+1}{1+r} + 0 = 1.$$

Also, $\dfrac{\partial P(a,r)}{\partial a} > 0$ if $\dfrac{\partial P(1r)}{\partial a} \geqslant 0$. Equivalently, we have

$$\frac{1-r}{1+r} - \frac{2r}{(1-r)(1+r)^2} - \frac{2r^3}{1-r} - \frac{2r}{1-r} \geqslant 0, \tag{30}$$

which, when simplified, gives

$$2r^5 + 4r^4 + 3r^3 + 5r^2 + 5r - 1 \leqslant 0. \tag{31}$$

Inequality (28) holds for $r \in (0,1)$ only if $r \leqslant R_5$, where R_5 is the real root of $2r^5 + 4r^4 + 3r^3 + 5r^2 + 5r - 1 = 0$. The sharpness of constants R_5 can be shown by adopting the style of proof of Theorems 1 and 2. Also, the proof of the second part of Theorem 3 easily follows by replacing $\dfrac{a+r}{1+ar}$ in (29) with $\left(\dfrac{a+r}{1+ar}\right)^2$ and then following the same line of proof. This completes the proof of Theorem 3. □

Theorem 4. *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ *is an harmonic mapping of the disk* $\mathbb{D}$*, such that* $|g'(z)| \leqslant |h'(z)|$ *and* $|h(z)| < 1$ *for* $z \in \mathbb{D}$*. Then*

$$M^n_{h,g}(r) = |h(z)| + \sum_{n=1}^{\infty} \frac{|h^{(n)}(z)|}{n!} r^n + \sum_{n=1}^{\infty} \frac{|g^{(n)}(z)|}{n!} r^n \leqslant 1 \tag{32}$$

for $|z| = r \leqslant R_n = \dfrac{\sqrt{33}-5}{4}$. *The constant* R_n *cannot be improved.*

Proof. From Theorem 4 we have: $|g'(z)| \leqslant |h'(z)|$. By letting $z = re^{i\theta}$ in Lemma 2, we get

$$\frac{|h^{(n)}(z)|}{n!} \leqslant \frac{1-|h(z)|^2}{(1+r)(1-r)^n} \quad \text{and} \quad \frac{|g^{(n)}(z)|}{n!} \leqslant \frac{1-|h(z)|^2}{(1+r)(1-r)^n}. \tag{33}$$

From (32) and by Lemma 1, (33), and (13), we have the following:

$$\begin{aligned}
M^n_{h,g}(r) &= |h(z)| + \sum_{n=1}^{\infty} \frac{|h^{(n)}(z)|}{n!} r^n + \sum_{n=1}^{\infty} \frac{|g^{(n)}(z)|}{n!} r^n \leqslant \\
&\leqslant |h(z)| + 2\frac{1-|h(z)|^2}{1+r} \sum_{n=1}^{\infty} \frac{r^n}{(1-r)^n} = \\
&= \frac{2r}{(1+r)(1-2r)}(1+|h(z)|)(1-|h(z)|) + |h(z)| \leqslant \\
&\leqslant \frac{2r}{(1+r)(1-2r)}\left(1+\frac{a+r}{1+ar}\right)(1-|h(z)|) + |h(z)| \leqslant \\
&\leqslant \frac{4r}{(1+r)(1-2r)}(1-|h(z)|) + |h(z)| = \\
&= \frac{4r}{(1+r)(1-2r)} + \left(1 - \frac{4r}{(1+r)(1-2r)}\right)|h(z)| \leqslant \\
&\leqslant \frac{4r}{(1+r)(1-2r)} + \left(1 - \frac{4r}{(1+r)(1-2r)}\right)\frac{a+r}{1+ar} = \\
&= \frac{r+a}{1+ar} + \frac{2r(1-r)(1-a^2)}{(1-2r)(1+ar)^2} = P(a,r),
\end{aligned}$$

where the last inequality holds for any $r \in [0, \dfrac{\sqrt{33}-5}{4}]$, since
$\dfrac{4r}{(1+r)(1-2r)} \leqslant 1$ if $r \in [0, \dfrac{\sqrt{33}-5}{4}]$.

First partial differentiation of $P(a,r)$ w.r.t. a yields

$$\frac{\partial P(a,r)}{\partial a} = \frac{1-r^2}{(1+ar)^2} - \frac{4r(1-r)(a+r)}{(1-2r)(1+ar)^3}.$$

After elementary Computations of $P(a,r)$ for $a \in [0,1)$ and $r \in [0,1)$, we find that $P(a,r){>}0$. Since $a < 1$, then

$$M^n_{h,g} \leqslant P(a,r) < P(1,r) = \frac{r+1}{1+r} = 1.$$

After differentiating $\dfrac{\partial P(a,r)}{\partial a}$, we find that $\dfrac{\partial^2 P(a,r)}{\partial a^2} \leqslant 0$ for $a \in [0,1)$ and $r \in (0,1)$. Hence, $\dfrac{\partial P(a,r)}{\partial a} > \dfrac{\partial P(1,r)}{\partial a}$ for $a < 1$. Now, $\dfrac{\partial P(a,r)}{\partial a} > 0$ if $\dfrac{\partial P(1,r)}{\partial a} \geqslant 0$, which can equivalently be written as

$$\frac{1-r}{1+r} - \frac{4r(1-r)}{(1-2r)(1+r)^2} \geqslant 0. \tag{34}$$

Simplifying (34), we get $2r^2 + 5r - 1 \leqslant 0$ and this holds for $r \in (0,1)$ only when $r \leqslant R_n = \dfrac{\sqrt{33}-5}{4}$.

To prove the sharpness of the number R_n, consider $f(z) = h(z) + \overline{g(z)}$ as in (16). Then we have that $h^{(n)}(z) = \dfrac{-n!(1-a^2)a^{n-1}}{(1-az)^{n+1}}$ and $g^{(n)}(z) = \lambda \dfrac{-n!(1-a^2)a^{n-1}}{(1-az)^{n+1}}$. For this function, we find that

$$\begin{aligned} &|h(r)| + \sum_{n=1}^{\infty} \frac{|h^{(n)}(r)|}{n!} r^n + \sum_{n=1}^{\infty} \frac{|g^{(n)}(r)|}{n!} r^n = \\ &= \frac{a-r}{1-ar} + \frac{2(1-a^2)}{1-ar} \sum_{n=1}^{\infty} \frac{a^{n-1} r^n}{(1-ar)^n} = \frac{a-r}{1-ar} + \frac{2(1-a^2)r}{(1-ar)(1-2ar)}. \end{aligned}$$

The last expression is larger than or equal to 1 if and only if

$$r \geqslant \frac{\sqrt{1+16a+16a^2} - (1+4a)}{4a}.$$

Since a could be chosen close to 1^-, we have $r \geqslant \dfrac{\sqrt{33}-5}{4}$. This shows that the constant R_n cannot be improved, and thus, the proof of Theorem 4 is complete. □

Theorem 5. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a sense-preserving harmonic mapping of the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$ and $g'(0) = 0$. Then*

$$|h(z)| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leqslant 1 \quad \text{for } r \leqslant R_7 = 0.2215\ldots \tag{35}$$

where $r = |z|$*. The inequality is sharp and the constant* R_7 *is the minimum positive root of the equation* $3r^2 + 6r + 2(1 - r^2)\ln(1 - r) = 1$*.*

Remark. *The Bohr-type inequalities for Theorems 5 and 6 are sharp for the function* $f(z) = h(z) + \overline{g(z)}$*, where*

$$h(z) = \frac{a - z}{1 - az} \quad \textit{and} \quad g'(z) = zh'(z), \quad 0 \leqslant a < 1. \tag{36}$$

Proof of Theorem 5. Let $|a_0| = a$ and $h(z) \in \mathcal{B}$. Then we have $|a_n| \leqslant 1 - |a_0|^2 = 1 - a^2$, $n \geqslant 1$. Hence, from (35), using Lemma 1 and Lemma 6, we obtain

$$\begin{aligned} |h(z)| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leqslant \\ \leqslant \frac{r + a}{1 + ar} + (1 - a^2)\frac{r}{1 - r} + \sum_{n=2}^{\infty} \left(\frac{n-1}{n}\right) |a_{n-1}| r^n \leqslant \\ \leqslant \frac{r + a}{1 + ar} + \frac{r(1 - a^2)}{1 - r} + (1 - a^2)\left(\frac{r + (1 - r) ln(1 - r)}{1 - r}\right). \end{aligned} \tag{37}$$

Now, let $\varphi(a, r) = \dfrac{r + a}{1 + ar} + \dfrac{r(1 - a^2)}{1 - r} + (1 - a^2)\Big(\dfrac{r + (1 - r)\ln(1 - r)}{1 - r}\Big)$, so that $|h(z)| + \sum\limits_{n=1}^{\infty} |a_n| r^n + \sum\limits_{n=2}^{\infty} |b_n| r^n \leqslant \varphi(a, r)$. Differentiating $\varphi(a, r)$ partially w.r.t. a twice provides

$$\frac{\partial \varphi(a, r)}{\partial a} = \frac{1 - r^2}{(1 + ar)^2} - \frac{2ar}{1 - r} - \frac{2a(r + (1 - r))\ln(1 - r))}{1 - r}$$

and

$$\frac{\partial^2 \varphi(a, r)}{\partial a^2} = -\frac{2r(1 - r^2)}{(1 + ar)^3} \frac{2r}{1 - r} - \frac{2(r + (1 - r))\ln(1 - r))}{1 - r}.$$

Clearly, $\dfrac{\partial \varphi(a, r)}{\partial a} > 0$ for $a \in [0, 1)$. Thus,

$$|h(z)| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leqslant \varphi(a,r) < \varphi(1, r) = \frac{r + 1}{1 + r} = 1.$$

Also, it is easily seen that $\dfrac{\partial^2\varphi(a,r)}{\partial a^2} \leqslant 0$ for $a \in [0,1)$. Hence, $\dfrac{\partial\varphi(a,r)}{\partial a} > \dfrac{\partial\varphi(1,r)}{\partial a} \geqslant 0$, that is,

$$\frac{1-r}{(1+r)} - \frac{2r}{1-r} - \frac{2(r+(1-r))\ln(1-r))}{1-r} \geqslant 0. \tag{38}$$

Equation (38) holds for $r \leqslant 0.2215\ldots$.

To complete the proof, we need to show that the constant $0.2215\ldots$ is sharp. To do this, consider (from (36)) the function

$$h(z) = \frac{a-z}{1-az} = a - \frac{1-a^2}{a}\sum_{n=1}^{\infty} a^n z^n.$$

Since $g'(z) = zh'(z) = -(1-a^2)\sum_{n=2}^{\infty}(n-1)a^{n-2}z^{n-1}$, then

$$g(z) = -(1-a^2)\sum_{n=2}^{\infty}\frac{n-1}{n}a^{n-2}z^n.$$

From this, we get $|a_n| = (1-a^2)a^{n-1}$ and $|b_n| = (1-a^2)\dfrac{n-1}{n}a^{n-2}$. Therefore,

$$\begin{aligned}
|h(-r)| + \sum_{n=1}^{\infty}|a_n|r^n + \sum_{n=2}^{\infty}|b_n|r^n &= \\
= \frac{a+r}{1+ar} + \sum_{n=1}^{\infty}(1-a^2)a^{n-1}r^n + \sum_{n=2}^{\infty}\frac{(1-a^2)(n-1)}{n}a^{n-2}r^n &= \\
= \frac{a+r}{1+ar} + \frac{(1-a^2)r}{1-ar} + \frac{(1-a^2)}{a^2}\,\frac{ar+(1-ar)\ln(1-ar)}{1-ar}&.
\end{aligned} \tag{39}$$

Expression (39) is greater than or equal to one, if and only if

$$\begin{aligned}
r(1+a)(1+ar) + (1+a)(1+ar)a^{-2}[ar+(1-ar)\ln(1-ar)] &- \\
-(1-r)(1-ar) \geqslant 0&.
\end{aligned} \tag{40}$$

As $a < 1$, then, as $a \to 1$, (40) becomes $3r^2+6r+2(1-r^2)\ln(1-r)-1 \geqslant 0$, and this holds if only $r \geqslant R_7 = 0.2215\ldots$, where R_7 is the minimum positive root of $3R^2+6R+2(1-R^2)\ln(1-R)=1$. This shows that the constant R_7 cannot be improved. Hence, the proof is complete.

Theorem 6. *Suppose* $f(z)=h(z)+\overline{g(z)}=\sum_{n=0}^{\infty}a_nz^n+\overline{\sum_{n=1}^{\infty}b_nz^n}$ *is a sense-preserving harmonic mapping of the disk* $\mathbb{D}$ *with* $h(z)\in\mathcal{B}$ *and* $g'(0)=0$. *Then*

$$M(h',g)=|h(z)|+r|h'(z)|+\sum_{n=2}^{\infty}|a_n|r^n+\sum_{n=2}^{\infty}|b_n|r^n\leqslant 1, r=|z| \qquad (41)$$

for $r\leqslant R_8=0.25487\ldots$. *The constant* R_8 *cannot be improved.*

Proof. Adopting the lines of proof of (20), we have

$$|h(z)|+r|h'(z)|\leqslant\frac{a+r}{1+ar}+\frac{r}{1-r^2}\left(1-\left(\frac{a+r}{1+ar}\right)^2\right), \qquad (42)$$

which holds for $r\in[0,\sqrt{2}-1]$, since $\frac{1-r^2}{2r}\geqslant 1$ if $r\in[0,\sqrt{2}-1]$. Since $h(z)\in\mathcal{B}$, we have $|a_n|\leqslant 1-|a_0|^2$, $n\geqslant 1$, therefore, using Lemma 6 for the second summation and adopting (42) in (41), we have the following:

$$\begin{aligned}M(h',g)&=|h(z)|+r|h'(z)|+\sum_{n=2}^{\infty}|a_n|r^n+\sum_{n=2}^{\infty}|b_n|r^n\leqslant\\&\leqslant\frac{a+r}{1+ar}+\frac{r}{1-r^2}\left(1-\left(\frac{a+r}{1+ar}\right)^2\right)+\frac{(1-a^2)r^2}{1-r}+\\&+(1-a^2)\left(\frac{r+(1-r)\ln(1-r)}{1-r}\right)=\\&=\frac{r+a}{1+ar}+\frac{(1-a^2)r}{(1+ar)^2}+\frac{(1-a^2)r^2}{1-r}+\\&+(1-a^2)\left(\frac{r+(1-r)\ln(1-r)}{1-r}\right). \qquad (43)\end{aligned}$$

Let $\varphi(a,r)$ be the right-hand side of (43). Then its partial derivative w.r.t. a becomes

$$\frac{\partial\varphi(a,r)}{\partial a}=\frac{1-r^2}{(1+ar)^2}-\frac{2r(a+r)}{(1+ar)^3}-\frac{2ar^2}{1-r}-$$

$$-\frac{2a(r+(1-r)\ln(1-r))}{1-r}. \quad (44)$$

Elementary computations reveal that $\dfrac{\partial\varphi(a,r)}{\partial a} > 0$ for $a \in [0,1)$ and $r \in (0,1)$. Since $a < 1$, $M(h', g) \leqslant \varphi(a,r) < \varphi(1,r) = 1$.

Also, for $a \in [0,1)$, we find that $\dfrac{\partial^2\varphi(a,r)}{\partial a^2} \leqslant 0$. Thus, since $a < 1$,

$$\frac{\partial\varphi(a,r)}{\partial a} > \frac{\partial\varphi(1,r)}{\partial a} = \frac{1-r}{1+r} - \frac{2r}{(1+ar)^2} - \frac{2r^2}{1-r} - \\ - \frac{2(r+(1-r)\ln(1-r))}{1-r} \geqslant 0. \quad (45)$$

Inequality (45) holds in $r \in$ (0,1) only if $r \leqslant 0.25487\ldots$. To complete the proof, we need to show that number $R_8 = 0.25487\ldots$ is sharp. Now, consider $f(z) = h(z) + \overline{g(z)}$ as in (36), and for this function we find that

$$h'(z) = -\frac{1-a^2}{(1-ar)^2}, |a_n| = (1-a^2)a^{n-1} \quad \text{and} \quad |b_n| = (1-a^2)\frac{n-1}{n}a^{n-2}.$$

Therefore,

$$|h(-r)| + r|h'(-r)| + \sum_{n=2}^{\infty}|a_n|r^n + \sum_{n=2}^{\infty}|b_n|r^n = \\ = \frac{a+r}{1+ar} + \frac{r(1-a^2)}{(1+ar)^2} + \frac{a(1-a^2)r^2}{1-ar} + \\ + \frac{(1-a^2)}{a^2}\frac{ar+(1-ar)\ln(1-ar)}{1-ar}. \quad (46)$$

The expression in (46) is larger than or equal to one, if and only if

$$(1+a)r(1-ar) + a(1+a)r^2(1+ar^2) + a^{-2}(1+a)(1+ar)^2[ar+ \\ + (1-ar)\ln(1-ar)] - (1-r)(1-a^2r^2) \geqslant 0. \quad (47)$$

Since $a < 1$, a could be chosen close to 1^-; thus, (47) becomes $2r^4 + 5r^3 + 5r^2 + 5r + 2(1-r^2)(1+r)\ln(1-r) - 1 \geqslant 0$, and this holds for $r \in (0,1)$ if only $r \geqslant R_8 = 0.25487\ldots$, where R_8 is the minimum positive root of $2r^4 + 5r^3 + 5r^2 + 5r + 2(1-r^2)(1+r)\ln(1-r) = 1$. Hence, this shows that the number R_8 cannot be improved. $\square$

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