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ON A BOUNDARY-VALUE PROBLEM FOR THE POISSON EQUATION AND THE CAUCHY–RIEMANN EQUATION IN A LENS

Abstract. In this paper, we consider the Dirichlet boundary-value problem for complex partial differential equations in a lens. With the help of the harmonic Green function, the Dirichlet boundary-value problem is solved explicitly for the Poisson equation in a lens. In particular, the boundary behaviors at corner points are considered. In addition, we study the explicit solvability of the Dirichlet boundary-value problem for the homogeneous Cauchy–Riemann equation in the lens.

Key words: *boundary-value problem, Dirichlet problem, Poisson Equation, Cauchy–Riemann Equation, lens*

2020 Mathematical Subject Classification: *30E25, 31A10, 35C15, 35F15, 35J25, 45E05*

1. Introduction. The theory of boundary-value problems for partial differential equations is a key area in mathematical analysis and applied mathematics, focusing on the solutions of partial differential equations subject to specified conditions on the boundaries of the domain in which they are defined. In recent years, advancements in analytic methods have further enhanced our ability to solve complex boundary-value problems, leading to significant improvements in integral representation formulas used in mathematics, mathematical physics, and engineering. Moreover, the theory of boundary-value problems for partial differential equations has profound connections to other areas of mathematics, including complex analysis, functional analysis, differential equations. It also plays a crucial role in solving problems in applied mathematics, physics and engineering, such as in analysis of wave phenomena, quantum mechanics, electricity and magnetism, etc, [1], [2], [3], [6], [7], [8].

The Dirichlet boundary-value problem is a fundamental concept in the field of differential equations and mathematical analysis. It is a type of boundary-value problem where the solution to a partial differential equation is required to satisfy specific conditions on the boundary of the domain. The main tools for solving the Dirichlet boundary-value problem for complex partial differential equations are the integral representation formulas and the Green function, which are discussed in detail in [5], [6].

Many results for boundary-value problems of complex partial differential equations have been obtained in some special areas, which are referred to in [1]– [17].H. Begehr and T. Vaitekhovich introduced in 2014 the area that was the intersection of two circles [2].This area is called a lens and is defined by $D = \mathbb{D} \cap D_r(m)$, where $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ and $D_r(m) = \{z \in \mathbb{C}\colon\ |z - m| < r\}$, $0 < r < 1 < m$, $r^2 + 1 = m^2$. They investigated the Schwarz problem in lenses of different radii. Also, the Dirichlet problem in a lens and a lune was presented by M. Akel and S. Mondal [1]. In [6], we investigated the Schwarz boundary-value problem and the Dirichlet boundary-value problem for the inhomogeneous Cauchy–Riemann equation in a new lens. This new lens is formed by the intersection of two circles of same radius.

In this paper, we investigate the Dirichlet boundary-value problem for the Poisson equation and the homogeneous Cauchy–Riemann equation in a lens using integral representation formulas. In other words, we deal with solving the Dirichlet boundary-value problem for complex partial differential equations for the first and second order. In the following, we introduce the lens area:

Let M be the lens in the complex plane $\mathbb{C}$ defined by [6],

$$M = \left\{z \in \mathbb{C}\colon\ |z-1| < \sqrt{2},\ |z+1| < \sqrt{2}\right\},$$

where $C_1 = \left\{z \in \mathbb{C}\colon\ |z-1| = \sqrt{2}\right\}$ and $C_2 = \left\{z \in \mathbb{C}\colon\ |z+1| = \sqrt{2}\right\}$ are two circles of the same radius. The boundary ∂M of the domain M consists of two circular arcs, see Figure 1.

In this paper, we provide an explicit solution and solvability conditions for the Dirichlet boundary-value problem for the Poisson equation in the lens M. In the next step, we consider the Dirichlet boundary-value problem for the homogeneous Cauchy–Riemann equation. We also discuss the solvability condition for the Dirichlet problem for the homogeneous Cauchy–Riemann equation, using the Dirichlet problem for the Poisson equation.

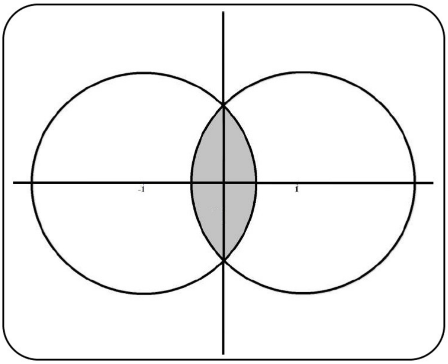

Figure 1: The lens M

2. Dirichlet problem for the Poisson equation in M. In this section, we study the Dirichlet boundary-value problem for the Poisson equation in the lens domain. In order to treat the Dirichlet boundary-value problem for second order complex partial differential equations, some special kernel functions (the Green functions) have to be constructed. The harmonic Green function for the lens M is given by the following formula (see [5], [6]):

$$G_1(z,\zeta) = \log\left|\frac{\bar{\zeta}(z-1)-z-1}{\zeta-z}\frac{\bar{\zeta}(z+1)+z-1}{\zeta z+1}\right|^2.$$

For $z \in \partial M \cap C_1$, that is, $|z-1| = \sqrt{2}$, we have

$$\partial_{v_z} G_1(z,\zeta) = \left(\left(\frac{z-1}{\sqrt{2}}\right)\partial_z + \left(\frac{\bar{z}-1}{\sqrt{2}}\right)\partial_{\bar{z}}\right) G_1(z,\zeta),$$

and for $z \in \partial M \cap C_2$, that is, $|z+1| = \sqrt{2}$, we have

$$\partial_{v_z} G_1(z,\zeta) = \left(\left(\frac{z+1}{\sqrt{2}}\right)\partial_z + \left(\frac{\bar{z}+1}{\sqrt{2}}\right)\partial_{\bar{z}}\right) G_1(z,\zeta).$$

The next theorem contains a representation formula for a class of functions via the Green function, which is used to solve the Dirichlet boundary-value problem for Poisson equation (see [4], [5]).

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be a regular domain, and let G_1 be the harmonic Green function for Ω. Then any $\omega \in C^2(\Omega;\mathbb{C}) \cap C^1(\overline{\Omega};\mathbb{C})$ can be* *represented by:*

$$\omega(z) = -\frac{1}{4\pi}\int\limits_{\partial\Omega} \omega(\zeta)\partial_{v_\zeta} G_1(z,\zeta)\mathrm{d}t_\zeta - \frac{1}{\pi}\int\limits_{\Omega} \omega_{\zeta\bar{\zeta}}(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

where v is the outward normal derivative on $\partial\Omega$ and t is the arc length parameter.

Thus, according to Theorem 1, the explicit form of the Green representation formula for the lens domain is as following:

$$\omega(z) = \frac{1}{2\pi i}\int\limits_{\partial M\cap C_1} \omega(\zeta)\left(\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta-1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} - 1\right)\frac{d\zeta}{\zeta-1} +$$
$$+ \frac{1}{2\pi i}\int\limits_{\partial M\cap C_2} \omega(\zeta)\left(\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta+1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} - 1\right)\frac{d\zeta}{\zeta+1} -$$
$$- \frac{1}{\pi}\int\limits_{\Omega} \omega_{\zeta\bar{\zeta}}(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \quad (1)$$

In fact, formula (1) provides a solution to the Dirichlet problem.

Theorem 2. *The Dirichlet problem for the Poisson equation in M*

$$\omega_{z\bar{z}} = f, \quad z\in M, \quad f\in C(M;\mathbb{C}),$$
$$\omega = \gamma, \text{ on } \partial M, \ \gamma\in C(\partial M;\mathbb{C}), \quad (2)$$

is uniquely solvable and the solution is given by

$$\omega(z) = \frac{1}{2\pi i}\int\limits_{\partial M\cap C_1} \gamma(\zeta)\left(\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta-1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} - 1\right)\frac{d\zeta}{\zeta-1} +$$
$$+ \frac{1}{2\pi i}\int\limits_{\partial M\cap C_2} \gamma(\zeta)\left(\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta+1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} - 1\right)\frac{d\zeta}{\zeta+1} -$$
$$- \frac{1}{\pi}\int\limits_{\Omega} f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad (3)$$

where $\zeta = \xi + i\eta$.

Proof. By the properties of the Green function and the harmonicity of the boundary integrals, ω is seen to be a solution to the Poisson equation [4].

So, it remains to verify that the boundary conditions are satisfied for the boundary integrals. Studying the boundary behavior of the boundary integral implies computations on the different parts of the boundary ∂M.
So, for $z \in \partial M \cap C_1$:
Case 1: $\zeta \in C_1$,

$$\frac{\bar{z}(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} = \frac{(\frac{z+1}{z-1})(\frac{2}{\zeta-1})}{(\frac{\zeta+1}{\zeta-1})(\frac{z+1}{z-1})+1} = \frac{2(z+1)}{(z+1)(\zeta+1)+(z-1)(\zeta-1)} = \frac{z+1}{\zeta z+1}.$$

Case 2: $\zeta \in C_2$,

$$\frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} = \frac{-z+1}{\zeta z+1},$$
$$\frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} = \frac{-z-1}{\zeta-z}.$$

Thus, on $\partial M \cap C_1$ we have

$$\lim_{z\to\zeta}\omega(z) = \lim_{z\to\zeta}\frac{1}{2\pi i}\int\limits_{\partial M\cap C_1}\gamma(\zeta)\left[\frac{\zeta-1}{\zeta-z}+\frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta-1} =$$
$$= \lim_{z\to\zeta}\frac{1}{2\pi i}\int\limits_{C_1}\Gamma(\zeta)\left[\frac{\zeta-1}{\zeta-z}+\frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta-1},$$

where

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & \zeta \in \partial M \cap C_1, \\ 0, & \zeta \in C_1 \backslash (\partial M \cap C_1). \end{cases}$$

By considering the properties of the Poisson kernel for C_1 [6]

$$\lim_{z\to\zeta}\omega(z) = \gamma(\zeta),$$

we see that $\zeta \in \partial M \cap C_1$ up to the tips $\pm i$ of the lens M, because Γ fails to be continuous there unless γ accidentally vanishes at these points.

By the same way, $z \in \partial M \cap C_2$.
Case 1: $\zeta \in C_1$,

$$\frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} = \frac{z+1}{\zeta z+1},$$
$$\frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} = \frac{-z+1}{\zeta-z}.$$

Case 2: $\zeta \in C_2$,

$$\frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} = \frac{-z+1}{\zeta z+1}.$$

Thus, on $\partial M \cap C_2$, we have

$$\lim_{z\to\zeta}\omega(z) = \lim_{z\to\zeta}\frac{1}{2\pi i}\int\limits_{\partial M\cap C_2}\gamma(\zeta)\left[\frac{\zeta+1}{\zeta-z}+\frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+1} =$$
$$= \lim_{z\to\zeta}\frac{1}{2\pi i}\int\limits_{C_2}\Upsilon(\zeta)\left[\frac{\zeta+1}{\zeta-z}+\frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+1},$$

where

$$\Upsilon(\zeta) = \begin{cases}\gamma(\zeta), & \zeta\in\partial M\cap C_2,\\ 0, & \zeta\in C_2\backslash(\partial M\cap C_2).\end{cases}$$

By considering the properties of the Poisson kernel for C_2,

$$\lim_{z\to\zeta}\omega(z) = \gamma(\zeta),$$

we see that $\zeta \in \partial M \cap C_2$ up to the tips $\pm i$ of the lens M, because Υ fails to be continuous there unless γ accidentally vanishes at these points.

Now consider the boundary behavior at the tips $\pm i$. We represent the constant function 1 as

$$1 = \frac{1}{2\pi i}\int\limits_{\partial M\cap C_1}\left[\frac{\zeta-1}{\zeta-z}+\frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta-1)}{\zeta z+1}+\frac{\bar{z}(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1}-1\right]\frac{d\zeta}{\zeta-1}+$$
$$+\frac{1}{2\pi i}\int\limits_{\partial M\cap C_2}\left[\frac{\zeta+1}{\zeta-z}+\frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta+1)}{\zeta z+1}+\frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1}-1\right]\frac{d\zeta}{\zeta+1}.$$

Multiplying this relation by $\gamma(\pm i)$ and subtracting the resulting equation from $\omega(z)$ shows for $z \in \partial M \cap C_1$,

$$\lim_{z\to\pm i}(\omega(z)-\gamma(\pm i)) = \lim_{z\to\pm i}\frac{1}{2\pi i}\int\limits_{\partial M\cap C_1}\tilde{\gamma}(\zeta)\left[\frac{\zeta-1}{\zeta-z}+\frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta-1},$$

where $\tilde{\gamma}(\zeta) = \gamma(\zeta) - \gamma(\pm i)$ and $\tilde{\gamma}(\pm i) = 0$,

$$\lim_{z\to\pm i}\omega(z) = \gamma(\pm i).$$

Similarly, for $z \in \partial M \cap C_2$,

$$\lim_{z\to\pm i}\left(\omega(z)-\gamma(\pm i)\right)=\lim_{z\to\pm i}\frac{1}{2\pi i}\int\limits_{\partial M\cap C_2}\hat{\gamma}(\zeta)\left[\frac{\zeta+1}{\zeta-z}+\frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+1},$$

where $\hat{\gamma}(\zeta)=\gamma(\zeta)-\gamma(\pm i)$ and $\hat{\gamma}(\pm i)=0$,

$$\lim_{z\to\pm i}\omega(z)=\gamma(\pm i).$$

This completes the proof. □

3. Dirichlet problem for the Cauchy–Riemann equation in M. In this section, we investigate the Dirichlet boundary-value problem for the homogeneous Cauchy–Riemann equation in the lens domain. To solve Dirichlet problem for analytic functions in the following representation formula is important.

Theorem 3. *Any* $\omega \in C^1(M;\mathbb{C})\cap C(\overline{M};\mathbb{C})$ *can be represented as*

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{\partial M}\omega(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z+1}\right]d\zeta-\frac{1}{\pi}\int\limits_{M}\omega_{\bar{\zeta}}(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z+1}\right]d\xi d\eta,$$

where $\zeta=\xi+i\eta$.

Proof. The Cauchy-Pompeiu formula applied to $z\in M$ and $-\frac{1}{z}\notin M$, respectively, gives the following equalities:

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{\partial M}\omega(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int\limits_{M}\omega_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta-z},$$

$$0=\frac{1}{2\pi i}\int\limits_{\partial M}\omega(\zeta)\frac{z d\zeta}{\zeta z+1}-\frac{1}{\pi}\int\limits_{M}\omega_{\bar{\zeta}}(\zeta)\frac{z d\xi d\eta}{\zeta z+1},$$

adding the above relations leads to the claimed representation formula. □

This Integral representation formula serves to solve the Dirichlet boundary value problem for the homogeneous Cauchy–Riemann equation in the lens.

Theorem 4. *The Dirichlet problem for the homogeneous Cauchy-Riemann equation in M*

$$\begin{cases} \omega_{\bar{z}} = 0, & in \quad M, \\ \omega = \gamma, & on \quad \partial M, \ \gamma \in C(\partial M; \mathbb{C}), \end{cases} \tag{4}$$

with given $\gamma \in C(\partial M; \mathbb{C})$*, is solvable if and only if*

$$\frac{1}{2\pi i} \int\limits_{\partial M} \gamma(z) \left[\frac{\bar{z}-1}{\zeta(\bar{z}-1)-\bar{z}-1} + \frac{\bar{z}+1}{\zeta(\bar{z}+1)+\bar{z}-1} \right] d\zeta = 0, \tag{5}$$

and the unique solution can be represented as

$$\omega(z) = \frac{1}{2\pi i} \int\limits_{\partial M} \gamma(z) \left[\frac{1}{\zeta - z} + \frac{z}{\zeta z + 1} \right] d\zeta. \tag{6}$$

Proof. Let ω, as defined by (6), be a solution to the Dirichlet problem. Then we know that

$$\lim_{z \to \zeta} \omega(z) = \gamma(\zeta), \quad \zeta \in \partial M. \tag{7}$$

That the condition (5) is necessary can be shown as follows. Consider a new function

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial M} \gamma(\zeta) \left[\frac{\bar{z}-1}{\zeta(\bar{z}-1)-\bar{z}-1} + \frac{\bar{z}+1}{\zeta(\bar{z}+1)+\bar{z}-1} \right] d\zeta. \tag{8}$$

Then we have

$$\begin{aligned} \omega(z) - f(z) &= \frac{1}{2\pi i} \int\limits_{\partial M} \gamma(\zeta) \left[\frac{1}{\zeta - z} + \frac{z}{\zeta z + 1} \right] d\zeta - \\ &- \frac{1}{2\pi i} \int\limits_{\partial M} \gamma(\zeta) \left[\frac{\bar{z}-1}{\zeta(\bar{z}-1)-\bar{z}-1} + \frac{\bar{z}+1}{\zeta(\bar{z}+1)+\bar{z}-1} \right] d\zeta = \\ &= \frac{1}{2\pi i} \int\limits_{\partial M} \gamma(\zeta) \left[\frac{1}{\zeta - z} - \frac{\bar{z}-1}{\zeta(\bar{z}-1)-\bar{z}-1} + \frac{z}{\zeta z+1} - \frac{\bar{z}+1}{\zeta(\bar{z}+1)+\bar{z}-1} \right] d\zeta = \\ &= \frac{1}{2\pi i} \int\limits_{\partial M \cap C_1} \gamma(\zeta) \left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta-1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} - 1 \right] \frac{d\zeta}{\zeta-1} + \\ &+ \frac{1}{2\pi i} \int\limits_{\partial M \cap C_2} \gamma(\zeta) \left[\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta+1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} - 1 \right] \frac{d\zeta}{\zeta+1}. \end{aligned} \tag{9}$$

So, by Theorem 2,

$$\lim_{z\to\zeta}[\omega(z)-f(z)]=\gamma(\zeta),\ \ \zeta\in\partial M. \tag{10}$$

By (7) and (10), we have

$$\lim_{z\to\zeta}f(z)=0,\ \ \zeta\in\partial M.$$

Then, from the maximum principle for analytic functions $f(z)=0$ for $z\in M$, which is given as condition (5).

On the other hand, if the condition (5) is satisfied, then the analytic function ω, defined by (6), can be expressed as

$$\omega(z)-f(z)=\frac{1}{2\pi i}\int\limits_{\partial M}\gamma(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z+1}\right]d\zeta-$$

$$-\frac{1}{2\pi i}\int\limits_{\partial M}\gamma(\zeta)\left[\frac{\bar z-1}{\zeta(\bar z-1)-\bar z-1}+\frac{\bar z+1}{\zeta(\bar z+1)+\bar z-1}\right]d\zeta=$$

$$=\frac{1}{2\pi i}\int\limits_{\partial M}\gamma(\zeta)\left[\frac{1}{\zeta-z}-\frac{\bar z-1}{\zeta(\bar z-1)-\bar z-1}+\frac{z}{\zeta z+1}-\frac{\bar z+1}{\zeta(\bar z+1)+\bar z-1}\right]d\zeta.$$

Hence, by Theorem 2, we get

$$\lim_{z\to\zeta}\omega(z)=\gamma(\zeta),\ \ \zeta\in\partial M.$$

This completes the proof of Theorem 4. □

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