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BPP AND FP OF CYCLIC $\mathcal{G}$-$(\varphi-\psi)$-WEAK CONTRACTIVE MAPPINGS IN GRAPHICAL METRIC SPACES AND THEIR CONSEQUENCES

Abstract. The main goal of this article is first to express a cyclic $\mathcal{G}$-$(\varphi-\psi)$-weak contractive mapping, and second to present the existence of their best proximity points and fixed points. Several consequences are as well prepared to show the efficiency of our main results. One of the most important issues of this work is that it can also involve all former papers introduced by taking comparable and ε-close members.

Key words: *cyclic $\mathcal{G}$-$(\varphi-\psi)$-weak contractive mappings, orbitally $\mathcal{G}$-continuous, property $\mathcal{P}$, unconditionally Cauchy property, best proximity point.*

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1. Introduction. The metric fixed point (FP) theory has played a significant key in nonlinear analysis since 1922, when Banach introduced his famous contraction principle (see [3], [6], [21]). Note that this theory is extended in two ways by many researchers, but most use their generalizations in one direction, which includes applications in mathematics, such as proving the existence of solution for integral equations, solving optimization problems, analysing iterative methods and algorithms, etc. One way is to present a new contractive mapping to state and prove some FP theorems. For instance, one of the most practical contractions, defined by Dutta and Choudury [8], is $(\varphi-\psi)$-weak contraction. They also derived an attractive FP theorem as follows:

Theorem 1. [8] *If $(\mathcal{X},\mathsf{d})$ is a complete MS and $\mathcal{F}\colon \mathcal{X}\to\mathcal{X}$ is a mapping fulfilling*

$$\psi\big(\mathsf{d}(\mathcal{F}\mathfrak{a},\mathcal{F}\mathfrak{b})\big)\leqslant\psi\big(\mathsf{d}(\mathfrak{a},\mathfrak{b})\big)-\varphi\big(\mathsf{d}(\mathfrak{a},\mathfrak{b})\big)$$

for any $\mathfrak{a},\mathfrak{b} \in \mathcal{X}$, where $\psi, \varphi \colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ are nondecreasing and continuous functions and $\varphi(f) = 0 = \psi(f)$ iff $f = 0$ for $f \in \mathbb{R}^{\geqslant 0}$, then $\mathcal{F}$ possesses a unique FP.

Note that this contraction involves many other contractions collected by Rhoades (1977) [21], which is the reason why we take it as the base of this paper. For example, if we take $\psi(f) = f$ and $\lim_{f\to\infty} \varphi(f) = \infty$, then we can obtain the same main results of Rhoades's paper [22].

Another type of contractive mappings playing an important role in FP theory, called a cyclic contractive mapping, is due to Kirk et al. [15]. A mapping $\mathcal{F} \colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is named cyclic if $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{F}(\mathcal{B}) \subseteq \mathcal{A}$. They also established a version of the Banach theorem as follows:

Theorem 2. [15] *Assume $\mathcal{A}, \mathcal{B} \neq \varnothing$ are closed subsets of a complete MS $(\mathcal{X}, \mathsf{d})$ and $\mathcal{F} \colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic mapping fulfilling*

$$\mathsf{d}(\mathcal{F}\mathfrak{a}, \mathcal{F}\mathfrak{b}) \leqslant \alpha \mathsf{d}(\mathfrak{a}, \mathfrak{b})$$

for any $\mathfrak{a} \in \mathcal{A}$ and $\mathfrak{b} \in \mathcal{B}$, where $\alpha \in (0, 1)$. Then $\mathcal{A} \cap \mathcal{B} \neq \varnothing$ and $\mathcal{F}$ possess a unique FP in $\mathcal{A} \cap \mathcal{B}$.

On the other hand, another way to develop FP theory is to define a new version of a metric space (MS) due to a change in metric or set conditions. For example, in 2004, Ran and Reurings [20] considered a partial order (PO) in an MS and discussed the existence of FP(s) for contractive mappings regarding comparable elements.

Theorem 3. [20] *Assume $(\mathcal{X}, \preceq)$ is a PO set, $(\mathcal{X}, \mathsf{d})$ is a complete MS, and $\mathcal{F} \colon \mathcal{X} \to \mathcal{X}$ is a nondecreasing mapping fulfilling*

$$\mathsf{d}(\mathcal{F}\mathfrak{a}, \mathcal{F}\mathfrak{b}) \leqslant \alpha \mathsf{d}(\mathfrak{a}, \mathfrak{b})$$

for any $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ with $\mathfrak{a} \preceq \mathfrak{b}$, where $\alpha \in [0, 1)$. Also, assume that one of the following conditions holds:

- *$\mathcal{F}$ is continuous;*
- *when a nondecreasing sequence $\mathfrak{a}_\mathfrak{n}$ converges to a $\mathfrak{a} \in \mathcal{X}$, we have $\mathfrak{a}_\mathfrak{n} \preceq \mathfrak{a}$.*

If there is $\mathfrak{a}_0 \in \mathcal{X}$ satisfying $\mathfrak{a}_0 \preceq \mathcal{F}\mathfrak{a}_0$, then $\mathcal{F}$ possesses a FP. Further, if each two FP(s) are comparable, then the FP is unique.

Note that we say $\mathcal{F}$ in Theorem 3 is nondecreasing when $\mathfrak{a} \preceq \mathfrak{b}$ implies $\mathcal{F}\mathfrak{a} \preceq \mathcal{F}\mathfrak{b}$ for all $\mathfrak{a},\mathfrak{b} \in \mathcal{X}$. Also, setting $\mathcal{X} = \mathbb{R}$, we have the same usual order $\leqslant$ in $\mathbb{R}$ and the classic definition of a nondecreasing mapping. It is claimed that this theorem has many applications. For example, in 2005, Nieto and Rodrıguez-López [16] used this definition and FP theorem to solve some ordinary differential equations. Moreover, in 2011, Abkar and Gabeleh [1] fused Theorems 2 and 3 and gained the following FP result:

Theorem 4. [1, Theorem 3.1, (2011)] *Assume that $(\mathcal{X}, \preceq)$ is a PO set, $\mathcal{A}, \mathcal{B} \neq \varnothing$ are subsets of an MS $(\mathcal{X}, \mathsf{d})$, $\mathcal{A}$ is complete and $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic mapping fulfilling*

$$\mathsf{d}(\mathcal{F}\mathfrak{a}', \mathcal{F}^2\mathfrak{a}) \leqslant \alpha \mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a})$$

for each $(\mathfrak{a}, \mathfrak{a}') \in \mathcal{A} \times \mathcal{A}$ with $\mathfrak{a} \preceq \mathfrak{a}'$, where $\alpha \in (0, 1)$ and $\mathcal{F}^2$ is nondecreasing on $\mathcal{A}$. Also, assume that one of the following conditions holds:

- *$\mathcal{F}$ is continuous;*
- *when a nondecreasing sequence $\mathfrak{a}_n$ converges to a $\mathfrak{a} \in \mathcal{A}$, we have $\mathfrak{a}_n \preceq \mathfrak{a}$.*

If there is $\mathfrak{a}_0 \in \mathcal{A}$ satisfying $\mathfrak{a}_0 \preceq \mathcal{F}^2\mathfrak{a}_0$, then $\mathcal{A} \cap \mathcal{B} \neq \varnothing$ and $\mathcal{F}$ possesses a FP in $\mathcal{A} \cap \mathcal{B}$. Further, if $\mathfrak{a}_{n+1} = \mathcal{F}(\mathfrak{a}_n)$, then $\mathfrak{a}_{2n} \to p^\star$.

In 2012, they also combined Theorems 1–4 regarding $\psi(f) = f$ and presented the following FP theorem:

Theorem 5. [2, Theorems 2.4 and 2.5, (2012)] *Assume that $(\mathcal{X}, \preceq)$ is a PO set, $\mathcal{A}, \mathcal{B} \neq \varnothing$ are subsets of an MS $(\mathcal{X}, \mathsf{d})$, $\mathcal{A}$ is complete, and $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic mapping fulfilling*

$$\mathsf{d}(\mathcal{F}\mathfrak{a}', \mathcal{F}^2\mathfrak{a}) \leqslant \mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a}) - \varphi\big(\mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a})\big)$$

for each $(\mathfrak{a}, \mathfrak{a}') \in \mathcal{A} \times \mathcal{A}$ with $\mathfrak{a} \preceq \mathfrak{a}'$, where $\varphi\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ is a nondecreasing and continuous function, $\varphi(f) = 0 = \psi(f)$ iff $f = 0$ for $f \in \mathbb{R}^{\geqslant 0}$ and $\lim_{f\to\infty} \varphi(f) = \infty$, and $\mathcal{F}^2$ is nondecreasing on $\mathcal{A}$. Also, assume that one of the following conditions holds:

- *$\mathcal{F}$ is continuous;*
- *when a nondecreasing sequence $\mathfrak{a}_n$ converges to a $\mathfrak{a} \in \mathcal{A}$, we have $\mathfrak{a}_n \preceq \mathfrak{a}$.*

If there is $\mathfrak{a}_0 \in \mathcal{A}$ satisfying $\mathfrak{a}_0 \preceq \mathcal{F}^2\mathfrak{a}_0$, then $\mathcal{A} \cap \mathcal{B} \neq \varnothing$ and $\mathcal{F}$ possesses a FP in $\mathcal{A} \cap \mathcal{B}$. Further, if $\mathfrak{a}_{n+1} = \mathcal{F}\mathfrak{a}_n$, then $\mathfrak{a}_{2n} \to p^\star$.

It is clear that if we take $\varphi(f) = (1-\alpha)f$, then Theorem 5 coincides with Theorem 4, which is why the authors claimed that the last theorem contains many former FP results.

In spite of the fact that the FP theory is an important tool for finding FP of a mapping $\mathcal{F}$ defined on $\mathcal{A} \subseteq \mathcal{X}$, a non-self mapping $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ does not, of course, possess a FP. Therefore, one tries to obtain a member $\mathfrak{a}$ that is closest to $\mathcal{F}\mathfrak{a}$. Hence, the best proximity point (BPP) results became famous in applied mathematics. Assume $\mathcal{A}$,$\mathcal{B} \neq \emptyset$ are two subsets of an MS $(\mathcal{X}, \mathsf{d})$, $\mathsf{dist}(\mathcal{A}, \mathcal{B}) = \inf\{\mathsf{d}(\mathfrak{a}, \mathfrak{b})\colon \mathfrak{a} \in \mathcal{A}, \mathfrak{b} \in \mathcal{B}\}$ and $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a mapping. The BPP(s) of $\mathcal{F}$ is any point $\mathfrak{a} \in \mathcal{A}$ in which $\mathsf{d}(\mathfrak{a},\mathcal{F}\mathfrak{a}) = \mathsf{dist}(\mathcal{A}, \mathcal{B})$. In 2006, Eldred and Veeremani [9] presented the existence of BPP(s) of cyclic contractive mappings on uniformly convex Banach spaces. Also, Suzuki et al. [25] explained the existence of such point of cyclic contractive mappings in an MS by virtue of an unconditionally Cauchy (UC) property.

Definition 1. [25] *Let $\mathcal{A}, \mathcal{B} \neq \emptyset$ be two subsets of an MS $(\mathcal{X}, \mathsf{d})$. The pair $(\mathcal{A}, \mathcal{B})$ is said to possess the UC property when for two sequences $\{\mathfrak{a}_\mathfrak{n}\}$ and $\{\mathfrak{a}'_\mathfrak{n}\}$ in $\mathcal{A}$ and a sequence $\{\mathfrak{b}_\mathfrak{n}\}$ in $\mathcal{B}$, $\lim_{\mathfrak{n}\to\infty} \mathsf{d}(\mathfrak{a}_\mathfrak{n}, \mathfrak{b}_\mathfrak{n}) = \lim_{\mathfrak{n}\to\infty} \mathsf{d}(\mathfrak{a}'_\mathfrak{n},\mathfrak{b}_\mathfrak{n}) = \mathsf{dist}(\mathcal{A}, \mathcal{B})$ we have $\lim_{\mathfrak{n}\to\infty} \mathsf{d}(\mathfrak{a}, \mathfrak{a}'_\mathfrak{n}) = 0$.*

Lemma 1. [25] *Assume $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two subsets of an MS $(\mathcal{X}, \mathsf{d})$ and $(\mathcal{A}, \mathcal{B})$ has the UC property. Also, assume that $\{\mathfrak{a}_\mathfrak{n}\}$ and $\{\mathfrak{b}_\mathfrak{n}\}$ are two sequences in $\mathcal{A}$ and $\mathcal{B}$, respectively, provided that either*

$$\lim_{\mathfrak{m}\to\infty} \sup_{\mathfrak{n}\geqslant\mathfrak{m}} \mathsf{d}(\mathfrak{a}_\mathfrak{m}, \mathfrak{b}_\mathfrak{n}) = \mathsf{dist}(\mathcal{A}, \mathcal{B}) \quad \textit{or} \quad \lim_{\mathfrak{n}\to\infty} \sup_{\mathfrak{m}\geqslant\mathfrak{n}} \mathsf{d}(\mathfrak{a}_\mathfrak{m}, \mathfrak{b}_\mathfrak{n}) = \mathsf{dist}(\mathcal{A}, \mathcal{B}).$$

Then $\{\mathfrak{a}_\mathfrak{n}\}$ is Cauchy.

The theory of BPP of various mappings in different type of MS(s) has been continued by many researchers (see also [4], [11], [12], [13], [19], [23], [24] and references therein). On the other hand, if $\mathcal{A} \cap \mathcal{B} = \emptyset$ in Theorem 2, then $\mathcal{F}\mathfrak{a} = \mathfrak{a}$ has no answer. Hence, we may think about an approximate solution $\mathfrak{a} \in \mathcal{A} \cup \mathcal{B}$ so that the error $\mathsf{dist}(\mathfrak{a}, \mathcal{F}\mathfrak{a})$ be minimum. As $\mathcal{F}$ is cyclic on $\mathcal{A} \cup \mathcal{B}$, we have $\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{a}) \geqslant \mathsf{dist}(\mathcal{A}, \mathcal{B})$. Hence, Abkar and Gabeleh introduced some useful tools for finding BPP of cyclic contractive and cyclic φ-contractive mapping, respectively.

Theorem 6. [1, Theorem 4.1, (2011)] *Assume $(\mathcal{X}, \preceq)$ is a PO set, $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of an MS $(\mathcal{X}, \mathsf{d})$ and $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$*

is a cyclic mapping fulfilling

$$\mathsf{d}(\mathcal{F}\mathfrak{a}', \mathcal{F}^2\mathfrak{a}) \leqslant \alpha \mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a}) - (1-\alpha)\mathsf{dist}(\mathcal{A}, \mathcal{B})$$

for each $(\mathfrak{a}, \mathfrak{a}') \in \mathcal{A} \times \mathcal{A}$ *with* $\mathfrak{a} \preceq \mathfrak{a}'$*, where* $\alpha \in (0, 1)$ *and* $\mathcal{F}^2$ *is nondecreasing on* $\mathcal{A}$*. Also, assume that the following condition holds:*

- *When a nondecreasing sequence* $\mathfrak{a}_\mathfrak{n}$ *converges to a* $\mathfrak{a}$ *in* $\mathcal{A}$*, we have* $\mathfrak{a}_\mathfrak{n} \preceq \mathfrak{a}$*.*

If there is $\mathfrak{a}_0 \in \mathcal{A}$ *satisfying* $\mathfrak{a}_0 \preceq \mathcal{F}^2\mathfrak{a}_0$*,* $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_\mathfrak{n}$ *for* $n \geqslant 0$ *and* $\{\mathfrak{a}_{2\mathfrak{n}}\}$ *possesses a convergent subsequence in* $\mathcal{A}$*, then* $\mathcal{F}$ *has a BPP in* $\mathcal{A}$*.*

Theorem 7. [2, Theorem 3.4, (2012)] *Assume* $(\mathcal{X}, \preceq)$ *is a PO set,* $\mathcal{A}, \mathcal{B} \neq \varnothing$ *are two closed subsets of an MS* $(\mathcal{X}, \mathsf{d})$ *and* $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ *is a cyclic mapping fulfilling*

$$\mathsf{d}(\mathcal{F}\mathfrak{a}', \mathcal{F}^2\mathfrak{a}) \leqslant \mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a}) - \varphi\big(\mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a})\big) + \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big)$$

for each $(\mathfrak{a}, \mathfrak{a}') \in \mathcal{A} \times \mathcal{A}$ *with* $\mathfrak{a} \preceq \mathfrak{a}'$*, where* $\varphi\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ *is a strictly increasing function and* $\mathcal{F}^2$ *is nondecreasing on* $\mathcal{A}$*. Also, assume that the following condition holds:*

- *When a nondecreasing sequence* $\mathfrak{a}_\mathfrak{n}$ *converges to a* $\mathfrak{a}$ *in* $\mathcal{A}$*, we have* $\mathfrak{a}_\mathfrak{n} \preceq \mathfrak{a}$*.*

If there is $\mathfrak{a}_0 \in \mathcal{A}$ *satisfying* $\mathfrak{a}_0 \preceq \mathcal{F}^2\mathfrak{a}_0$*,* $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_\mathfrak{n}$ *for* $n \geqslant 0$ *and* $\{\mathfrak{a}_{2\mathfrak{n}}\}$ *possesses a convergent subsequence in* $\mathcal{A}$*, then* $\mathcal{F}$ *has a BPP in* $\mathcal{A}$*.*

To complete the discussion about PO sets and FP theory: in 2008, Jachymski [14] defined a graphical MS and developed several concepts and FP theorems. After that, many researchers working on both FP theory and BPP theorems extended Jachymski's idea in various directions regarding different spaces and contractions (see [5], [10], [17], [18]). Note that the results of these references can well expand the results regarding a PO.

Assume that $\mathcal{G}$ is a graph. A link is an edge of $\mathcal{G}$ with distinct ends. Also, a loop is an edge of $\mathcal{G}$, with identical ends. Parallel edges of $\mathcal{G}$ are two or more links of $\mathcal{G}$ with same pairs of ends. Suppose that $(\mathcal{X}, \mathsf{d})$ is an MS and $\mathcal{G}$ is a directed graph in which $\mathsf{V}(\mathcal{G})$ is the vertex set coinciding with $\mathcal{X}$ and $\mathsf{E}(\mathcal{G})$ is the edge set containing all loops. Assume that $\mathcal{G}$ has no parallel edges. Then $(\mathcal{X}, \mathsf{d})$ is named an MS with graph $\mathcal{G}$ (or GMS). What is more, assume $\mathcal{G}^{-1}$ is a directed graph obtained from $\mathcal{G}$

by changing directions of the edges of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ is an undirected graph obtained from $\mathcal{G}$ by removing the directions of the edges $\mathcal{G}$. Evidently, $\mathsf{V}(\mathcal{G}^{-1}) = \mathsf{V}(\tilde{\mathcal{G}}) = \mathsf{V}(\mathcal{G}) = \mathcal{X}$, $\mathsf{E}(\mathcal{G}^{-1}) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathcal{X} \times \mathcal{X} : (\mathfrak{b}, \mathfrak{a}) \in \mathsf{E}(\mathcal{G})\}$ and $\mathsf{E}(\tilde{\mathcal{G}}) = \mathsf{E}(\mathcal{G}) \cup \mathsf{E}(\mathcal{G}^{-1})$. For details in graph theory and discussion about a graphical MS, FP, and BPP, see [7], [14] and references therein.

Combining Theorems 1–5 and 6–7 and considering a GMS instead an MS or an MS with a PO, we prove the existence of FP and BPP of a class of contractions, called cyclic $\mathcal{G}$-$(\varphi - \psi)$-weak, which extends many previous papers (see [1], [2], [8], [22], [25]). Note that the process used in Theorems 8, 9, and 10 can be served as a survey in the future works to prove BPP and FP theorems, named graphical version, in a class bigger than PO sets, comparable elements, and ε-close members. Some consequences are also derived from the main theorems that can show the efficiency of obtained theorems. To do this, some of symbols and definitions needed are given below.

- Assume Ψ is the class of all continuous and nondecreasing functions $\psi, \varphi : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$, where φ and ψ are positive on $(0, +\infty)$ and $\psi(0) = \varphi(0) = 0$;
- Assume $\mathcal{A}, \mathcal{B} \neq \varnothing$ are two subset of a GMS $(\mathcal{X}, \mathsf{d})$.

$$\mathsf{dist}(\mathcal{A}, \mathcal{B}) = \inf \left\{ \mathsf{d}(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a} \in \mathcal{A}, \mathfrak{b} \in \mathcal{B} \right\}.$$

- Assume that $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a mapping. We mean $\mathcal{C}_{\mathcal{F}}$ by the set of all points $\mathfrak{a} \in \mathcal{X}$ provided that $(\mathcal{F}^{\mathfrak{m}}\mathfrak{a}, \mathcal{F}^{\mathfrak{n}}\mathfrak{a})$ is an edge of $\tilde{\mathcal{G}}$ for each $\mathfrak{m}, \mathfrak{n} \in \mathbb{N} \cup \{0\}$; that is,

$$\mathcal{C}_{\mathcal{F}} = \left\{ \mathfrak{a} \in \mathcal{X} : (\mathcal{F}^{\mathfrak{m}}\mathfrak{a}, \mathcal{F}^{\mathfrak{n}}\mathfrak{a}) \in \mathsf{E}(\tilde{\mathcal{G}}) \quad \mathfrak{m}, \mathfrak{n} = 0, 1, \dots \right\}.$$

  Note that $\mathcal{C}_{\mathcal{F}}$ may become an empty set.

It is clear that Definition 1 and Lemma 1 are valid in a GMS. Additionally, a type of continuity in a GMS named orbitally $\mathcal{G}$-continuous and a type of $\mathcal{G}$ named **C**-graph will be needed in the next parts.

Definition 2. ([10], [14]) *Let $(\mathcal{X}, \mathsf{d})$ be a GMS. A mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is known as an orbitally $\mathcal{G}$-continuous mapping on $\mathcal{X}$ whenever $\mathcal{F}^{\mathfrak{b}_{\mathfrak{n}}}\mathfrak{a} \to \mathfrak{b}$ implies $\mathcal{F}(\mathcal{F}^{\mathfrak{b}_{\mathfrak{n}}}\mathfrak{a}) \to \mathcal{F}\mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ and sequences $\{\mathfrak{b}_{\mathfrak{n}}\}$ of natural numbers, so that $(\mathcal{F}^{\mathfrak{b}_{\mathfrak{n}}}\mathfrak{a}, \mathcal{F}^{\mathfrak{b}_{\mathfrak{n}}+1}\mathfrak{a}) \in \mathsf{E}(\mathcal{G})$ for every $\mathfrak{n} \in \mathbb{N}$.*

Definition 3. ([10], [14]) *Assume that $(\mathcal{X}, \mathsf{d})$ is a GMS. $\mathcal{G}$ is named a* **C***-graph on $\mathcal{X}$ if the following feature holds:*

- *If* $\mathfrak{a} \in \mathcal{X}$ *and* $\{\mathfrak{a}_\mathfrak{n}\}$ *is a sequence in* $\mathcal{X}$ *provided that* $\mathfrak{a}_\mathfrak{n} \to \mathfrak{a}$ *and* $(\mathfrak{a}_{\mathfrak{n}+1}, \mathfrak{a}_\mathfrak{n}) \in \mathsf{E}(\mathcal{G})$ *for each* $\mathfrak{n} \in \mathbb{N}$*, then there is a subsequence* $\{\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}}\}$ *of* $\{\mathfrak{a}_\mathfrak{n}\}$ *provided that* $(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}}, \mathfrak{a}) \in \mathsf{E}(\mathcal{G})$ *for any* $\mathfrak{i} \in \mathbb{N}$*.*

2. FP results. First, following the idea of Theorems 1-5, we state the definition of a $(\varphi - \phi)$-weak contraction in a GMS.

Definition 4. *Assume that* $(\mathcal{X}, \mathsf{d})$ *is a GMS. A mapping* $\mathcal{F}\colon \mathcal{X} \to \mathcal{X}$ *is acknowledged as a* $\mathcal{G}$-$(\varphi - \psi)$*-weak contractive mapping if*

($\mathcal{G}_1$) $\mathcal{F}^2$ *keeps the edges of* $\mathcal{G}$*, i.e.* $(\mathfrak{a}, \mathfrak{b}) \in \mathsf{E}(\mathcal{G})$ *implies* $(\mathcal{F}^2\mathfrak{a}, \mathcal{F}^2\mathfrak{b}) \in \mathsf{E}(\mathcal{G})$ *for each* $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$*;*

($\mathcal{G}_2$) *for any* $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ *with* $(\mathfrak{a}, \mathfrak{b}) \in \mathsf{E}(\mathcal{G})$*,*

$$\psi(\mathsf{d}(\mathcal{F}\mathfrak{a}, \mathcal{F}^2\mathfrak{b})) \leqslant \psi(\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{b})) - \varphi(\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{b})) \tag{1}$$

in which $\psi, \varphi \in \Psi$*.*

Now, we prove the main result of this part.

Theorem 8. *Assume* $\mathcal{A}, \mathcal{B} \neq \varnothing$ *are two subsets of a GMS* $(\mathcal{X}, \mathsf{d})$*,* $\mathcal{A}$ *is complete, and* $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ *is a cyclic* $\mathcal{G}$-$(\varphi - \psi)$*-weak contractive mapping on* $\mathcal{A}$*. Also, assume that either* $\mathcal{F}$ *is orbitally* $\mathcal{G}$*-continuous on* $\mathcal{A}$ *or* $\mathcal{G}$ *is a* **C***-graph on* $\mathcal{A}$*. Then* $\mathcal{F}$ *possesses a FP in* $\mathcal{A} \cap \mathcal{B}$ *whenever there exists* $\mathfrak{a}_0 \in \mathcal{A}$ *with* $\mathfrak{a}_0 \in \mathcal{C}_\mathcal{F}$*.*

Proof. As $\mathfrak{a}_0 \in \mathcal{C}_\mathcal{F}$, we have $(\mathfrak{a}_0, \mathcal{F}^2\mathfrak{a}_0) \in \mathsf{E}(\mathcal{G})$. On the other hand, since $\mathcal{F}$ is a cyclic $\mathcal{G}$-$(\varphi - \psi)$-weak contractive mapping on $\mathcal{A}$, it follows from ($\mathcal{G}_1$) on $\mathcal{A}$ that $(\mathfrak{a}_{2\mathfrak{n}}, \mathfrak{a}_{2\mathfrak{n}+2}) \in \mathsf{E}(\mathcal{G})$ for $\mathfrak{n} = 0, 1, \ldots$, so $\mathfrak{a}_{2\mathfrak{n}} = \mathcal{F}^{2\mathfrak{n}}\mathfrak{a}_0$. Now, if $\mathcal{F}^2\mathfrak{a}_0 = \mathfrak{a}_0$, we obtain

$$\begin{aligned}
\psi\big(\mathsf{d}(\mathfrak{a}_0, \mathcal{F}\mathfrak{a}_0)\big) &= \psi\Big(\mathsf{d}\big(\mathcal{F}^2\mathfrak{a}_0, \mathcal{F}(\mathcal{F}^2\mathfrak{a}_0)\big)\Big) \\
&\leqslant \psi\big(\mathsf{d}(\mathcal{F}^2\mathfrak{a}_0, \mathcal{F}\mathfrak{a}_0)\big) - \varphi\big(\mathsf{d}(\mathcal{F}^2\mathfrak{a}_0, \mathcal{F}\mathfrak{a}_0)\big) \\
&= \psi\big(\mathsf{d}(\mathfrak{a}_0, \mathcal{F}\mathfrak{a}_0)\big) - \varphi\big(\mathsf{d}(\mathfrak{a}_0, \mathcal{F}\mathfrak{a}_0)\big),
\end{aligned}$$

which induces that $\varphi\big(\mathsf{d}(\mathfrak{a}_0, \mathcal{F}\mathfrak{a}_0)\big) = 0$, and as $\varphi \in \Psi$, $\mathsf{d}(\mathfrak{a}_0, \mathcal{F}\mathfrak{a}_0) = 0$. Hence, $\mathfrak{a}_0 = \mathcal{F}\mathfrak{a}_0$; that is, $\mathfrak{a}_0$ is a FP of $\mathcal{F}$ and the proof ends. Consequently, take $\mathcal{F}^2\mathfrak{a}_0 \neq \mathfrak{a}_0$. Since $(\mathfrak{a}_{2\mathfrak{n}}, \mathfrak{a}_{2\mathfrak{n}+2}) \in \mathsf{E}(\mathcal{G})$ for any $\mathfrak{n} \in \mathbb{N} \cup \{0\}$ and by (1) on $\mathcal{A}$, we get

$$\begin{aligned}
\psi\big(\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}}, \mathfrak{a}_{2\mathfrak{n}+1})\big) &= \psi\big(\mathsf{d}(\mathcal{F}\mathfrak{a}_{2\mathfrak{n}}, \mathcal{F}^2\mathfrak{a}_{2\mathfrak{n}-2})\big) \\
&\leqslant \psi\big(\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}}, \mathfrak{a}_{2\mathfrak{n}-1})\big) - \varphi\big(\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}}, \mathfrak{a}_{2\mathfrak{n}-1})\big)
\end{aligned}$$

$$\leqslant \psi\big(d(\mathfrak{a}_{2n}, \mathfrak{a}_{2n-1})\big) = \psi\big(d(\mathcal{F}\mathfrak{a}_{2n-2}, \mathcal{F}^2\mathfrak{a}_{2n-2})\big)$$
$$\leqslant \psi\big(d(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1})\big) - \varphi\big(d(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1})\big),$$

which yields $\psi\big(d(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1})\big) \leqslant \psi\big(d(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1})\big)$. Since ψ is a non-decreasing function, we conclude that $\{d(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1})\}$ is a decreasing sequence. Assume that $d(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1}) \to \mathfrak{u}$. Since $\varphi \in \Psi$; we have $\varphi(d(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1})) \to \varphi(\mathfrak{u}) = 0$ and so $\mathfrak{u} = 0$. Hence, there is $\mathcal{N}_1 \in \mathbb{N}$ provided that $d(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1}) \leqslant \varepsilon$ for each $\mathfrak{n} \geqslant \mathcal{N}_1$. Now, we claim that there is $\mathcal{N}_2 \in \mathbb{N}$, such that $d(\mathfrak{a}_{2m}, \mathfrak{a}_{2n+1}) < \varepsilon$ for all $\mathfrak{m} \geqslant \mathfrak{n} \geqslant \mathcal{N}_2$. To the contrary, assume that there exists $\varepsilon_0 > 0$ and integers $\mathfrak{m}_i > \mathfrak{n}_i \geqslant i$ for every $i \geqslant 1$, such that $d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2n_i+1}) \geqslant \varepsilon_0$ and $d(\mathfrak{a}_{2m_i-2}, \mathfrak{a}_{2n_i+1}) < \varepsilon_0$. Thus, we obtain

$$\begin{aligned}\varepsilon_0 &\leqslant d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2n_i+1})\\ &\leqslant d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2m_i-1}) + d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2m_i-2}) + d(\mathfrak{a}_{2m_i-2}, \mathfrak{a}_{2n_i+1})\\ &\leqslant d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2m_i-1}) + d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2m_i-2}) + \varepsilon_0.\end{aligned}$$

Assume that $i \to \infty$. Then $d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2n_i+1}) \to \varepsilon_0$. In addition, from

$$d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2n_i+1}) \leqslant d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2m_i-1}) + d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2n_i}) + d(\mathfrak{a}_{2n_i}, \mathfrak{a}_{2n_i+1}),$$
$$d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2n_i}) \leqslant d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2m_i}) + d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2n_i+1}) + d(\mathfrak{a}_{2n_i+1}, \mathfrak{a}_{2n_i}),$$

we have $d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2n_i}) \to \varepsilon_0$. Consequently, by (1) and the continuity of ψ and φ, and since $\mathfrak{a}_0 \in \mathcal{C}_{\mathcal{F}}$ implies that $(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2n_i}) \in E(\mathcal{G})$, we obtain

$$\begin{aligned}\psi(\varepsilon_0) &= \lim_{i\to\infty} \psi\big(d(\mathfrak{a}_{2m_i}, \mathfrak{a}_{2n_i+1})\big)\\ &\leqslant \lim_{i\to\infty} \psi\big(d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2n_i})\big) - \lim_{i\to\infty} \varphi\big(d(\mathfrak{a}_{2m_i-1}, \mathfrak{a}_{2n_i})\big) \leqslant \psi(\epsilon_0) - \varphi(\epsilon_0),\end{aligned}$$

which conclude that $\varphi(\epsilon_0) = 0$ and so $\varepsilon_0 = 0$: this is impossible. Setting $\mathcal{N} = \max\{\mathcal{N}_1, \mathcal{N}_2\}$ for any $\mathfrak{m} > \mathfrak{n} \geqslant \mathcal{N}$, we get

$$d(\mathfrak{a}_{2m}, \mathfrak{a}_{2n}) \leqslant d(\mathfrak{a}_{2m}, \mathfrak{a}_{2n+1}) + d(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1}) < 2\varepsilon.$$

It means that $\{\mathfrak{a}_{2n}\}$ is a Cauchy sequence on $\mathcal{A}$, and by the completeness of $\mathcal{A}$, $\mathfrak{a}' \in \mathcal{A}$ exists so that $\mathfrak{a}_{2n} \to \mathfrak{a}'$. Now, we show that $\mathfrak{a}'$ is a FP of the mapping $\mathcal{F}$. To do this, note first that from $\mathfrak{a}_0 \in \mathcal{C}_{\mathcal{F}}$, we have $(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1}) \in E(\mathcal{G})$ for each $\mathfrak{n} \in \mathbb{N}$. If $\mathcal{F}$ is orbitally $\mathcal{G}$-continuous on $\mathcal{A}$, then $\mathfrak{a}_{2n} \to \mathfrak{a}'$ implies $\mathcal{F}(\mathfrak{a}_{2n}) \to \mathcal{F}\mathfrak{a}'$, and, as a result, we have $d(\mathfrak{a}', \mathcal{F}\mathfrak{a}') = \lim_{n\to\infty} d(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1}) = 0$. Thus, $\mathcal{F}\mathfrak{a}' = \mathfrak{a}'$. Otherwise, let $\mathcal{G}$ be a **C**-graph. As $\mathfrak{a}_{2n} \to \mathfrak{a}'$, there is a strictly increasing sequence $\{n_i\}$ of positive integers, such that $(\mathfrak{a}_{2n_i}, \mathfrak{a}') \in \mathsf{E}(\mathcal{G})$ for each $i \in \mathbb{N}$. Now, it follows from (1) that

$$\begin{aligned}\lim_{i\to\infty} \psi\big(\mathsf{d}(\mathfrak{a}_{2n_i+1}, \mathcal{F}\mathfrak{a}')\big) &= \lim_{i\to\infty} \psi\big(\mathsf{d}(\mathcal{F}\mathfrak{a}', \mathcal{F}^2\mathfrak{a}_{2n_i-1})\big) \\ &\leqslant \lim_{i\to\infty} \psi\big(\mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a}_{2n_i-1})\big) - \lim_{i\to\infty} \varphi\big(\mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a}_{2n_i-1})\big) \\ &\leqslant \lim_{i\to\infty} \psi\big(\mathsf{d}(\mathfrak{a}', \mathfrak{a}_{2n_i})\big) - \lim_{i\to\infty} \varphi\big(\mathsf{d}(\mathfrak{a}', \mathfrak{a}_{2n_i})\big) = 0,\end{aligned}$$

which induces that $\lim_{i\to\infty} \psi\big(\mathsf{d}(\mathfrak{a}_{2n_i+1}, \mathcal{F}\mathfrak{a}')\big) = 0$. Since $\psi \in \Psi$, we get $\lim_{i\to\infty} \mathsf{d}(\mathfrak{a}_{2n_i+1}, \mathcal{F}\mathfrak{a}') = 0$, implying $\mathsf{d}(\mathfrak{a}', \mathcal{F}\mathfrak{a}') = \lim_{n\to\infty} \mathsf{d}(\mathfrak{a}_{2n_i}, \mathfrak{a}_{2n_i+1}) = 0$. Thus, $\mathcal{F}\mathfrak{a}' = \mathfrak{a}'$ and the proof ends. □

3. BPP results. First, following the idea of Theorems 6 and 7, we state the definition of a cyclic $(\varphi - \phi)$-weak contraction in a GMS.

Definition 5. *Assume $(\mathcal{X}, \mathsf{d})$ is a GMS. A mapping $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic $\mathcal{G}$-$(\varphi - \psi)$-contractive mapping when*

$$\psi\big(\mathsf{d}(\mathcal{F}\mathfrak{a}, \mathcal{F}^2\mathfrak{b})\big) \leqslant \psi\big(\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{b})\big) - \varphi\big(\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{b})\big) + \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big) \qquad (2)$$

for any $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{A} \times \mathcal{A}$ with $(\mathfrak{a}, \mathfrak{b}) \in \mathsf{E}(\mathcal{G})$, where $\psi, \varphi \in \Psi$.

Now, we prove the first fundamental theorem of this part.

Theorem 9. *Assume $\mathcal{A}, \mathcal{B} \neq \varnothing$ are two closed subsets of a GMS $(\mathcal{X}, \mathsf{d})$, $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic $\mathcal{G}$-$(\varphi - \psi)$-weak contractive mapping on $\mathcal{A}$, and $\mathcal{F}^2$ keeps the edges of $\mathcal{G}$ on $\mathcal{A}$. Also, assume that $\mathcal{C}_{\mathcal{F}}|_{\mathcal{A}} \neq \varnothing$ and $\mathfrak{a}_{n+1} = \mathcal{F}\mathfrak{a}_n$. If $\mathcal{G}$ is **C**-graph on $\mathcal{A}$ and $\{\mathfrak{a}_{2n}\}$ possesses a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ has a BPP $\mathfrak{a}' \in \mathcal{A}$.*

Proof. By the assumption, $\mathcal{C}_{\mathcal{F}}|_{\mathcal{A}} \neq \varnothing$. Thus, assume $\mathfrak{a}_0 \in \mathcal{C}_{\mathcal{F}}$ with $\mathfrak{a}_0 \in \mathcal{A}$. Then $(\mathfrak{a}_0, \mathcal{F}^2\mathfrak{a}_0) \in \mathsf{E}(\mathcal{G})$ and as $\mathcal{F}^2$ keeps the edges of $\mathcal{G}$ on $\mathcal{A}$, we have $(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+2}) \in \mathsf{E}(\mathcal{G})$ for $n = 0, 1, \ldots$, in which $\mathfrak{a}_{2n} = \mathcal{F}^{2n}\mathfrak{a}_0$. Since $(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+2}) \in \mathsf{E}(\mathcal{G})$ for any $n \in \mathbb{N} \cup \{0\}$ and by (2) on $\mathcal{A}$, we have

$$\begin{aligned}\psi\big(\mathsf{d}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1})\big) &\leqslant \psi\big(\mathsf{d}(\mathcal{F}\mathfrak{a}_{2n}, \mathcal{F}^2\mathfrak{a}_{2n-2})\big) \\ &\leqslant \psi\big(\mathsf{d}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n-1})\big) - \varphi\big(\mathsf{d}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n-1})\big) + \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big).\end{aligned} \qquad (3)$$

Now, as $\varphi\big(\mathsf{d}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n-1})\big) \geqslant \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big)$ for each $n \in \mathbb{N}$, we conclude that

$$\psi\big(\mathsf{d}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1})\big) \leqslant \psi\big(\mathsf{d}(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1})\big).$$

Since $\psi \in \Psi$, it is a nondecreasing function, and as a result, we get $\{\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}-2},\mathfrak{a}_{2\mathfrak{n}-1})\}$ is a decreasing sequence. Assume $\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}-2},\mathfrak{a}_{2\mathfrak{n}-1}) \to p^\star$. By (3) and since $\varphi \in \psi$, we have

$$\lim_{\mathfrak{n}\to\infty} \varphi\big(\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}},\mathfrak{a}_{2\mathfrak{n}-1})\big) = \varphi(p^\star) \leqslant \varphi\big(\mathsf{dist}(\mathcal{A},\mathcal{B})\big),$$

which concludes $\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}-2},\mathfrak{a}_{2\mathfrak{n}-1}) \to \mathsf{dist}(\mathcal{A},\mathcal{B})$. Now, assume that $\{\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}}\}$ is a subsequence of $\{\mathfrak{a}_{2\mathfrak{n}}\}$ converging to a $\mathfrak{a}' \in \mathcal{A}$. Then we have

$$\mathsf{dist}(\mathcal{A},\mathcal{B}) \leqslant \mathsf{d}(\mathfrak{a}',\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1}) \leqslant \mathsf{d}(\mathfrak{a}',\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}}) + \mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}},\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1}).$$

Thus, it follows by taking limit that $\lim\limits_{\mathfrak{i}\to\infty} \mathsf{d}(\mathfrak{a}',\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1}) = \mathsf{dist}(\mathcal{A},\mathcal{B})$. Since $\mathcal{F}^2$ keeps the edges of $\mathcal{G}$ and $\mathcal{G}$ is a $\mathbf{C}$-graph, $(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}},\mathfrak{a}') \in \mathsf{E}(\mathcal{G})$ for any $\mathfrak{i} \in \mathbb{N}$. Now, using (2), we obtain

$$\begin{aligned}\psi\big(\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}},\mathcal{F}\mathfrak{a}')\big) &= \psi\big(\mathsf{d}(\mathcal{F}\mathfrak{a}',\mathcal{F}^2\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-2})\big) \\ &\leqslant \psi\big(\mathsf{d}(\mathfrak{a}',\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1})\big) - \varphi\big(\mathsf{d}(\mathfrak{a}',\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1})\big) + \varphi\big(\mathsf{dist}(\mathcal{A},\mathcal{B})\big) \\ &\leqslant \psi\big(\mathsf{d}(\mathfrak{a}',\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1})\big).\end{aligned} \tag{4}$$

As ψ is a nondecreasing function, from (4) we have:

$$\mathsf{dist}(\mathcal{A},\mathcal{B}) \leqslant \mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}},\mathcal{F}\mathfrak{a}') \leqslant \mathsf{d}(\mathfrak{a}',\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1}).$$

Hence, $\mathsf{d}(\mathfrak{a}',\mathcal{F}\mathfrak{a}') = \lim\limits_{\mathfrak{i}\to\infty} \mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}},\mathcal{F}\mathfrak{a}') = \mathsf{dist}(\mathcal{A},\mathcal{B})$, which means that $\mathfrak{a}' \in \mathcal{A}$ is a BPP and this completes the proof. □

Now, to show that Theorem 9 can expand all existing theorems in the graph-type literature, we list two following results by taking special type of the functions ψ and φ. First, taking $\psi(f) = f$, we have next corollary:

Corollary 1. *Assume $\mathcal{A},\mathcal{B} \neq \varnothing$ are two closed subsets of a GMS $(\mathcal{X},\mathsf{d})$, $\mathcal{F}\colon \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$ is a cyclic $\mathcal{G}$-φ-weak contractive mapping, i.e.*

$$\mathsf{d}(\mathcal{F}\mathfrak{a},\mathcal{F}^2\mathfrak{b}) \leqslant \mathsf{d}(\mathfrak{a},\mathcal{F}\mathfrak{b}) - \varphi\big(\mathsf{d}(\mathfrak{a},\mathcal{F}\mathfrak{b})\big) + \varphi\big(\mathsf{dist}(\mathcal{A},\mathcal{B})\big)$$

for any $(\mathfrak{a},\mathfrak{b}) \in \mathcal{A}\times\mathcal{A}$ with $(\mathfrak{a},\mathfrak{b}) \in \mathsf{E}(\mathcal{G})$ in which $\varphi \in \Psi$, and $\mathcal{F}^2$ keeps the edges of $\mathcal{G}$ on $\mathcal{A}$. Also, assume that $\mathcal{C}_\mathcal{F}|_\mathcal{A} \neq \varnothing$ and $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_\mathfrak{n}$. If $\mathcal{G}$ is $\mathbf{C}$-graph on $\mathcal{A}$ and $\{\mathfrak{a}_{2\mathfrak{n}}\}$ possesses a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ has a BPP $\mathfrak{a}' \in \mathcal{A}$.

Second, taking $\psi(f) = f$ and $\varphi(f) = (1-\alpha)f$ in Theorem 9 (or setting just $\varphi(f) = (1-\alpha)f$ in Corollary 1), we have next corollary:

Corollary 2. *Assume $\mathcal{A}, \mathcal{B} \neq \varnothing$ are two closed subsets of a GMS $(\mathcal{X}, \mathsf{d})$, $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic $\mathcal{G}$-weak contractive mapping, i.e.,*

$$\mathsf{d}(\mathcal{F}\mathfrak{a}, \mathcal{F}^2\mathfrak{b}) \leqslant \alpha \mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{b}) + (1-\alpha)\mathsf{dist}(\mathcal{A}, \mathcal{B})$$

for every $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{A} \times \mathcal{A}$ with $(\mathfrak{a}, \mathfrak{b}) \in \mathsf{E}(\mathcal{G})$ in which $\alpha \in (0,1)$, and $\mathcal{F}^2$ keeps the edges of $\mathcal{G}$ on $\mathcal{A}$. Also, assume that $\mathcal{C}_{\mathcal{F}}|_{\mathcal{A}} \neq \varnothing$ and $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_{\mathfrak{n}}$. If $\mathcal{G}$ is **C***-graph on $\mathcal{A}$ and $\{\mathfrak{a}_{2\mathfrak{n}}\}$ possesses a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ has a BPP $\mathfrak{a}' \in \mathcal{A}$.*

Several consequences of our first fundamental result can also be obtained for a special type of the graphs. First, take $\mathcal{G} = \mathcal{G}_0$ in which $\mathcal{G}_0$ is a complete graph; that is, $\mathcal{G}_0$ is a graph with $\mathsf{V}(\mathcal{G}_0) = \mathcal{X}$ and $\mathsf{E}(\mathcal{G}_0) = \mathcal{X} \times \mathcal{X}$.

Corollary 3. *Let $\mathcal{A}, \mathcal{B} \neq \varnothing$ be two closed subsets of a GMS $(\mathcal{X}, \mathsf{d})$ and $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclic $\mathcal{G}_0$-$(\varphi - \psi)$-weak contractive mapping. Also, assume that $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_{\mathfrak{n}}$. If $\{\mathfrak{a}_{2\mathfrak{n}}\}$ possesses a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ has a BPP $\mathfrak{a}' \in \mathcal{A}$.*

Next, assume that $(\mathcal{X}, \preceq)$ is a PO set and $\mathcal{G}_1$ is a graph on $\mathcal{X}$, where $\mathsf{V}(\mathcal{G}_1) = \mathcal{X}$ and $\mathsf{E}(\mathcal{G}_1) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathcal{X} \times \mathcal{X} \colon \mathfrak{a} \preceq \mathfrak{b}\}$. If $\mathcal{G} = \mathcal{G}_1$ in Theorem 9, then we obtain the second consequence.

Corollary 4. *Assume $(\mathcal{X}, \preceq)$ is a PO set, $\mathcal{A}, \mathcal{B} \neq \varnothing$ are two closed subsets of a GMS $(\mathcal{X}, \mathsf{d})$, $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic $\mathcal{G}_1$-$(\varphi - \psi)$-weak contractive mapping, and $\mathcal{F}^2$ is nondecreasing on $\mathcal{A}$. Also, assume that $x_0 \in \mathcal{A}$ with $x_0 \preceq \mathcal{F}^2 x_0$ exists and $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_{\mathfrak{n}}$. If $\mathcal{G}_1$ is* **C***-graph on $\mathcal{A}$ and $\{\mathfrak{a}_{2\mathfrak{n}}\}$ possesses a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ has a BPP $\mathfrak{a}' \in \mathcal{A}$.*

Again, assume that $(\mathcal{X}, \preceq)$ is a PO set. $\mathcal{G}_2$ is a graph on $\mathcal{X}$, where $\mathsf{V}(\mathcal{G}_2) = \mathcal{X}$ and $\mathsf{E}(\mathcal{G}_2) = \{(\mathfrak{a},\mathfrak{b}) \in \mathcal{X} \times \mathcal{X} : \mathfrak{a} \preceq \mathfrak{b} \text{ or } \mathfrak{b} \preceq \mathfrak{a}\}$; i.e., $\mathfrak{a}$ and $\mathfrak{b}$ are comparable members of $\mathcal{X}$. If $\mathcal{G} = \mathcal{G}_2$ in Theorem 9, then we obtain the next consequence.

Corollary 5. *Assume $(\mathcal{X}, \preceq)$ is a PO set, $\mathcal{A}, \mathcal{B} \neq \varnothing$ are two closed subsets of a GMS $(\mathcal{X}, \mathsf{d})$, and $\mathcal{F}\colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic $\mathcal{G}_2$-$(\varphi - \psi)$-weak contractive mapping. Also, assume that if $\mathfrak{a}$ and $\mathfrak{b}$ are comparable for $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$, then $\mathcal{F}^2\mathfrak{a}$ and $\mathcal{F}^2\mathfrak{b}$ are comparable. Moreover, assume $x_0 \in \mathcal{A}$ exists, such that x_0 and $\mathcal{F}^2 x_0$ are comparable, and $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_{\mathfrak{n}}$. If $\mathcal{G}_2$ is* **C***-graph on $\mathcal{A}$ and $\{\mathfrak{a}_{2\mathfrak{n}}\}$ possesses a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ has a BPP $\mathfrak{a}' \in \mathcal{A}$.*

Finally, assume that $\varepsilon > 0$ is fixed. Remember that $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ are named ε-close when $\mathsf{d}(\mathfrak{a}, \mathfrak{b}) \preceq \varepsilon$. Define the ε-graph $\mathcal{G}_3$ by $\mathsf{V}(\mathcal{G}_3) = \mathcal{X}$

and $\mathsf{E}(\mathcal{G}_3) = \{(\mathfrak{a},\mathfrak{b}) \in \mathcal{X} \times \mathcal{X} \colon \mathsf{d}(\mathfrak{a},\mathfrak{b}) \preceq \varepsilon\}$. We see that $\mathsf{E}(\mathcal{G}_3)$ contains all loops. Now, if $\mathcal{G} = \mathcal{G}_3$ in Theorem 9, then the last consequence is obtained.

Corollary 6. *Assume $\mathcal{A}, \mathcal{B} \neq \varnothing$ are two closed subsets of a GMS $(\mathcal{X}, \mathsf{d})$, $\mathcal{F} \colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic $\mathcal{G}_3$-$(\varphi - \psi)$-weak contractive mapping. Also, assume that if $\mathfrak{a}$ and $\mathfrak{b}$ are ε-close for $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$, then $\mathcal{F}^2\mathfrak{a}$ and $\mathcal{F}^2\mathfrak{b}$ are ε-close. Moreover, assume that $x_0 \in \mathcal{A}$ exists, such that x_0 and $\mathcal{F}^2 x_0$ are ε-close and $\mathfrak{a}_{\mathfrak{n}+1} = \mathcal{F}\mathfrak{a}_{\mathfrak{n}}$. If $\mathcal{G}_3$ is $\mathbf{C}$-graph on $\mathcal{A}$ and $\{\mathfrak{a}_{2\mathfrak{n}}\}$ possesses a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ possesses a BPP $\mathfrak{a}' \in \mathcal{A}$.*

The second fundamental result of this part is the next theorem regarding a graph instead a partial order, extending Theorem 3.5 of Abkar and Gabeleh [2].

Theorem 10. *Assume $\mathcal{A}, \mathcal{B} \neq \varnothing$ are subsets of a GMS $(\mathcal{X}, \mathsf{d})$, $\mathcal{A}$ is complete, both $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ have the UC property, $\mathcal{F} \colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic $\mathcal{G}$-$(\varphi - \psi)$-weak contractive mapping on both $\mathcal{A}$ (and $\mathcal{B}$) in which $\mathcal{F}$ and $\mathcal{F}^2$ keep the edges of $\mathcal{G}$ on $\mathcal{A}$. If either $\mathcal{F}$ is orbitally $\mathcal{G}$-continuous on $\mathcal{A}$ or $\mathcal{G}$ is a $\mathbf{C}$-graph on $\mathcal{A}$, then $\mathcal{F}$ has a BPP $p^\star \in \mathcal{A}$ whenever $\mathfrak{a}_0 \in \mathcal{A}$ with $\mathfrak{a}_0 \in \mathcal{C}_{\mathcal{F}}$.*

Proof. Assume that $\mathfrak{a}_0 \in \mathcal{C}_{\mathcal{F}}$ with $\mathfrak{a}_0 \in \mathcal{A}$. As both $\mathcal{F}$ and $\mathcal{F}^2$ keep the edges of $\mathcal{G}$ on $\mathcal{A}$ and $(\mathfrak{a}_0, \mathcal{F}^2\mathfrak{a}_0) \in \mathsf{E}(\mathcal{G})$, we obtain

$$(\mathfrak{a}_{2\mathfrak{n}}, \mathfrak{a}_{2\mathfrak{n}+2}) \in \mathsf{E}(\mathcal{G}) \quad \text{and} \quad (\mathfrak{a}_{2\mathfrak{n}+1}, \mathfrak{a}_{2\mathfrak{n}+3}) \in \mathsf{E}(\mathcal{G}) \quad \text{for} \quad \mathfrak{n} = 0, 1, \ldots .$$

Similar to the proof of Theorem 9, we have

$$\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}}, \mathfrak{a}_{2\mathfrak{n}+1}) \to \mathsf{dist}(\mathcal{A}, \mathcal{B}) \;\; \text{and} \;\; \mathsf{d}(\mathfrak{a}_{2\mathfrak{n}+2}, \mathfrak{a}_{2\mathfrak{n}+1}) \to \mathsf{dist}(\mathcal{A}, \mathcal{B}).$$

Now, as $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ have the UC property, we can obtain

$$\mathsf{d}(\mathfrak{a}_{2\mathfrak{m}}, \mathfrak{a}_{2\mathfrak{m}+2}) \to 0 \;\; \text{and} \; \mathsf{d}(\mathfrak{a}_{2\mathfrak{m}+1}, \mathfrak{a}_{2\mathfrak{m}+3}) \to 0,$$

respectively. Let $\varepsilon > 0$ be arbitrary. We show that there is a $\mathfrak{n} \in \mathbb{N}$ that

$$\mathsf{D}(\mathfrak{a}_{2m}, \mathfrak{a}_{2\mathfrak{n}+1}) < \varepsilon \tag{5}$$

for any $\mathfrak{m} > \mathfrak{n} \geqslant \mathcal{N}$, where $\mathsf{D}(\mathfrak{g}, \mathfrak{h}) = \mathsf{d}(\mathfrak{g}, \mathfrak{h}) - \mathsf{dist}(\mathcal{A}, \mathcal{B})$ for each $(\mathfrak{g}, \mathfrak{h}) \in \mathcal{A} \times \mathcal{B}$. To the contrary, assume (5) is not valid. Then there are $\varepsilon_0 > 0$ and $\mathfrak{m}_\mathfrak{i} > \mathfrak{n}_\mathfrak{i} \geqslant \mathfrak{i}$ for each $\mathfrak{i} \geqslant 1$, so that $\mathsf{D}(\mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}}, \mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}) \geqslant \varepsilon_0$ and $\mathsf{D}(\mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}-2}, \mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}) < \varepsilon_0$. Then we have

$$\varepsilon_0 \leqslant \mathsf{D}(\mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}}, \mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}) \leqslant \mathsf{d}(\mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}-2}, \mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}}) + \mathsf{D}(\mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}-2}, \mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}) \leqslant \mathsf{d}(\mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}-2}, \mathfrak{a}_{2\mathfrak{m}_\mathfrak{i}}) + \varepsilon_0.$$

Now, taking limit as $i \to \infty$, we obtain

$$\lim_{i\to\infty} D(a_{2m_i}, a_{2n_i+1}) = \varepsilon_0.$$

Since $\mathcal{F}$ and $\mathcal{F}^2$ preserves the edges of $\mathcal{G}$ on $\mathcal{A}$,

$$\begin{aligned}\psi\big(d(a_{2m_i+2}, a_{2n_i+3})\big) &= \psi\Big(d\big(\mathcal{F}a_{2m_i+1}, \mathcal{F}^2 a_{2n_i+1}\big)\Big)\\ &\psi\big(d(a_{2m_i+1}, a_{2n_i+2})\big) - \varphi\big(d(a_{2m_i+1}, a_{2n_i+2})\big) + \varphi\big(\mathrm{dist}(\mathcal{A},\mathcal{B})\big)\\ &\leqslant \psi\big(d(a_{2m_i+1}, a_{2n_i+2})\big) = \psi\Big(d\big(\mathcal{F}a_{2m_i}, \mathcal{F}^2 a_{2n_i}\big)\Big)\\ &\leqslant \psi\big(d(a_{2m_i}, a_{2n_i+1})\big) - \varphi\big(d(a_{2m_i}, a_{2n_i+1})\big) + \varphi\big(\mathrm{dist}(\mathcal{A},\mathcal{B})\big)\\ &\leqslant \psi\big(d(a_{2m_i}, a_{2n_i+1})\big). \quad (6)\end{aligned}$$

As ψ is a nondecreasing function, we have $d(a_{2m_i+2}, a_{2n_i+3}) \leqslant d(a_{2m_i}, a_{2n_i+1})$ and hence,

$$\begin{aligned}D(a_{2m_i}, a_{2n_i+1}) &\leqslant d(a_{2m_i}, a_{2m_i+2}) + D(a_{2m_i+2}, a_{2n_i+3}) + d(a_{2n_i+1}, a_{2n_i+3})\\ &\leqslant d(a_{2m_i}, a_{2m_i+2}) + D(a_{2m_i}, a_{2n_i+1}) + d(a_{2n_i+1}, a_{2n_i+3}).\end{aligned}$$

Now, taking limit as $i \to \infty$, we get $D(a_{2m_i+2}, a_{2n_i+3}) \to \varepsilon_0$; that is,

$$\lim_{i\to\infty} d(a_{2m_i+2}, a_{2n_i+3}) = \varepsilon_0 + \mathrm{dist}(\mathcal{A},\mathcal{B}).$$

Using (6) and continuity of functions ψ and φ, we obtain

$$\begin{aligned}\psi\big(\varepsilon_0 + \mathrm{dist}(\mathcal{A},\mathcal{B})\big) &= \lim_{i\to\infty} \psi\big(d(a_{2m_i+2}, a_{2n_i+3})\big)\\ &\leqslant \lim_{i\to\infty} \psi\big(d(a_{2m_i}, a_{2n_i+1})\big) - \lim_{i\to\infty} \varphi\big(d(a_{2m_i}, a_{2n_i+1})\big) + \varphi\big(\mathrm{dist}(\mathcal{A},\mathcal{B})\big)\\ &= \psi\big(\varepsilon_0 + \mathrm{dist}(\mathcal{A},\mathcal{B})\big) - \varphi\big(\varepsilon_0 + \mathrm{dist}(\mathcal{A},\mathcal{B})\big) + \varphi\big(\mathrm{dist}(\mathcal{A},\mathcal{B})\big),\end{aligned}$$

implying $\varphi\big(\varepsilon_0 + \mathrm{dist}(\mathcal{A},\mathcal{B})\big) \leqslant \varphi\big(\mathrm{dist}(\mathcal{A},\mathcal{B})\big)$, which is a contradiction. Therefore, (5) is valid. Hence, $\lim_{m\to\infty} \sup_{n\geqslant m} D(a_{2m}, a_{2n+1}) = 0$. As $(\mathcal{A},\mathcal{B})$ has the UC property, and by Lemma 1, $\{a_{2n}\}$ is a Cauchy sequence in $\mathcal{A}$. As $\mathcal{A}$ is complete, $\{a_{2n}\}$ converges to a $p^\star \in \mathcal{A}$. Now, let us show that $p^\star$ is same BPP.

First, as $a_0 \in \mathcal{C}_{\mathcal{F}}$, we have $(a_{2n}, a_{2n+1}) \in E(\mathcal{G})$ for each $n \in \mathbb{N}$. Assume $\mathcal{F}$ is orbitally $\mathcal{G}$-continuous on $\mathcal{A}$. Then $a_{2n} \to p^\star$ implies that $\mathcal{F}a_{2n} \to \mathcal{F}p^\star$. Hence,

$$d(p^\star, \mathcal{F}p^\star) = \lim_{n\to\infty} d(a_{2n}, a_{2n+1}) = \mathrm{dist}(\mathcal{A},\mathcal{B})$$

and $p^\star$ is a BPP. Otherwise, assume $\mathcal{G}$ is a $\mathbf{C}$-graph. As $\mathfrak{a}_{2\mathfrak{n}} \to p^\star$, there is a strictly increasing sequence $\{\mathfrak{n}_\mathfrak{i}\}$ of natural numbers so that $(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}}, p^\star) \in \mathsf{E}(\mathcal{G})$ for any $\mathfrak{i} \in \mathbb{N}$. Since $\mathcal{F}$ satisfies (2), we have

$$\begin{aligned}\lim_{\mathfrak{i}\to\infty} \psi\big(\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}, \mathcal{F}p^\star)\big) &\leqslant \lim_{\mathfrak{i}\to\infty} \psi\big(\mathsf{d}(\mathcal{F}p^\star, \mathcal{F}^2\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1})\big) \\ &\leqslant \lim_{\mathfrak{i}\to\infty} \psi\big(\mathsf{d}(p^\star, \mathcal{F}\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1})\big) - \lim_{\mathfrak{i}\to\infty} \varphi\big(\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}-1})\big) + \varphi(\mathsf{dist}(\mathcal{A}, \mathcal{B})) \\ &\leqslant \lim_{\mathfrak{i}\to\infty} \psi\big(\mathsf{d}(p^\star, \mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}})\big) - \lim_{\mathfrak{i}\to\infty} \varphi\big(\mathsf{d}(p^\star, \mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}})\big) + \varphi(\mathsf{dist}(\mathcal{A}, \mathcal{B})) \\ &= \psi(\mathsf{dist}(\mathcal{A}, \mathcal{B})),\end{aligned}$$

which induces that

$$\lim_{\mathfrak{i}\to\infty} \psi\big(\mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}, \mathcal{F}p^\star)\big) = \psi(\mathsf{dist}(\mathcal{A}, \mathcal{B})).$$

Thus, $\lim_{\mathfrak{i}\to\infty} \mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}, \mathcal{F}p^\star) = \mathsf{dist}(\mathcal{A}, \mathcal{B})$, which causes

$$\mathsf{d}(p^\star, \mathcal{F}p^\star) = \lim_{\mathfrak{n}\to\infty} \mathsf{d}(\mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}}, \mathfrak{a}_{2\mathfrak{n}_\mathfrak{i}+1}) = \mathsf{dist}(\mathcal{A}, \mathcal{B}),$$

i.e., $p^\star$ is a BPP and the proof ends. □

Example 1. Take $\mathcal{X} = \mathbb{R}^2$ and the usual metric

$$\mathsf{d}\big((\mathfrak{a}_1, \mathfrak{b}_1), (\mathfrak{a}_2, \mathfrak{b}_2)\big) = \sqrt{(\mathfrak{a}_1 - \mathfrak{a}_2)^2 + (\mathfrak{b}_1 - \mathfrak{b}_2)^2}$$

for $(\mathfrak{a}_1, \mathfrak{b}_1)$,$(\mathfrak{a}_2, \mathfrak{b}_2) \in \mathbb{R}^2$ and set

$$\mathcal{A} = \big\{(\mathfrak{a}, 1) \colon \mathfrak{a} \in [0, 1]\big\} \ , \ \mathcal{B} = \big\{(\mathfrak{b}, 0) \colon \mathfrak{b} \in [0, 1]\big\}.$$

Then $\mathsf{dist}(\mathcal{A}, \mathcal{B}) = 1$. Also, define $\varphi, \psi \colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ by

$$\psi(\kappa) = \begin{cases} \kappa^2, & 0 \leqslant \kappa < \dfrac{1}{2}, \\ \dfrac{\kappa}{2}, & \kappa \geqslant \dfrac{1}{2}, \end{cases} \quad \text{and} \quad \varphi(\kappa) = \begin{cases} \dfrac{\kappa^2}{4}, & 0 \leqslant \kappa < \dfrac{1}{2}, \\ \dfrac{\kappa}{8}, & \kappa \geqslant \dfrac{1}{2}. \end{cases}$$

Moreover, define $\mathcal{F} \colon \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ by

$$\mathcal{F}(\mathfrak{a}, 1) = \begin{cases} (0, 0), & 0 \leqslant \mathfrak{a} < 1, \\ (\dfrac{2}{3}, 0), & \mathfrak{a} = 1, \end{cases}$$

for $(\mathfrak{a}, 1) \in \mathcal{A}$ and

$$\mathcal{F}(\mathfrak{b}, 0) = \begin{cases} (0, 1), & 0 \leqslant \mathfrak{b} < 1, \\ (\frac{2}{3}, 1), & \mathfrak{b} = 1, \end{cases}$$

for $(\mathfrak{b}, 0) \in \mathcal{B}$. Note that for $(1, 1), (\frac{1}{2}, 1) \in \mathbb{R}^2$, we have

$$\mathsf{d}\big((1, 1), (\frac{1}{2}, 1)\big) = \frac{1}{2}$$

and, again, by (2), we have

$$\begin{aligned} \psi\Big(\mathsf{d}\big(\mathcal{F}(1, 1), \mathcal{F}^2(\frac{1}{2}, 1)\big)\Big) &= \frac{\sqrt{13}}{6} > \frac{3\sqrt{2} - 1}{8} \\ &= \psi\Big(\mathsf{d}\big((1, 1), \mathcal{F}(\frac{1}{2}, 1)\big)\Big) - \varphi\Big(\mathsf{d}\big((1, 1), \mathcal{F}(\frac{1}{2}, 1)\big)\Big) + \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big). \end{aligned}$$

Consequently, (2) is not true for the mapping $\mathcal{F}$ when we take a usual metric on $\mathcal{A}$. Now, take a graph $\mathcal{G}$ by $\mathsf{V}(\mathcal{G}) = \mathbb{R}^2$ and

$$\begin{aligned} \mathsf{E}(\mathcal{G}) = \Big\{\big((\mathfrak{a}_1, \mathfrak{a}_2), (\mathfrak{a}_1, \mathfrak{a}_2)\big) : (\mathfrak{a}_1, \mathfrak{a}_2) \in \mathbb{R}^2\Big\} \cup \Big\{\big((0, 1), (1, 1)\big), \big((1, 1), (0, 1)\big), \\ \big((0, 0), (1, 0)\big), \big((1, 0), (0, 0)\big)\Big\}. \end{aligned}$$

Then $(\mathbb{R}^2, \mathsf{d})$ is a complete GMS endowed by $\mathcal{G}$. Evidently, $\mathcal{F}$ is orbitally $\mathcal{G}$-continuous. Also, it is clear for $\mathfrak{a}, \mathfrak{b} \in [0, 1]$ that

$$\begin{aligned} \psi\Big(\mathsf{d}\big(\mathcal{F}(\mathfrak{a}, 1), \mathcal{F}^2(\mathfrak{a}, 1)\big)\Big) \leqslant \psi\Big(\mathsf{d}\big((\mathfrak{a}, 1), \mathcal{F}(\mathfrak{a}, 1)\big)\Big) - \varphi\Big(\mathsf{d}\big((\mathfrak{a}, 1), \mathcal{F}(\mathfrak{a}, 1)\big)\Big) \\ + \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big) \end{aligned}$$

and

$$\begin{aligned} \psi\Big(\mathsf{d}\big(\mathcal{F}(\mathfrak{b}, 0), \mathcal{F}^2(\mathfrak{b}, 0)\big)\Big) \leqslant \psi\Big(\mathsf{d}\big((\mathfrak{b}, 0), \mathcal{F}(\mathfrak{b}, 0)\big)\Big) - \varphi\Big(\mathsf{d}\big((\mathfrak{b}, 0), \mathcal{F}(\mathfrak{b}, 0)\big)\Big) \\ + \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big). \end{aligned}$$

Moreover,

$$\begin{aligned} \psi\Big(\mathsf{d}\big(\mathcal{F}(0, 1), \mathcal{F}^2(1, 1)\big)\Big) \leqslant \psi\Big(\mathsf{d}\big((0, 1), \mathcal{F}(1, 1)\big)\Big) - \varphi\Big(\mathsf{d}\big((0, 1), \mathcal{F}(1, 1)\big)\Big) \\ + \varphi\big(\mathsf{dist}(\mathcal{A}, \mathcal{B})\big) \end{aligned}$$

and

$$\psi\Big(\mathsf{d}\big(\mathcal{F}(0,0),\mathcal{F}^2(1,0)\big)\Big) \leqslant \psi\Big(\mathsf{d}\big((0,0),\mathcal{F}(1,0)\big)\Big) - \varphi\Big(\mathsf{d}\big((0,0),\mathcal{F}(1,0)\big)\Big) + \varphi\big(\mathsf{dist}(\mathcal{A},\mathcal{B})\big).$$

Thus, (2) is valid for the mapping $\mathcal{F}$ on $\mathcal{A}$ (and $\mathcal{B}$). Therefore, all assumptions of Theorem 10 are fulfilled and $\mathcal{F}$ has a BPP, being $\vartheta = (0,1)$ and $\gamma = (0,0)$.

At the end, it is worth recalling that all Corollaries 3–6 also hold for Theorems 8 and 10 with a similar statement, regarding special types of graph $\mathcal{G}_j$ with $j = 0,1,2,3$.

4. Application. Now, we can state several applications on the integral-type of the cyclic $\mathcal{G}$-(φ–ψ)-weak contractions. Take μ the Lebesgue measure on a Borel σ-algebra of a metric subspace $\mathbb{R}^{\geqslant 0}$, $L = [a_1, a_2]$ a Borel set, and $\int_{a_1}^{a_2} \phi(x)dx$ the Lebesgue integral of a function ϕ on L. Additionally, assume Γ is a total class of $\phi\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ provided that the followings holds:

(Γ1) ϕ is Lebesgue-integrable on $\mathbb{R}^{\geqslant 0}$;

(Γ2) $\int_0^{\upsilon} \phi(x)dx > 0$ and it is finite for each $\upsilon > 0$.

Theorem 11. *Assume $\mathcal{A},\mathcal{B} \neq \varnothing$ are subsets of a GMS $(\mathcal{X},\mathsf{d})$, $\mathcal{A}$ is complete, both $(\mathcal{A},\mathcal{B})$ and $(\mathcal{B},\mathcal{A})$ have the UC property, $\mathcal{F}\colon \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$ is a mapping, such that*

$$\int_0^{\mathsf{d}(\mathcal{F}\mathfrak{a},\mathcal{F}^2\mathfrak{b})} \chi(x)dx \leqslant \int_0^{\mathsf{d}(\mathfrak{a},\mathcal{F}\mathfrak{b})} \chi(x)dx - \int_0^{\mathsf{d}(\mathfrak{a},\mathcal{F}\mathfrak{b})} \varpi(x)dx + \int_0^{\mathsf{d}(\mathcal{A},\mathcal{B})} \varpi(x)dx \qquad (7)$$

for all $\mathfrak{a},\mathfrak{b} \in \mathcal{A}$ (and $\mathfrak{a},\mathfrak{b} \in \mathcal{B}$) with $(\mathfrak{a},\mathfrak{b}) \in \mathsf{E}(\mathcal{G})$, where $\chi,\varpi \in \Gamma$, and $\mathcal{F}$ and $\mathcal{F}^2$ keep the edges of $\mathcal{G}$ on $\mathcal{A}$. If $\mathcal{F}$ is an orbitally $\mathcal{G}$-continuous mapping on $\mathcal{A}$ or $\mathcal{G}$ is a $\mathbf{C}$-graph on $\mathcal{A}$, then $\mathcal{F}$ has a BPP $p^\star \in \mathcal{A}$ whenever $\mathfrak{a}_0 \in \mathcal{A}$ with $\mathfrak{a}_0 \in \mathcal{C}_{\mathcal{F}}$.

Proof. Setting $\psi(\kappa) = \int_0^{\kappa} \chi(x)dx$ and $\varphi(\kappa) = \int_0^{\kappa} \varpi(x)dx$ in Theorem 10, we obtain the assertion from Theorem 10. □

In Theorem 11, assume that $\mathcal{A} = \mathcal{B} = \mathcal{X}$. Then we reach the following FP result.

Theorem 12. *Assume* $(\mathcal{X}, \mathsf{d})$ *is a complete GMS and* $\mathcal{F}\colon \mathcal{X} \to \mathcal{X}$ *is a mapping, such that*

$$\int_0^{\mathsf{d}(\mathcal{F}\mathfrak{a}, \mathcal{F}^2\mathfrak{b})} \chi(x)dx \leqslant \int_0^{\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{b})} \chi(x)dx - \int_0^{\mathsf{d}(\mathfrak{a}, \mathcal{F}\mathfrak{b})} \varpi(x)dx \tag{8}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ *with* $(\mathfrak{a}, \mathfrak{b}) \in \mathsf{E}(\mathcal{G})$, *where* $\chi, \varpi \in \Gamma$ *and* $\mathcal{F}$ *and* $\mathcal{F}^2$ *keep the edges of* $\mathcal{G}$ *on* $\mathcal{X}$. *If* $\mathcal{F}$ *is an orbitally* $\mathcal{G}$*-continuous mapping on* $\mathcal{X}$ *or* $\mathcal{G}$ *is a* **C**-*graph on* $\mathcal{X}$, *then* $\mathcal{F}$ *has a FP* $p^\star \in \mathcal{X}$ *whenever* $\mathfrak{a}_0 \in \mathcal{X}$ *with* $\mathfrak{a}_0 \in \mathcal{C}_{\mathcal{F}}$.

Remark 1. *Substituting* $\mathcal{G}$ *in Theorems 11 and 12 with the special graphs mentioned in the previous section, we are able to show the validity of the assertions for these types of the graphs.*

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