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ON CALCULATING SUMS OF SOME DOUBLE SERIES

Abstract. We consider systems of two equations consisting of a polynomial and an entire function. By calculating the resultant of the polynomial and the entire function in two different ways, we can obtain a relation for double numerical series. The formula of A. M. Kytmanov and E. K. Myshkina was used as the first method for calculating the resultant. For the second method, we chose the formula for product of values of one function in the roots of another. A family of sums of some types of double numerical series absent in known references was found. We also demonstrate an approach to finding sums of lower dimension (one-dimensional sums) that arise when calculating the resultant of the original system of functions.

Key words: *sum of double numerical series, resultant, entire function*

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1. Introduction. The purpose of this work is to calculate the sums of double numerical series that, to our knowledge, are absent in literature.

In works [7], [11], [16], power sums of roots of systems of non-algebraic equations consisting of entire or meromorphic functions of finite order of growth are investigated. These power sums are associated with certain residue integrals that are not Grothendieck residues. As an application, the sums of some multiple series were found. Using formulas for finding power sums of negative powers of roots of systems of equations in [4], examples of calculating some multiple series were considered in [4].

In this article, we use a different method to obtain relations for sums of double numerical series. We will use the formulas for the resultant from [12] and [13]. Using the concept of a resultant, in [6] we proposed an approach to calculating sums of some types of multiple numerical series. By calculating the resultant of a polynomial and an entire function in two different ways, it is possible to obtain a relation for multiple numerical series in [6]. As the second method for finding the resultant, we choose the formula for calculating the product of the values of one function at the roots of another.

We apply this approach to systems of two equations; the function of one of them is a polynomial with zero constant term. In [6] it was assumed that the constant term of the polynomial is non-zero. Another difference between this article and [6] is that we construct a new system of equations to find sums of smaller dimensions obtained by calculating the resultant of a system of entire functions, and, also, calculate its resultant.

Note that using the concept of the resultant of a system, one can study the roots of systems of transcendental equations [10].

The relevance of this article is given by the fact that in applied problems, for example, in the equations of chemical kinetics, there are functions and systems of equations consisting of exponential polynomials [2].

2. Resultant of a polynomial and an entire function. For the given polynomials f and g, the classical resultant $R(f,g)$ can be defined in various ways using Sylvester's determinant [1], the Bezout – Caley method [5], or formulas for the product [1]

$$R(f,g) = \prod_{\{x\colon\, f(x)=0\}} g(x)\,.$$

The first step in finding the resultant of two entire functions was the work [14], where the case when one of the functions is entire and another one is a polynomial (or an entire function with a finite number of zeros) was considered. For entire functions, the question of localization of real zeros was considered in the classical works of N. G. Chebotarev [3, p. 28-56], as well as in [9] (we refer to the collected works of N. G. Chebotarev, since his original works are inaccessible). In [8], the results of [14] are generalized for the case when one of the entire functions satisfies certain severe constraints but may have infinite number of zeros.

The work [13] presents a construction of the resultant of two entire functions in the complex plane. The advantage of this formula is that it makes possible to answer the question whether or not entire functions have common zeros without calculating the zeros themselves. In [12], the results of [13] are generalized.

Consider the system of equations consisting of a polynomial $f(z)$ of degree m and an entire function $g(z)$:

$$\begin{cases} f(z) = a_0 + a_1 z + \ldots + a_{m-1} z^{m-1} + z^m, \\ g(z) = b_0 + b_1 z + \ldots + b_n z^n + \ldots . \end{cases} \tag{1}$$

Theorem 1. [13] *The resultant $R(f,g)$ of the system of functions of form (1), where $m = 2$, is calculated by the formula*

$$R(f,g) = \sum_{k=0}^{\infty} b_k^2 a_0^k + \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} b_t b_s a_0^t S_{s-t}, \tag{2}$$

where the power sums S_j of the roots of the polynomial $f(z)$ are computed by the recurrent Newton formulas.

Results for higher-order values of m are given in [12] and [15].

Theorem 2. [12] *The resultant $R(f,g)$ of the system of functions of form (1), where $m = 3$, is calculated by the formula*

$$\begin{aligned} R(f,g) = {} & \sum_{k=0}^{\infty} b_k^3 (-a_0)^k + \\ & + \sum_{s=0}^{\infty} \sum_{t=s+1}^{\infty} (-a_0)^s \left(\frac{1}{2} b_s b_t^2 \left(S_{t-s}^2 - S_{2t-2s} \right) + b_s^2 b_t S_{t-s} \right) + \\ & + \sum_{s=0}^{\infty} \sum_{t=s+1}^{\infty} \sum_{p=t+1}^{\infty} b_s b_t b_p (-a_0)^s \left[S_{t-s} \cdot S_{p-s} - S_{t+p-2s} \right]. \end{aligned} \tag{3}$$

Note that the series on the right-hand sides of formulas (2) and (3) converge absolutely due to the fact that b_k are the coefficients of the expansion of an entire function.

3. The main result. These methods lead to the calculation of the sums of some double numerical series that are absent in well-known references. The difference between this article and [6] is that here we consider the case when the coefficient a_0 in the expansion (1) of the function $f(z)$ is equal to zero. Also, a new idea (in contrast to [6]) for finding sums of double numerical series is that to find sums of a smaller dimension (one-dimensional), obtained by calculating the resultant of the original system of entire functions, a new system of equations is constructed and its resultant is also calculated. Let us demonstrate our approach using the following example.

Consider the system of equations

$$\begin{cases} f(z) = (z^2 - 4)\, z = z^3 - 4z, \\ g(z) = e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots . \end{cases}$$

Calculating the resultant $R(f, g)$, using Theorem 2 on the one hand (taking into account that $a_0 = 0$), and using the definition of the resultant in the form of a formula for the product on the other hand, we obtain the relation:

$$b_0^3 + \sum_{t=1}^{\infty} \left(\frac{1}{2} b_0 b_t^2 \left(S_t^2 - S_{2t} \right) + b_0^2 b_t S_t \right) + \sum_{t=1}^{\infty} \sum_{p=t+1}^{\infty} b_0 b_t b_p \left[S_t \cdot S_p - S_{t+p} \right] =$$
$$= e^{-2} \cdot e^2 \cdot e^0.$$

Since S_j are the power sum of the roots of the polynomial $f(z)$, we have

$$S_t = 2^t + (-2)^t, \quad S_p = 2^p + (-2)^p,$$
$$S_t \cdot S_p - S_{t+p} = 2^{t+p} + 2^{t+p} (-1)^p + 2^{t+p} (-1)^t + 2^{t+p} (-1)^{t+p} -$$
$$- 2^{t+p} - (-2)^{t+p} = 2^{t+p} (-1)^p + 2^{t+p} (-1)^t =$$
$$= 2^{t+p} \left((-1)^p + (-1)^t \right).$$

In particular, $S_t^2 - S_{2t} = 2^{2t+1} (-1)^t$. Thus, we have

$$\sum_{t=1}^{\infty} \left(\frac{1}{2} \left(\frac{1}{t!} \right)^2 2^{2t+1} (-1)^t + \frac{1}{t!} \left(2^t + (-2)^t \right) \right) +$$
$$+ \sum_{t=1}^{\infty} \sum_{p=t+1}^{\infty} \frac{1}{t!} \cdot \frac{1}{p!} 2^{t+p} \left((-1)^p + (-1)^t \right) = 0,$$

or

$$\sum_{t=1}^{\infty} \sum_{p=t+1}^{\infty} \frac{1}{t!} \cdot \frac{1}{p!} 2^{t+p} \left((-1)^p + (-1)^t \right) = - \sum_{t=1}^{\infty} \frac{(-1)^t \, 2^{2t}}{(t!)^2} - \sum_{t=1}^{\infty} \frac{2^t}{t!} - \sum_{t=1}^{\infty} \frac{(-2)^t}{t!}. \tag{4}$$

The first sum on the right-hand side of (4) is [17, formula 5.2.10.1]: a well-known special function. This is a Bessel function of the first kind, namely $J_0(4) - 1$. The second and third one-dimensional sums on the right-hand side of (4) are also well-known and their values are

$$\sum_{t=1}^{\infty} \frac{2^t}{t!} = e^2 - 1, \quad \sum_{t=1}^{\infty} \frac{(-2)^t}{t!} = \frac{1}{e^2} - 1.$$

Note that these values of the series can be obtained by considering two systems of equations: the first one consisting of a polynomial $f_1(z) = (z-2)\,z$ and an entire function $g(z) = e^z$, and the second one consisting of $f_2(z) = (z+2)\,z$ and $g(z) = e^z$, respectively. The idea is to calculate the resultant of new systems of equations and use the definition of the resultant as a product. This technique can be useful in finding sums of double series, considered in [12, example 3], the calculation of which currently is an open problem.

Thus, returning to (4), we obtain a statement about the sum of a double numerical series.

Example 1. The following holds:

$$\sum_{t=1}^{\infty}\sum_{p=t+1}^{\infty}\frac{2^{t+p}\left((-1)^t+(-1)^p\right)}{t!\,p!} = -J_0(4) - e^2 - \frac{1}{e^2} + 3. \tag{5}$$

4. Some generalizations of the obtained formulas. The obtained formulas for the sum of a double numerical series can be generalized. In order to do this, it is sufficient to consider a system of functions of $f(z)$ and $g(z)$ with two non-zero roots $z_1 = a$ and $z_2 = b$ of the function $f(z)$ and its root $z_3 = 0$:

$$\begin{cases} f(z) = z(z-a)(z-b) = z^3 - (a+b)\,z^2 + abz, \\ g(z) = e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots. \end{cases}$$

Calculating the resultant $R(f,g)$, using Theorem 2 on the one hand (taking into account that $a_0 = 0$), and using the definition of the resultant as a product on the other hand, we obtain the relation

$$\sum_{t=1}^{\infty}\left(\frac{1}{2}\left(\frac{1}{t!}\right)^2\left(S_t^2 - S_{2t}\right) + \frac{1}{t!}S_t\right) + \sum_{t=1}^{\infty}\sum_{p=t+1}^{\infty}\frac{1}{t!}\cdot\frac{1}{p!}\left[S_t\cdot S_p - S_{t+p}\right] = e^a e^b - 1.$$

As before, S_j are the power sum of the roots of the polynomial $f(z)$, that is

$$S_t = a^t + b^t, \quad S_p = a^p + b^p,$$
$$S_t\cdot S_p - S_{t+p} = a^{t+p} + a^t b^p + a^p b^t + b^{t+p} - a^{t+p} - b^{t+p} = a^t b^p + a^p b^t,$$

in particular, $S_t^2 - S_{2t} = 2a^t b^t$. Thus, we have

$$\sum_{t=1}^{\infty}\left(\frac{a^t b^t}{(t!)^2}+\frac{a^t+b^t}{t!}\right)+\sum_{t=1}^{\infty}\sum_{p=t+1}^{\infty}\frac{a^t b^p + a^p b^t}{t!\,p!} = e^a e^b - 1,$$

or

$$\sum_{t=1}^{\infty}\sum_{p=t+1}^{\infty}\frac{a^t b^p + a^p b^t}{t!\,p!} = e^a e^b - 1 - \sum_{t=1}^{\infty}\frac{a^t b^t}{(t!)^2} - \sum_{t=1}^{\infty}\frac{a^t}{t!} - \sum_{t=1}^{\infty}\frac{b^t}{t!}. \tag{6}$$

Further on, we assume that a and b are generally different (but not excluding their coincidence). Based on [17, formula 5.2.10.1], we obtain the value of the series $\sum_{t=1}^{\infty}\frac{a^t b^t}{(t!)^2}$ that is equal to $J_0\left(2\sqrt{a\,|b|}\right)-1$ for $a>0$, $b<0$; is equal to $J_0\left(2\sqrt{|a|\,b}\right)-1$ for $a<0, b>0$; and, finally, is equal to $I_0\left(2\sqrt{ab}\right)-1$ in other cases. Here $I_0\left(2\sqrt{ab}\right)$ is the modified Bessel function of the first kind (the Bessel function of the imaginary argument). It is known that

$$\sum_{t=1}^{\infty}\frac{a^t}{t!}+\sum_{t=1}^{\infty}\frac{b^t}{t!} = e^a - 1 + e^b - 1.$$

Thus, returning to (6), we can find the sum of the double numerical series as follows:

Proposition 1. *The following holds:*

$$\sum_{t=1}^{\infty}\sum_{p=t+1}^{\infty}\frac{a^t b^p + a^p b^t}{t!\,p!} =$$

$$= \begin{cases} e^{a+b} - I_0\left(2\sqrt{ab}\right) - e^a - e^b + 2, & a>0,\ b>0,\\ e^{a+b} - I_0\left(2\sqrt{ab}\right) - \frac{1}{e^{|a|}} - \frac{1}{e^{|b|}} + 2, & a<0,\ b<0,\\ e^{a+b} - J_0\left(2\sqrt{a\,|b|}\right) - e^a - \frac{1}{e^{|b|}} + 2, & a>0,\ b<0,\\ e^{a+b} - J_0\left(2\sqrt{|a|\,b}\right) - \frac{1}{e^{|a|}} - e^b + 2, & a<0,\ b>0. \end{cases}$$

In particular, if $b=a$, then

$$\sum_{t=1}^{\infty}\sum_{p=t+1}^{\infty}\frac{2a^{t+p}}{t!\,p!} = \begin{cases} e^{2a} - I_0(2a) - 2e^a + 2, & a>0,\\ e^{2a} - I_0(2\,|a|) - \frac{2}{e^{|a|}} + 2, & a<0; \end{cases}$$

if $b = -a$*, then*

$$\sum_{t=1}^{\infty}\sum_{p=t+1}^{\infty}\frac{a^{t+p}\left((-1)^t+(-1)^p\right)}{t!\,p!}=\begin{cases}-J_0\left(2a\right)-e^a-\frac{1}{e^a}+3, & a>0,\\ -J_0\left(2\left|a\right|\right)-\frac{1}{e^{|a|}}-e^{-a}+3, & a<0.\end{cases}$$

Note that for $a = 2$, from the last relation we obtain equality (5).

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