Probl. Anal. Issues Anal. Vol. 14 (32), No 2, 2025, pp. 103–119

DOI: 10.15393/j3.art.2025.17170

UDC 517.98

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$\mathcal{I}^{\mathcal{K}}$-SEQUENTIAL AND $\mathcal{I}^{\mathcal{K}}$-FRÉCHET-URYSOHN SPACES

Abstract. Notions of $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn and $\mathcal{I}^{\mathcal{K}}$-sequential spaces are studied by letting ideals $\mathcal{I}$, $\mathcal{K}$ of subsets of natural numbers to play measurable role in the well-established concepts of Fréchet-Urysohn and sequential spaces. Relation among those spaces in new and old setting have been established by introducing $\mathcal{I}^{\mathcal{K}}$-quotient maps and $\mathcal{I}^{\mathcal{K}}$-covering maps.

Key words: *$\mathcal{I}^{\mathcal{K}}$-quotient map, $\mathcal{I}^{\mathcal{K}}$-covering map, $\mathcal{I}^{\mathcal{K}}$-sequential space, $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space*

2020 Mathematical Subject Classification: *40A35, 54C05, 54D55*

1. Introduction. In 1973, J. R. Boone and F. Siwiec [1] introduced the concept of sequentially quotient maps, which are the convergent sequence analogs of the bi-quotient maps of Michael [18]. The notions of sequential spaces and sequentially open subsets of a space were introduced by Franklin [8]. In [17], the notions of statistically Fréchet-Urysohn and statistically sequential spaces have been defined and studied in detail in [25]. Statistical convergence introduced by H. Fast [6] is an extension of the concept of convergence of sequence of real numbers. During last four decades, many mathematicians explored and generalized that concepts in various directions ([3], [13], [17], [22], etc.). Two interesting generalizations of statistical convergence are $\mathcal{I}$ and $\mathcal{I}^*$-convergence [12]. After a long time, in the year 2011, M. Macaj and M. Sleziak introduced the concept of $\mathcal{I}^{\mathcal{K}}$-convergence, as a generalization of $\mathcal{I}^*$-convergence. In 2022, C. Choudhury and S. Debnath [2] defined the notions of $\mathcal{I}^{\mathcal{K}}$-supremum, $\mathcal{I}^{\mathcal{K}}$-infimum, $\mathcal{I}^{\mathcal{K}}$-limit superior and $\mathcal{I}^{\mathcal{K}}$-limit inferior and studied their relations. Recently, the concept of $\mathcal{I}^{\mathcal{K}}$-convergence of sequences of bi-complex numbers was introduced and explored its properties [11]. Several properties of $\mathcal{I}^{\mathcal{K}}$-convergence of functions have been studied in [4], [20], [21].

Here are some basic definitions and findings provided as a ready references that will be used in the sequel.

An ideal $\mathcal{I}$ on an arbitrary set X is a family $\mathcal{I} \subset 2^X$ that is closed under finite unions and taking subsets [14]. An ideal $\mathcal{I}$ is called trivial if $\mathcal{I} = \varnothing$ or X in $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset 2^X$ is called admissible if it contains all the singleton sets [14]. The class of all finite subsets of $\mathbb{N}$ is an admissible ideal on $\mathbb{N}$, denoted by Fin.

Various examples of non-trivial admissible ideals are given in [12]. Suppose $\mathcal{I}$, $\mathcal{K}$ are ideals on $\mathbb{N}$. A sequence (x_n) in a topological space X is said to be $\mathcal{I}$-convergent to l in X if for any open set U containing l, $\{n \in \mathbb{N} \colon x_n \notin U\} \in \mathcal{I}$ [15]. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}^*$-convergent to $l \in X$ if there exists a set $M \in \mathcal{F}(\mathcal{I})$, such that the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_n = x_n$, $n \in M$, and $y_n = l$, $n \in \mathbb{N}\setminus M$ is Fin-convergent to l. In addition, $\mathcal{I}^{\mathcal{K}}$-convergence is defined by replacing Fin by an arbitrary ideal $\mathcal{K}$ on $\mathbb{N}$.

Let us recall the notion of sequential spaces. A subset C of a topological space X is called sequentially closed if no sequence in C converges to a point in $X\setminus C$. A topological space X is said to be sequential if each sequentially closed subset of X is closed [8]. Every first countable space is a sequential space. Suppose X, Y are topological spaces and $f : X \to Y$ is an onto map; f is called a quotient map provided a subset U is open in Y if and only if $f^{-1}(U)$ is open in X, and f is called a sequentially quotient map provided a subset U is sequentially open in Y if and only if $f^{-1}(U)$ is sequentially open in X [1]. f is said to be sequence-covering if whenever (y_i) is a sequence in Y converging to some point l in Y, there exists a sequence (x_i) of points $x_i \in f^{-1}(y_i)$ for all $i \in \mathbb{N}$ and $p \in f^{-1}(l)$, such that (x_n) converges to p [1]. Every sequence-covering mapping is sequentially quotient. A topological space X is said to be Fréchet-Urysohn if for each subset C of X and $x \in \bar{C}$, there exists a sequence in C converging to x [8]. Every Fréchet-Urysohn space is sequential, but the reverse implication may not hold [8].

Before entering into the main discussion, let us take a look at some of the ones that will be followed throughout the article:

- A sequence is a mapping whose domain is a cofinal subset of $\mathbb{N}$. Let $x = (x_n)_{n\in L}$ be a sequence in a topological space X and M be a cofinal subset of L. Then call $(x_n)_{n\in M}$ a subsequence of $x = (x_n)_{n\in L}$.
- Nonthin subsets of natural numbers were introduced by J. A. Fridy [9] in terms of natural density [10]. Inspired by the notion of nonthin

sets, $\mathcal{I}$-nonthin subsets of natural numbers are defined in [23]. A sequence $(x_n)_{n\in A}$ in X is said to be $\mathcal{I}$-thin if $A \in \mathcal{I}$, otherwise it is called $\mathcal{I}$-nonthin, where $A \subset \mathbb{N}$ and $\mathcal{I}$ is an ideal on $\mathbb{N}$ [23].

- For $M \subset \mathbb{N}$, $\mathcal{I}|_M = \{A \cap M; A \in \mathcal{I}\}$ is an ideal on M [16]. $\mathcal{I}|_M$ is nontrivial if $M \notin \mathcal{I}$.
- $\mathcal{I}$, $\mathcal{K}$ stand for nontrivial admissible ideal on $\mathbb{N}$, unless otherwise stated.
- all mappings are onto.

2. Main Results. In this section, the notion of $\mathcal{I}^{\mathcal{K}}$-sequential space is introduced, and we show that $\mathcal{I}^{\mathcal{K}}$-sequential space may not be sequential.

Definition 1. *The $\mathcal{I}^{\mathcal{K}}$-closure of a subset C of a topological space X is denoted by $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = \{x \in X\colon$ there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in A}$ in X, such that $(\mathcal{I}|_A)^{\mathcal{K}}$-converges to $x\}$.*

Theorem 1. *Let $\mathcal{K} \subset \mathcal{I}$. For any subset C of a topological space X, $C \subset \overline{C}^{\mathcal{I}^{\mathcal{K}}} \subset \bar{C}$, where $\bar{C}$ is the closure of C. Furthermore, if X is first countable, $\bar{C} = \overline{C}^{\mathcal{I}^{\mathcal{K}}}$.*

Proof. Let $x \in \overline{C}^{\mathcal{I}^{\mathcal{K}}}$. Then there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in A}$ in C, which $(\mathcal{I}|_A)^{\mathcal{K}}$-converges to x. Therefore, there exists $M \in \mathcal{F}(\mathcal{I}|_A)$, such that the sequence $(y_n)_{n\in A}$ given by $y_n = x_n$ if $n \in M$ and $y_n = x$ if $n \in A\backslash M$ is $\mathcal{K}$-convergent to x. For any open set U containing x, $\{n \in A\colon y_n \in U\} \in \mathcal{F}(\mathcal{K}|_A)$. Since $\mathcal{K} \subset \mathcal{I}$, the set $\{n \in A\colon y_n \in U\} \in \mathcal{F}(\mathcal{I}|_A)$ and, so, $\{n \in A\colon x_n \in U\} \in \mathcal{F}(\mathcal{I}|_A)$. Therefore, there is $p \in A$, such that $p \in \{n \in A\colon x_n \in U\}$. Then $x_p \in C \cap U$ and, hence, $x \in \bar{C}$. Suppose X is first countable and $x \in \bar{C}$. Then there exists a sequence (x_n) in C, such that (x_n) is convergent to x. Since $\mathcal{I}$ and $\mathcal{K}$ are admissible ideals on $\mathbb{N}$, (x_n) is $\mathcal{K}$-convergent and, so, (x_n) $\mathcal{I}^{\mathcal{K}}$-converges to x. Thus, $x \in \overline{C}^{\mathcal{I}^{\mathcal{K}}}$. □

Definition 2. *A subset C of a topological space X is called $\mathcal{I}^{\mathcal{K}}$-closed if $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = C$.*

Theorem 2. *For any subset H of a topological space X, the following are equivalent:*

(a) H is $\mathcal{I}^{\mathcal{K}}$-open.

(b) for any $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in X with $(x_n)_{n\in L}$, $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in H$, $\{n \in L\colon x_n \in H\} \notin \mathcal{K}$.

(c) *for any $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in X with $(x_n)_{n\in L}i$, $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in H$, $|\{n \in L\colon x_n \in H\}| = \omega$.*

Proof. $(a) \implies (b)$ Suppose H is $\mathcal{I}^{\mathcal{K}}$-open and $(x_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in X, such that $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in H$. If possible, let $M = \{n \in L\colon x_n \in H\} \in \mathcal{K}$. Then $M \neq L$ and, so, $X \neq H$. Let $p \in X\backslash H$. Define a sequence $(y_n)_{n\in L}$ in X given by $y_n = p$, $n \in M$, and $y_n = x_n$, $n \notin L\backslash M$. Clearly, $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x. Since $X\backslash H$ is $\mathcal{I}^{\mathcal{K}}$-closed and $(y_n)_{n\in L}$ is a sequence in $X\backslash H$, $x \in X\backslash H$, which leads to a contradiction. Hence, $\{n \in L\colon x_n \in H\} \notin \mathcal{K}$.
$(b) \implies (c)$ It is obvious, as the ideal $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$.
$(c) \implies (a)$ Suppose $X\backslash H$ is not $\mathcal{I}^{\mathcal{K}}$-closed in X. Then there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in $X\backslash H$, such that $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to some point $x \in H$. So, $\{n \in L\colon x_n \in H\}$ is an empty set, which leads to a contradiction. Hence, H is $\mathcal{I}^{\mathcal{K}}$-open in X. □

Definition 3. *A topological space is said to be $\mathcal{I}^{\mathcal{K}}$-sequential if every $\mathcal{I}^{\mathcal{K}}$-closed set is closed.*

Suppose C is an $\mathcal{I}^{\mathcal{K}}$-closed subset of a topological space X. If (x_n) is a sequence in C, such that (x_n) converges to $x \in X$, then (x_n) $\mathcal{K}|_{\mathbb{N}}$-converges to x. Since C is $\mathcal{I}^{\mathcal{K}}$-closed, $x \in C$. Thus, C is sequentially closed. If X is a sequential space, then C is closed. So, every sequential space is $\mathcal{I}^{\mathcal{K}}$-sequential. But the converse may not be true.

Theorem 3. *Every sequential space is $\mathcal{I}^{\mathcal{K}}$-sequential.*

Example 1. Suppose $\mathcal{I}$ is an ideal on $\mathbb{N}$ and $\mathcal{K}$ is a maximal ideal on $\mathbb{N}$. Consider the space $\sum(\mathcal{K})$ defined in [26, Example 2.7] as follows:
Take the set $Y = \mathbb{N} \cup \{\infty\}$, $\infty \notin \mathbb{N}$. A topology on Y consists of each $\{n\}$ and sets G containing ∞ of the form $G = \{\infty\} \cup (\mathbb{N}\backslash A)$, where $A \in \mathcal{K}$. Denote the set Y equipped with this topology by $\sum(\mathcal{K})$. Suppose G is an $\mathcal{I}^{\mathcal{K}}$-open subset of $\sum(\mathcal{K})$. Let us assume that $\infty \in G$. Consider the sequence (n), which $\mathcal{I}^{\mathcal{K}}$-converges to ∞ in $\sum(\mathcal{K})$. By Theorem 2, it follows that $\{n \in \mathbb{N} : n \in G\} \notin \mathcal{K}$. Therefore, $G\backslash\{\infty\} \notin \mathcal{K}$. Since $\mathcal{K}$ is a maximal ideal of $\mathbb{N}$, $\mathbb{N}\backslash G \in \mathcal{K}$. Therefore, $G = \{\infty\} \cup (\mathbb{N}\backslash(\mathbb{N}\backslash G))$ is open in $\sum(\mathcal{K})$. Hence, $\sum(\mathcal{K})$ is an $\mathcal{I}^{\mathcal{K}}$-sequential space. Moreover, $\sum(\mathcal{K})$ is a Hausdorff space, but not a k-space [26, Example 2.9]. Again, since every sequential space is a k-space [19], $\sum(\mathcal{K})$ is not a sequential space.

Definition 4. *Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be $\mathcal{I}^{\mathcal{K}}$-continuous if for every $\mathcal{I}$-nonthin sequence $(x_i)_{i\in P}$ in Y,* *which is $(\mathcal{I}|_P)^{\mathcal{K}}$-convergent to x, $(f(x_i))_{i\in P}$ $(\mathcal{I}|_P)^{\mathcal{K}}$-converges to $f(x)$.*

Theorem 4. *Let X and Y be topological spaces. A function $f\colon X \to Y$ is $\mathcal{I}^{\mathcal{K}}$-continuous if and only if $f^{-1}(B)$ is $\mathcal{I}^{\mathcal{K}}$-closed for every $\mathcal{I}^{\mathcal{K}}$-closed subset B of Y.*

Proof. Suppose f is an $\mathcal{I}^{\mathcal{K}}$-continuous function and B is an $\mathcal{I}^{\mathcal{K}}$-closed subset of Y. Let $x \in \overline{f^{-1}(B)}^{\mathcal{I}^{\mathcal{K}}}$. Then there is an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in $f^{-1}(B)$, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in X$. So $(f(x_n))_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $f(x)$. Since B is $\mathcal{I}^{\mathcal{K}}$-closed, $x \in f^{-1}(B)$.
Conversely, for every $\mathcal{I}^{\mathcal{K}}$-closed subset B of Y, $f^{-1}(B)$ is an $\mathcal{I}^{\mathcal{K}}$-closed subset of X. Suppose f is not $\mathcal{I}^{\mathcal{K}}$-continuous. Then there is an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in M}$ in X, which $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to $x \in X$, but $(f(x_n))_{n\in M}$ does not $(\mathcal{I}|_M)^{\mathcal{K}}$-converge to $f(x)$. For all $T \in \mathcal{F}(\mathcal{I}|_M)$, such that the sequence $(y_n)_{n\in M}$ given by $y_n = f(x_n)$, $n \in T$ and $y_n = f(x)$, $n \in M\backslash T$ does not $\mathcal{K}$-converge to $f(x)$. Therefore, there exists an open set U containing $f(x)$, such that $\{n \in M\colon y_n \notin U\} \notin \mathcal{K}|_M$. As $\{n \in M\colon f(x_n) \notin U\} \supset \{n \in M\colon y_n \notin U\}$, $P = \{n \in M\colon f(x_n) \notin U\} \notin \mathcal{K}|_M$. Again, $Y\backslash U$ is $\mathcal{I}^{\mathcal{K}}$-closed in Y, because $Y\backslash U$ is closed in Y. So $f^{-1}(Y\backslash U)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. Since $(x_n)_{n\in P}$ $(\mathcal{I}|_P)^{\mathcal{K}}$-converges to x, $x \in \overline{f^{-1}(Y\backslash U)}^{\mathcal{I}^{\mathcal{K}}} = f^{-1}(Y\backslash U)$. Therefore, $f(x) \in Y\backslash U$, which is a contradiction. □

Corollary 1. *Suppose Y and Z are topological spaces. The following are equivalent for a function $\phi\colon Y \to Z$:*

(a) *ϕ is $\mathcal{I}^{\mathcal{K}}$-continuous.*

(b) *$\phi^{-1}(F)$ is $\mathcal{I}^{\mathcal{K}}$-closed for every $\mathcal{I}^{\mathcal{K}}$-closed subset F of Z.*

(c) *$\phi^{-1}(G)$ is $\mathcal{I}^{\mathcal{K}}$-open for every $\mathcal{I}^{\mathcal{K}}$-open subset G of Z.*

3. $\mathcal{I}^{\mathcal{K}}$-quotient map and $\mathcal{I}^{\mathcal{K}}$-covering map. In this section, the notion of $\mathcal{I}^{\mathcal{K}}$-quotient map is introduced, which is an extension of $\mathcal{I}^{\mathcal{K}}$-continuous map. Also, the concept of $\mathcal{I}^{\mathcal{K}}$-covering map is defined and relation between $\mathcal{I}^{\mathcal{K}}$-quotient map and $\mathcal{I}^{\mathcal{K}}$-covering map are studied. Suppose $(x_n)_{n\in L}$ is any $\mathcal{I}$-nonthin sequence in a topological space X. $(x_n)_{n\in L}$ is said to be $\mathcal{I}$-eventually constant at x if $\{n \in L\colon x_n \neq x\} \in \mathcal{I}|_L$ [24]. Every eventually constant sequence is $\mathcal{I}$-eventually constant. But the reverse implication may not hold [24].

Definition 5. *A function $f\colon X \to Y$ is said to be $\mathcal{I}^{\mathcal{K}}$-presequential if for any $\mathcal{I}$-nonthin sequence $(y_n)_{n\in M}$ in Y with $(y_n)_{n\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to y*

and $(y_n)_{n\in M}$ *non* $\mathcal{I}$*-eventually constant at* y, $\cup\{f^{-1}(y_n)\colon n \in M, y_n \neq y\}$ *is not* $\mathcal{I}^{\mathcal{K}}$*-closed.*

Definition 6. *A mapping* $\phi\colon X \to Y$ *is said to be* $\mathcal{I}^{\mathcal{K}}$*-quotient provided that a set* G *is* $\mathcal{I}^{\mathcal{K}}$*-closed* ($\mathcal{I}^{\mathcal{K}}$-open) in Y if and only if $\phi^{-1}(G)$ is $\mathcal{I}^{\mathcal{K}}$-closed (resp. $\mathcal{I}^{\mathcal{K}}$-open) in X.

Theorem 5. *Suppose* $\mathcal{I}$, $\mathcal{K}$ *are ideals on* $\mathbb{N}$ *and* $\phi\colon X \to Y$ *is* $\mathcal{I}^{\mathcal{K}}$*-continuous function. Then the following are equivalent:*

(a) ϕ *is* $\mathcal{I}^{\mathcal{K}}$*-presequential.*

(b) *For each non* $\mathcal{I}^{\mathcal{K}}$*-closed subset* C *of* Y, $\phi^{-1}(C)$ *is non* $\mathcal{I}^{\mathcal{K}}$*-closed subset of* X.

(c) *For each non* $\mathcal{I}^{\mathcal{K}}$*-open subset* G *of* Y, $\phi^{-1}(G)$ *is non* $\mathcal{I}^{\mathcal{K}}$*-open subset of* X.

Proof. The condition (b) and (c) are equivalent by considering complement.

For (b) $\implies$ (a), let $\alpha = (\alpha_i)_{i\in M}$ be any $\mathcal{I}$-nonthin sequence in Y, such that $(\alpha_i)_{i\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to ξ and is non $\mathcal{I}$-eventually constant at ξ. If $L = \{i \in M\colon \alpha_i \neq \xi\}$, ξ is not equal to any $(\alpha_i)_{i\in L}$. Again, $\cup\{\phi^{-1}(\alpha_i)\colon i \in M \text{ and } \alpha_i \neq \xi\} = \phi^{-1}(Im\ \alpha\backslash\{\xi\})$. Since $Im\ \alpha\backslash\{\xi\}$ of Y is not $\mathcal{I}^{\mathcal{K}}$-closed, $\cup\{\phi^{-1}(\alpha_i)\colon i \in M \text{ and } \alpha_i \neq \xi\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed.

(a) $\implies$ (b) Suppose ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential and $\mathcal{I}^{\mathcal{K}}$-continuous. Let C be a non $\mathcal{I}^{\mathcal{K}}$-closed subset of Y. Then there exists an $\mathcal{I}$-nonthin sequence $\alpha = (\alpha_i)_{i\in M}$ in C, which is $(\mathcal{I}|_M)^{\mathcal{K}}$ converging to some point ξ in $Y\backslash C$. Therefore, ξ is not equal to any α_i. Since ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential, then the set $G = \phi^{-1}(Im\ \alpha)$ is not $\mathcal{I}^{\mathcal{K}}$-closed. Thus there exists a sequence $(\gamma_i)_{i\in L}$ in G with $L \subset M$, such that $\phi(\gamma_i) = \alpha_i$ for all $i \in L$. So, $(\gamma_i)_{i\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to a point η in $X\backslash G$. Since ϕ is $\mathcal{I}^{\mathcal{K}}$-continuous, so the sequence $(\alpha_i)_{i\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $\phi(\eta) = \xi$, $\eta \notin \phi^{-1}(C)$. As $(\alpha_i)_{i\in L}$ is in G, $(\alpha_i)_{i\in L}$ is in $\phi^{-1}(C)$. Therefore, $\phi^{-1}(C)$ is not an $\mathcal{I}^{\mathcal{K}}$-closed subset in X. □

Corollary 2. *A mapping is* $\mathcal{I}^{\mathcal{K}}$*-quotient if and only if it is both* $\mathcal{I}^{\mathcal{K}}$*-continuous and* $\mathcal{I}^{\mathcal{K}}$*-presequential.*

Definition 7. *A mapping* $\phi\colon X \to Y$ *is said to be* $\mathcal{I}^{\mathcal{K}}$*-covering if for every* $\mathcal{I}$*-nonthin sequence* $(\beta_i)_{i\in M}$ *in* Y *that* $(\mathcal{I}|_M)^{\mathcal{K}}$*-converges to* β *in* Y, *there exists an* $\mathcal{I}$*-nonthin sequence* $(\alpha_i)_{i\in M}$ *with* $\alpha_i \in \phi^{-1}(\beta_i)$, *for* $i \in M$ *and* $\alpha \in \phi^{-1}(\beta)$, *such that* $(\alpha_i)_{i\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$*-converges to* α.

Suppose $\phi\colon X \to Y$ is an $\mathcal{I}^{\mathcal{K}}$-covering mapping. If G is a non $\mathcal{I}^{\mathcal{K}}$-closed subset of Y, then there exists an $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in Y, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to some point say y, $y \notin G$. As ϕ is $\mathcal{I}^{\mathcal{K}}$-covering, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ of points $x_n \in \phi^{-1}(y_n)$ for all $n \in L$ and $x \in \phi^{-1}(y)$, such that $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x. But $x \notin \phi^{-1}(G)$. Therefore, $\phi^{-1}(G)$ is not $\mathcal{I}^{\mathcal{K}}$-closed.

So if ϕ is an $\mathcal{I}^{\mathcal{K}}$-continuous $\mathcal{I}^{\mathcal{K}}$-covering mapping, ϕ satisfies condition (b) of Theorem 5 and, so, ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential. Therefore, ϕ is $\mathcal{I}^{\mathcal{K}}$-quotient.

Theorem 6. *Every $\mathcal{I}^{\mathcal{K}}$-continuous $\mathcal{I}^{\mathcal{K}}$-covering mapping is $\mathcal{I}^{\mathcal{K}}$-quotient.*

Theorem 7. *A one-to-one $\mathcal{I}^{\mathcal{K}}$-quotient mapping is $\mathcal{I}^{\mathcal{K}}$-covering.*

Proof. Suppose $\phi\colon X \to Y$ is an one-to-one $\mathcal{I}^{\mathcal{K}}$-quotient mapping. Let $(\beta_i)_{i\in M}$ be an $\mathcal{I}$-nonthin sequence in Y, which $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to a point $\beta \in Y$. Without loss of generality, let us assume that $(\beta_i)_{i\in M}$ consists of distinct points. Let $\alpha_i = \phi^{-1}(\beta_i)$ and $\alpha = \phi^{-1}(\beta)$. If possible, let $(\alpha_i)_{i\in M}$ be not $(\mathcal{I}|_M)^{\mathcal{K}}$-convergent to α. For any set $P \in \mathcal{F}(\mathcal{I}|_M)$, consider a sequence $(\gamma_i)_{i\in M}$ given by $\gamma_i = \alpha_i$, $i \in P$, and $\gamma_i = \alpha$, $i \in M\backslash P$ is not $\mathcal{K}|_M$-convergent to α. Then there exists an open set W containing α, such that the set $L = \{i \in M\colon \gamma_i \notin W\} \notin \mathcal{K}|_M$. Thus the sequence $(\gamma_i)_{i\in L}$ is not in W. For each $i \in L\backslash P$, $\phi(\gamma_i) = \beta_i$, which shows that $(\phi(\gamma_i))_{i\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β. Since ϕ is $\mathcal{I}^{\mathcal{K}}$-quotient, ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential. Then $\cup\{\phi^{-1}(\beta_i)\colon i \in K \text{ and } \beta_i \neq \beta\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed. So, there exists an $\mathcal{I}$-nonthin sequence $(z_i)_{i\in T}$ in $\cup\{\phi^{-1}(\beta_i)\colon i \in K \text{ and } \beta_i \neq \beta\}$, which $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to some point l in X. There is a set $A \in \mathcal{F}(\mathcal{I}|_T)$, such that the sequence $(u_i)_{i\in T}$ is given by $u_i = z_i$, $i \in A$ and $u_i = l$, $i \in T\backslash A$ $\mathcal{K}|_T$-converges to l. Again, since ϕ is $\mathcal{I}^{\mathcal{K}}$-continuous, then $(\phi(u_i))_{i\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $\phi(l)$. For each $i \in A$, $\phi(u_i) = \beta_i$, and $(\beta_i)_{i\in A}$, $(\mathcal{I}|_A)^{\mathcal{K}}$-converges to β. Therefore, $\phi(l) = \beta$ and $l = \phi^{-1}(\beta) = \alpha$. As W is an open set containing $\alpha = l$, $\{i \in T\colon u_i \in W\} \notin \mathcal{K}|_T$. So, $\{i \in L\colon \alpha_i \in W\} \notin \mathcal{K}|_L$, which leads to a contradiction. Hence, $(\alpha_i)_{i\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to α. □

Corollary 3. *A one-to-one $\mathcal{I}^{\mathcal{K}}$-continuous mapping is $\mathcal{I}^{\mathcal{K}}$-quotient if and only if the mapping is $\mathcal{I}^{\mathcal{K}}$-covering.*

Theorem 8. *For an $\mathcal{I}^{\mathcal{K}}$-continuous mapping $h\colon X \to Y$, the following are equivalent:*

(a) *h is an $\mathcal{I}^{\mathcal{K}}$-quotient map.*

(b) *for each $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in Y, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β say, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_i)_{i\in T}$ with $\alpha_{m_i} \in h^{-1}(y_{n_i})$, such that $(\alpha_i)_{i\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $\alpha \in h^{-1}(\beta)$, where $T = \{m_1 < m_2 < \ldots\}$ and $\{n_1 < n_2 < \ldots\}$ are $\mathcal{I}$-nonthin subsets of L.*

(c) *for each β in the $\mathcal{I}^{\mathcal{K}}$-closure of a subset D of Y, there exists a point $\alpha \in h^{-1}(\beta)$, such that α is in the $\mathcal{I}^{\mathcal{K}}$-closure of $h^{-1}(D)$.*

Proof. $(a) \implies (b)$ Suppose $(y_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in Y, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β. Without loss of generality, let $y_n \neq \beta$ for each $n \in L$. So, $\{y_n : n \in L\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed. As h is $\mathcal{I}^{\mathcal{K}}$-presequential, $\cup\{h^{-1}(y_n) : n \in L\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed. Again, since h is $\mathcal{I}^{\mathcal{K}}$-continuous, $\cup\{h^{-1}(y_n) : n \in L\} \cup h^{-1}(\beta)$ is $\mathcal{I}^{\mathcal{K}}$-closed. Therefore, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_n)_{n\in M}$ in $\cup\{h^{-1}(y_n) : n \in L\}$ that $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to some point $\alpha \in h^{-1}(\beta)$. For each $n \in L$, $h^{-1}(y_n)$ is $\mathcal{I}^{\mathcal{K}}$-closed. Therefore, for each $n \in L$ there is at most an $\mathcal{I}$-thin subsequence $(\alpha_n)_{n\in M_1}$ of $(\alpha_n)_{n\in M}$, which belong to $h^{-1}(y_n)$. So, there exists an $\mathcal{I}$-nonthin set $P = \{n_1 < n_2 < \ldots < n_k < \ldots\} \subset L$, such that $T = \{i \in M : \alpha_i \in h^{-1}(y_{n_i})\} \notin \mathcal{I}$. Thus, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_i)_{i\in T}$ with $\alpha_{m_i} \in h^{-1}(y_{n_i})$, $T = \{m_1 < m_2 < \ldots\}$, such that $(\alpha_i)_{i\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $\alpha \in h^{-1}(\beta)$.

$(b) \implies (c)$ Let $\beta \in \overline{D}^{\mathcal{I}^{\mathcal{K}}}$. Without loss of generality, let $\beta \notin D$. There exists an $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in D, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β. Then there is an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in M}$ in X, such that $x_{m_k} \in h^{-1}(y_{n_k})$, where $M = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ and $\{n_1 < n_2 < \ldots < n_k < \ldots\} \subset L$ and $(x_n)_{n\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to $\alpha \in h^{-1}(\beta)$. Since $x_n \in h^{-1}(D)$, for each $n \in M$, so α is in the $\mathcal{I}^{\mathcal{K}}$-closure of $h^{-1}(D)$.

$(c) \implies (a)$ Suppose h is not an $\mathcal{I}^{\mathcal{K}}$-presequential mapping. Then there exists a non $\mathcal{I}^{\mathcal{K}}$-closed subset D of Y, such that $h^{-1}(D)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. Suppose β is a point in the $\mathcal{I}^{\mathcal{K}}$-closure of D and $\beta \notin D$. Then $h^{-1}(\beta) \notin h^{-1}(D)$. Since $h^{-1}(D)$ is $\mathcal{I}^{\mathcal{K}}$-closed, there does not exist a point $\alpha \in h^{-1}(\beta)$, such that $\alpha \in \overline{h^{-1}(D)}^{\mathcal{I}^{\mathcal{K}}}$. Hence, h is $\mathcal{I}^{\mathcal{K}}$-presequential. □

Example 2. Let $\mathcal{I} = \mathcal{K} = \mathcal{P}(2\mathbb{N}) \cup Fin$, $\mathcal{P}(2\mathbb{N})$ be the power set of $2\mathbb{N}$, and Fin be the class of all finite subsets of $\mathbb{N}$. Consider $I = [0,1]$ with the usual topology and for each $\alpha \in I$, $S_\alpha = \{x_{\alpha,n} : n \in \mathbb{N}\}$ and $S'_\alpha = S_\alpha \cup \{x_\alpha\}$. A topology τ on S'_α consists of each $\{x_{\alpha,n}\}$ and sets U containing x_α equals to $\{x_{\alpha,n} : n \geqslant n_0\} \cup \{x_\alpha\}$, for some $n_0 \in \mathbb{N}$. Suppose X is a topological sum of a collection $\{I, S'_\alpha : \alpha \in I\}$. Let $Y = (\oplus S_\alpha) \oplus I$ be the space with a topology τ_1 that consists of each $\{x_{\alpha,n}\}$ and sets U containing α of the form $\{x_{\alpha,n} : n \geqslant m\} \cup G$, where G is an open set containing α in I and $m \in \mathbb{N}$. Consider the map $f : X \to Y$ defined by $f(x) = x$, if $x = x_{\alpha,n} \in S_\alpha$ and $f(x) = \alpha$, if $x = x_\alpha$ or $x \in I$.

Suppose $S = (y_n)_{n\in M}$ is an $\mathcal{I}$-nonthin sequence in Y that $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to y. So $y \in I$. Let $S_1 = S \cap S_y$ and $S_2 = S \cap I$. Since S is an $\mathcal{I}$-nonthin sequence, either S_1 or S_2 must be $\mathcal{I}$-nonthin. Again, S_1 and S_2 are $\mathcal{I}^{\mathcal{K}}$-convergent in X with its image being an $\mathcal{I}$-nonthin subsequence of S. Hence, f is an $\mathcal{I}^{\mathcal{K}}$-quotient map. Now, suppose (p_n) is a sequence in I converging to α in I. A sequence $S = (z_n)$ in Y is defined by $z_n = x_{\alpha,n}$, if $n \in 4\mathbb{N} + 1$ and $z_n = p_n$, if $n \notin 4\mathbb{N} + 1$. Therefore, (z_n) converges to α in Y, so (z_n) $\mathcal{I}^{\mathcal{K}}$-converges to α. Let $S_1 = S \cap S_\alpha$ and $S_2 = S \cap I$. Then S_1 and S_2 are $\mathcal{I}$-nonthin sequences in X. Thus S_1 $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x_α and S_2 $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to α, where $L = 4\mathbb{N} + 1$, $T = \mathbb{N}\backslash 4\mathbb{N} + 1$. Since X is Hausdorff, so, corresponding to S, there is no $\mathcal{I}$-nonthin sequence in X, whose image is S. Hence, f is not an $\mathcal{I}^{\mathcal{K}}$-covering map.

Theorem 9. *$\mathcal{I}^{\mathcal{K}}$-quotient mappings are hereditarily $\mathcal{I}^{\mathcal{K}}$-quotient.*

Proof. Let $f : X \to Y$ is an $\mathcal{I}^{\mathcal{K}}$-quotient map and D is a subspace of Y. Consider $g = f|_{f^{-1}(D)}$ and the restriction map $g : f^{-1}(D) \to D$. Clearly, g is an $\mathcal{I}^{\mathcal{K}}$-continuous map. Consider an $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in D, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to y in D. Since f is $\mathcal{I}^{\mathcal{K}}$-quotient map, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in T}$ with $x_{m_i} \in f^{-1}(y_{n_i}) \in f^{-1}(D)$, such that $(x_n)_{n\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $x \in f^{-1}(y) \in f^{-1}(D)$, where $T = \{m_1 < m_2 < \ldots\}$ and $\{n_1 < n_2 < \ldots\}$ are $\mathcal{I}$-nonthin subset of L. Therefore, g is an $\mathcal{I}^{\mathcal{K}}$-quotient map. □

Example 3. Consider the space $X = [1, \omega_1]$ with the order topology, where ω_1 is the first uncountable ordinal and the space $Y = \{0, 1\}$ with topology $\{\varnothing, \{0\}, Y\}$. A function $f : X \to Y$ is defined by $f([1, \omega_1)) = \{0\}$ and $f(\omega_1) = 1$. Then f is a continuous quotient map. Again, no $\mathcal{I}$-nonthin sequence in $X\backslash\{\omega_1\}$ $\mathcal{I}^{\mathcal{K}}$-converges to ω_1. This implies that $[1,\omega_1)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. Therefore, the set $f^{-1}(\{0\}) = [1, \omega_1)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. But the set $\{0\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed in Y. Hence, f is not an $\mathcal{I}^{\mathcal{K}}$-quotient map.

Example 4. Consider the space $X = [1, \omega_1]$ with the discrete topology and the space $Y = [1, \omega_1]$ with order topology. Let $f : X \to Y$ be the identity map. Then f is continuous but not a quotient map. Suppose $(x_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in Y, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to

$x \in Y$. Then there exists $M \in \mathcal{F}(\mathcal{I}|_L)$, such that the sequence $(y_n)_{n\in L}$ given by $y_n = x_n$, $n \in M$, and $y_n = x$, if $n \in L\backslash M$ $\mathcal{K}$-converges to x. There exists an open set U_0 containing x, $y_n \notin U_0$ for each $y_n \neq x$, so $\{n \in L\colon y_n \neq x\} = \{n \in L\colon y_n \notin U_0\} \in \mathcal{K}$. Therefore, $\{n \in L\colon y_n \notin \{x\}\} \in \mathcal{K}$ and thus $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x in X. Hence, f is an $\mathcal{I}^{\mathcal{K}}$-quotient map.

Theorem 10.

(a) *Suppose $f\colon X \to Y$ is an $\mathcal{I}^{\mathcal{K}}$-continuous quotient map and X is an $\mathcal{I}^{\mathcal{K}}$-sequential space. Then Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space and the map f is $\mathcal{I}^{\mathcal{K}}$-quotient.*

(b) *If $f\colon X \to Y$ is $\mathcal{I}^{\mathcal{K}}$-quotient and Y is $\mathcal{I}^{\mathcal{K}}$-sequential, then f is quotient.*

Proof. (a) Let G be an $\mathcal{I}^{\mathcal{K}}$-open set in Y. Suppose $(x_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in X, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x in $f^{-1}(G)$. Since f is $\mathcal{I}^{\mathcal{K}}$-continuous, $(f(x_n))_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $f(x)$ in G. Again, since G is $\mathcal{I}^{\mathcal{K}}$-open, from Theorem 2 it follows that $|\{n \in L\colon f(x_n) \in G\}| = \omega$. Thus $|\{n \in L\colon x_n \in f^{-1}(G)\}| = \omega$. Therefore, $f^{-1}(G)$ is $\mathcal{I}^{\mathcal{K}}$-open in X. Now, let $H \subset Y$ and $f^{-1}(H)$ be $\mathcal{I}^{\mathcal{K}}$-open in X. As X is $\mathcal{I}^{\mathcal{K}}$-sequential, $f^{-1}(H)$ is open in X. Again, since f is a quotient map, H is open in Y. Therefore, H is $\mathcal{I}^{\mathcal{K}}$-open in Y. Hence, f is an $\mathcal{I}^{\mathcal{K}}$-quotient map.
(*b*) Suppose $U \subset Y$ and $f^{-1}(U)$ is open in X. Then $f^{-1}(U)$ is $\mathcal{I}^{\mathcal{K}}$-open in X. Since f is $\mathcal{I}^{\mathcal{K}}$-quotient, U is $\mathcal{I}^{\mathcal{K}}$-open in Y. Again, since Y is $\mathcal{I}^{\mathcal{K}}$-sequential, U is open in Y. Hence, f is a quotient map. □

Corollary 4. *Let X and Y be topological spaces. Suppose $g\colon X \to Y$ is a continuous function and X is an $\mathcal{I}^{\mathcal{K}}$-sequential space. Then g is quotient if and only if g is $\mathcal{I}^{\mathcal{K}}$-quotient and Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*

4. $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Definition 8. *A topological space X is said to be $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn if for each $A \subset X$ and each $x \in cl(A)$, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in A $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to the point x.*

Every Fréchet-Urysohn space [8] is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn. The disjoint topological sum of any family of $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn spaces is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Consider a nonempty subspace G of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space X and $x \in cl_G(D)$, where $D \subset G$. Then $cl_G(D) = G \cap cl_X(D)$ and, so, $x \in cl_X(D)$. Since X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in D

$(\mathcal{I}|_L)^{\mathcal{K}}$-converging to the point x. Therefore, subspace of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Theorem 11. *Every $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*

Proof. Suppose U is an $\mathcal{I}^{\mathcal{K}}$-open subset of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space X. Let $l \in \overline{(X\backslash U)}$. Then there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in $X\backslash U$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to l. Since $X\backslash U$ is $\mathcal{I}^{\mathcal{K}}$-closed, $l \in X\backslash U$. Therefore, $X\backslash U$ is a closed set. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-sequential space. □

Corollary 5. *Every Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn and every $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-sequential.*

Example 5 is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, which is sequential but not Fréchet-Urysohn.

Example 5. Consider the space $X = \{0\} \cup \bigcup\limits_{i=1}^{\infty} X_i$, $X_i = \{\frac{1}{i}\} \cup \{\frac{1}{i} + \frac{1}{i^2}, \frac{1}{i} + \frac{1}{i^2+1}, \frac{1}{i} + \frac{1}{i^2+2}, \ldots\}$. Then $X_i \cap X_k = \emptyset$, for $i \neq k$. A topology τ on X consists of each $\{\frac{1}{i} + \frac{1}{j}\}$ and for an element x of the form $\frac{1}{i}$, sets are given by $\{\frac{1}{i}\} \cup \{\frac{1}{i} + \frac{1}{k}, \frac{1}{i} + \frac{1}{k+1}, \ldots\}$, for $k = i^2, i^2+1, \ldots$ and sets containing 0 are obtained from X by removing a finite number of X_i's and a finite number of points in all of the remaining X_i's that have the form $\frac{1}{i} + \frac{1}{j}$ ([5], Example 1.6.19). Consider the ideals $\mathcal{I} = \mathcal{K} = \{A\colon A \cap \Delta_i$ are finite for all but finitely many i $\}$, where $\mathbb{N} = \bigcup\limits_{i=1}^{\infty} \Delta_i$ and each Δ_i is infinite and $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$.

Let $A \subset X$ and $a \in \bar{A}$. If $a = \frac{1}{i} + \frac{1}{j}$, then $a \in A$. If $a = \frac{1}{i}$, then there exists an infinite subset Y_i of X_i, such that $Y_i \subset A$. Consider a sequence (x_n) in A, where $x_n = \frac{1}{i} + \frac{1}{i^2+k_n}$, (k_n) is an increasing sequence of natural numbers. Therefore, (x_n) $\mathcal{I}^{\mathcal{K}}$-converges to a. If $a = 0$, then there exists an increasing sequence $C = (c_n)$ of natural numbers, such that $\bigcup\limits_{i\in C} Y_i \subset A$ and each Y_i is an infinite subset of X_i. For each $i \in C$, consider a sequence (x_j) in A, defined by $x_j = \frac{1}{i} + \frac{1}{i^2+l_{i,j}}$, $j \in \Delta_i$, and $(l_{i,j})$ is an increasing sequence of natural numbers. Then (x_j) $\mathcal{I}^{\mathcal{K}}$-converges to 0. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Moreover, from Example 1.6.19 in [5], X is sequential but not Fréchet-Urysohn.

Example 6. Let $S = (a_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements. Consider the space $X = S \cup \{\alpha\}$, $\alpha \notin S$. A topology τ on X consists of each $\{a_n\}$ and sets U containing α of the form $U = \{\alpha\} \cup \{a_n \colon n \in L\}$, where $\mathbb{N}\backslash L \in \mathcal{K}$.

Let $A \subset X$ and $a \in \bar{A}$. If $a \in S$, then $a \in A$. Then, taking the constant sequence (a), space becomes $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn. If $a = \alpha$ and $a \notin A$, then A is a subset of S. Assume that A is a $\mathcal{K}$-thin subsequence of S. Then $X \backslash A = U$ is an open neighborhood of $\alpha = a$. But $a \in \bar{A}$ and $A \cap U = \emptyset$, which leads to a contradiction. Therefore, A is a $\mathcal{K}$-nonthin subsequence of S. So, A $\mathcal{K}$-converges to a and, then, A $\mathcal{I}^{\mathcal{K}}$-converges to a. Therefore, X is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn. Again, by Theorem 11, X is $\mathcal{I}^{\mathcal{K}}$-sequential. It is obvious that $\alpha \in \bar{S}$. Let $(a_n)_{n\in L}$ be a subsequence of S. Consider a $\mathcal{K}$-thin subsequence $(a_n)_{n\in L_1}$ of $(a_n)_{n\in L}$. Let $U = X \backslash \{a_n : n \in L_1\}$. Then U is an open neighborhood of α. Therefore, $(a_n)_{n\in L}$ does not converge to α. So, no subsequence of S converges to α. Hence, X is not a Fréchet-Urysohn space.

Nowhere tall ideal plays an important role in the following theorem. An ideal $\mathcal{I}$ on a non-empty set X is nowhere tall if for any set $A \notin \mathcal{I}$, there exists $B \subset A$, such that $\mathcal{I}|_B$ is the collection of all finite subsets of B ([7], Definition 2.25).

Theorem 12. *$\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is Fréchet-Urysohn provided $\mathcal{K} \subset \mathcal{I}$ and $\mathcal{K}$ is a nowhere tall ideal on $\mathbb{N}$.*

Proof. Let X be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, $A \subset X$ and $a \in \bar{A}$. There exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to a. Then there is a set $M \in \mathcal{F}(\mathcal{I}|_L)$, such that the sequence $(y_n)_{n\in L}$ given by $y_n = x_n$, $n \in M$, and $y_n = a$, $n \in L \backslash M$ $\mathcal{K}|_L$-converges to a. Since $\mathcal{K} \subset \mathcal{I}$ and $M \notin \mathcal{I}$, $M \notin \mathcal{K}$. As $\mathcal{K}$ is a nowhere tall ideal, there exists a subset M_1 of M, such that $\mathcal{I}|_{M_1}$ is the collection of all finite subsets of M_1. Therefore, the sequence $(x_n)_{n\in M_1}$ converges to a and, so, X is a Fréchet-Urysohn space. □

Example 7. For each $i \in \mathbb{N}$, consider a sequence of distinct elements $S_i = \{x_{i,j} : j \in \mathbb{N}\}$. Let $S = \{a_i : i \in \mathbb{N}\}$ be a sequence of distinct elements. Consider the space $X = \cup\{S_i : i \in \mathbb{N}\} \cup S \cup \{\alpha\}$, $\alpha \notin \cup\{S_i : i \in \mathbb{N}\} \cup S$. A topology τ on X consists of each $\{x_{i,j}\}$ and sets containing a_i of the form $\{a_i\} \cup \{x_{i,j} : j \in T\}$, $\mathbb{N}\backslash T \in \mathcal{K}$ for each $i \in \mathbb{N}$, and sets containing α of the form $\{\alpha\} \cup \{a_i : i \in L\} \cup \{\{x_{i,j} : j \in T\} : i \in L\}$ for each $i \in \mathbb{N}$, where $\mathbb{N}\backslash L \in \mathcal{K}$ and $\mathbb{N}\backslash T \in \mathcal{K}$.

Consider an $\mathcal{I}^{\mathcal{K}}$-closed subset Y of X. Let $p \in \bar{Y}$. If $p \in \bigcup_{i=1}^{\infty} S_i$, $\{p\}$ is an open set. As $p \in \bar{Y}$, $p \in Y$. If $p = \alpha$, consider the subsequence $Y \cap S$ of S. Since $\alpha \in \bar{Y}$, $Y \cap S$ is a $\mathcal{K}$-nonthin subsequence and, so, $Y \cap S$

$\mathcal{I}^{\mathcal{K}}$-converges to α. Therefore, $p \in \overline{Y}^{\mathcal{I}^{\mathcal{K}}} = Y$. If $p \in S$, there exists $i_0 \in \mathbb{N}$, such that $a_{i_0} = p$. Consider the subsequence $Y \cap S_{i_0}$ of S_{i_0}. Since $p \in \bar{Y}$, $Y \cap S_{i_0}$ is a $\mathcal{K}$-nonthin subsequence of S_{i_0}. Therefore $Y \cap S_{i_0}$ $\mathcal{I}^{\mathcal{K}}$-converges to a_{i_0}. So, $p = a_{i_0} \in \overline{Y}^{\mathcal{I}^{\mathcal{K}}} = Y$. Hence, Y is a closed subset of X. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-sequential space.
It is obvious that X is Hausdorff and $\alpha \in \overline{X \backslash (S \cup \{\alpha\})}$. Let $E = (y_n)_{n \in L}$ be a sequence in $X \backslash (S \cup \{\alpha\})$, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converge to α. If for each $i \in \mathbb{N}$, $E_i = E \cap S_i$ is a $\mathcal{K}$-thin sequence of S_i, then take $U = \{\alpha\} \cup S \cup \{S_i \backslash E_i : i \in \mathbb{N}\}$. Then U is an open set containing α and $U \cap E = \varnothing$, which leads to a contradiction. Therefore, there exists $i_0 \in \mathbb{N}$, such that $E \cap S_{i_0}$ is a $\mathcal{K}$-nonthin subsequence of S_{i_0}. So $E \cap S_{i_0}$ $\mathcal{I}^{\mathcal{K}}$-converges to $a_{i_0} \neq \alpha$. Again, by assumption $E \cap S_{i_0}$ $\mathcal{I}^{\mathcal{K}}$-converges to α, which leads to a contradiction as X is Hausdorff. Therefore, no sequence in $X \backslash (S \cup \{\alpha\})$ $\mathcal{I}^{\mathcal{K}}$-converges to α. Hence, X is not an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Theorem 13. *A topological space X is hereditarily $\mathcal{I}^{\mathcal{K}}$-sequential if and only if the space is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn.*

Proof. Suppose $G \subset X$ and $x \in \bar{G}$. Without loss of generality, let $x \notin G$. Then G is not a closed set in X. Let $Y = G \cup \{x\}$. Therefore, G is not closed in Y. As Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space, G is not an $\mathcal{I}^{\mathcal{K}}$-closed set in Y. There exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n \in L}$ in G, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Conversely let X be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Again, subspace of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn and every $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-sequential. Therefore, the space X is hereditarily $\mathcal{I}^{\mathcal{K}}$-sequential. □

A mapping $f : X \to Y$ is said to be pseudo-open if for each $p \in Y$ and each neighbourhood O of $f^{-1}(p)$ in X, $p \in int(f(O))$ [5].

Theorem 14. *Let X, Y be topological spaces and let Y be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Then each $\mathcal{I}^{\mathcal{K}}$-covering mapping f from X onto Y is pseudo-open.*

Proof. Suppose f is not a pseudo-open map. Then there exists a point $z \in Y$ and an open subset O of X, such that $f^{-1}(z) \subset O$ and z is not an interior point of $f(O)$. So, $z \in \overline{Y \backslash f(O)}$. As Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, there exists an $\mathcal{I}$-nonthin sequence $(z_n)_{n \in L}$ in $Y \backslash f(O)$, such that $(z_n)_{n \in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to z. Also, since f is an $\mathcal{I}^{\mathcal{K}}$-covering mapping, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_i)_{i \in L}$ with $\alpha_i \in f^{-1}(z_i)$, for all $i \in L$ and $\alpha \in f^{-1}(z)$, such that $(\alpha_i)_{i \in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to α. Therefore, $\alpha \in O$

and $\{i \in L \colon \alpha_i \notin O\} \in \mathcal{K}|_L$. Then there exists $t \in L$, such that $\alpha_t \in O$ and, so, $z_t \in f(O)$, which leads to a contradiction. Hence, f is a pseudo-open map. □

Theorem 15. *Suppose $f \colon X \to Y$ is a quotient map, where X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Then Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space if and only if f is pseudo open.*

Proof. Let $G \subset Y$ and $p \in \bar{G}$. If possible, let $f^{-1}(p) \cap \overline{f^{-1}(G)} = \emptyset$. Then $f^{-1}(p) \subset X \setminus \overline{f^{-1}(G)} = O$ (say). As f is pseudo-open, then $p \in int f(O)$. Again, $int f(O) \subset int f(X \setminus f^{-1}(G)) = int(Y \setminus G) = Y \setminus \bar{G}$. Thus $p \in Y \setminus \bar{G}$, which leads to a contradiction. Therefore, there exists a point $q \in f^{-1}(p) \cap \overline{f^{-1}(G)}$. Since X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n \in L}$ in $f^{-1}(G)$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to the point q. Therefore, there exists an $\mathcal{I}$-nonthin sequence $(f(x_n))_{n \in L}$ in G, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to $f(q) = p$. Hence, Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Conversely let Y be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Suppose $p \in Y$ and O is an open neighbourhood of $f^{-1}(p)$. Let us assume that $p \notin int f(O)$. Then $p \in \overline{Y \setminus f(O)}$. Since Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, there exists an $\mathcal{I}$-nonthin sequence $S = (y_n)_{n \in L}$ in $Y \setminus f(O)$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to p. Again, since f is a quotient map, $\overline{f^{-1}(S)} \subset f^{-1}(\bar{S}) = f^{-1}(S) \cup f^{-1}(p)$. Since O is an open neighborhood of $f^{-1}(p)$ and $O \cap f^{-1}(S) = \emptyset$, $f^{-1}(p) \cap \overline{f^{-1}(S)} = \emptyset$ and so $f^{-1}(S)$ is closed. Therefore, $X \setminus f^{-1}(S) = f^{-1}(Y \setminus S)$ is open. Since f is quotient, $Y \setminus S$ is open, which leads to a contradiction. Hence $p \in int f(O)$ and so f is pseudo open. □

The article is concluded with the diagram (Figure 1), which shows interrelations among Fréchet-Urysohn, $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn, sequential, and $\mathcal{I}^{\mathcal{K}}$-sequential spaces.

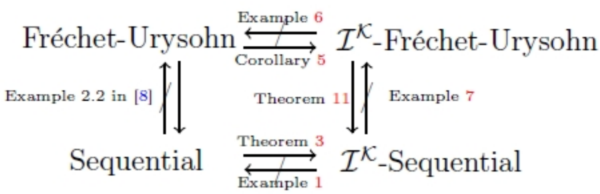

Figure 1: Relation among Fréchet-Urysohn, $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn, sequential, and $\mathcal{I}^{\mathcal{K}}$-sequential spaces

Acknowledgment. The authors are grateful to the referee for giving valuable suggestions that improved the presentation of the paper. The authors would like to acknowledge the Department of Mathematics, University of North Bengal, for providing infrastructural support.

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Received November 28, 2024.
In revised form, April 23, 2025.
Accepted April 30, 2025.
Published online June 4, 2025.

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