DOI: 10.15393/j3.art.2025.17210

UDC 517.98, 517.44

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ON HK-SOBOLEV SPACE OVER HYPERGROUP GELFAND PAIR

Abstract. In this article, we introduce the HK-Sobolev space $HK_{\zeta}^{\alpha,\natural}(\mathbf{G})$ over a Gelfand pair within the framework of a second countable hypergroup, employing the Fourier transform on the hypergroup. We discuss Kuelbs-Steadman space KS^p in Hypergroup and prove that $KS^p(\mathbf{G})$ is a Banach algebra under a suitable convolution. Additionally, we also address the dominated convergence theorem in the KS^p space over the hypergroup. Several Sobolev embedding-type results are discussed in the HK-Sobolev space $HK_{\zeta}^{\alpha,\natural}(\mathbf{G})$. Finally, we explore Rellich-Kondrashov theorem within this specific context.

Key words: *Sobolev space, Kuelbs-Steadman space, HK-Sobolev space, Hypergroup, Gelfand Pair, Rellich-Kondrachov theorem*

2020 Mathematical Subject Classification: *43A15, 43A62, 43A70, 43A75.*

1. Introduction. Sobolev spaces are a highly helpful tool in the theory of partial differential equations and have been subject of multiple research. While the concept of weak derivative is used to define Sobolev spaces, there exists an analogous concept for a class of Sobolev spaces that is associated with the Fourier transform. Moreover, Sobolev spaces are also defined on some classical spaces and algebraic structures like Riemannian manifolds [20], [21], locally compact group [16], [17], compact group [29], metric measure space [19], Heisenberg group [4], Bessel hypergroup, Laguerre hypergroup and the dual of the Laguerre hypergroup [1], [2], [3], etc. A. Behzadan and M. Holst define Sobolev space in [5] by using the Fourier Transform on $\mathbb{R}^n$, which is represented by $H^s(\mathbb{R}^n)$. Gorka et al. generalised this space on Hausdorff locally compact Abelian group in [16], [17]. In [18], the authors study Sobolev space on Locally compact group and discuss an analog of the Rellich compactness theorem in that context. In [29], the authors discuss Sobolev spaces on a compact group using the Fourier transform on compact groups. An Abelian group G can potentially be determined by the Gelfand pair $(G, \{e\})$. Mensah in [31] uses the spherical Fourier transformation of type δ to investigate the Sobolev space $H^s_{\delta,\gamma}(G, E)$, where G is a locally compact Hausdorff space and E is a complex Banach space.

Henstock-Kurzweil integral (HK integral), in short, is a kind of non-absolute integrals. One can see [13], [14], [15] and references therein for details of this integral. Recently, several function spaces have been constructed with HK-integrable functions. See [13], [22], [25], [26], [27] for several spaces of HK-integrable functions. Sobolev-like spaces of HK-integrable functions are called HK-Sobolev spaces and have been recently introduced on $\mathbb{R}^n$ (see [23]). H.M. Srivastava et al. discussed HK Sobolev spaces of the Newton-type in [35]. Hypergroups are a generalization of locally compact groups. In [33], [34], we present the study of variable Lebesgue space on locally compact group.

In this work, we continue our study of HK-Sobolev spaces and generalize the theory of HK-Sobolev spaces for hypergroup Gelfand pair in order to construct the Bessel potential like HK Sobolev space employing Fourier transform on Hypergroup and define an analog of the Rellich-Kondrachov theorem within this specific context. Now, we provide an overview and preliminary information on hypergroups and the harmonic analysis of hypergroup Gelfand pairs. For this, we cite [9].

We organize the subsequent sections of this paper in the following manner. In the Section 2, we recall several definitions and results that are used in the main part. In Section 3, we introduce Kuelbs-Steadman spaces $KS^p(\mathbf{G})$ on a second countable Hypergroup and establish Plancherel-type theorem on this space. We also show that $KS^p(\mathbf{G})$ space is complete under a convolution defined in this section. The next section discusses the HK-Sobolev space on hypergroup Gelfand pair $HK^{\alpha,\natural}_{\zeta}(\mathbf{G})$ with the help of Fourier transform on the hypergroup. Here we also establish several embedding results on $HK^{\alpha,\natural}_{\zeta}(\mathbf{G})$. In the last section, we prove Rellich-Kondrachov-type theorem on $HK^{\alpha,\natural}_{\zeta}(\mathbf{G})$ by defining dominated convergence theorem on $KS^p(\mathbf{G})$.

2. Preliminaries. Throughout this article, we use the following notation and conventions:

- $[a, b]$ represents an interval of $\mathbb{R}$.
- For a function space $\mathbf{A}$, its dual space is denoted by $\widehat{\mathbf{A}}$.

- $\mathbf{G}$ represents a locally compact Hausdorff space.
- The set of all complex-valued Radon measures on $\mathbf{G}$ is denoted by $\mathfrak{M}(\mathbf{G})$, while $C(\mathbf{G})$ represents the space of all complex-valued continuous functions on $\mathbf{G}$.
- The subset of $\mathfrak{M}(\mathbf{G})$ consisting of bounded measures is denoted by $\mathfrak{M}_b(\mathbf{G})$.
- The support of a measure $\mu \in \mathfrak{M}(\mathbf{G})$ is written as supp μ.
- The collection of all compact subsets of $\mathbf{G}$ is denoted by $\mathfrak{C}(\mathbf{G})$.
- The point measure at $t \in \mathbf{G}$ is indicated by δ_t.

A Hausdorff space $\mathbf{G}$ that is locally compact is said to be a hypergroup if there are a convolution $* : \mathfrak{M}_b(\mathbf{G}) \times \mathfrak{M}_b(\mathbf{G}) \to \mathfrak{M}_b(\mathbf{G})$, an involution $t \to t^-$ on $\mathbf{G}$, and an element e (called the identity element), such that the following conditions holds:

(i) $(\mathfrak{M}_b(\mathbf{G}), +, *)$ is a complex Banach algebra.

(ii) for all non-negative measures $\mu, \nu \in \mathfrak{M}_b(\mathbf{G})$, $\mu * \nu$ is also a non-negative measure and from $\mathfrak{M}_b(\mathbf{G}) \times \mathfrak{M}_b(\mathbf{G})$ into $\mathfrak{M}_b(\mathbf{G})$ there exists a continuous mapping $(\mu, \nu) \mapsto \mu * \nu$.

(iii) For every pair of values $t, s \in \mathbf{G}$, the probability measure $\delta_t * \delta_s$ has a compact support.

(iv) The mapping $(t, s) \mapsto \text{supp}\,(\mu * \nu)$ from $\mathbf{G} \times \mathbf{G}$ to $\mathfrak{C}(\mathbf{G})$ is continuous.

(v) $\delta_e * \delta_t = \delta_t = \delta_t * \delta_e$, for all $t \in \mathbf{G}$.

(vi) The mapping $t \mapsto t^-$ is a homeomorphism on $\mathbf{G}$, such that $(\delta_t * \delta_s)^- = = \delta_{s^-} * \delta_{t^-}$ for all $t, s \in \mathbf{G}$, where $(\delta_t * \delta_s)^-(\mathtt{f}) = (\delta_x * \delta_y)(\mathtt{f}^-)$ and $\mathtt{f}^-(t) = \mathtt{f}(t^-)$ for all continuous function $\mathtt{f}$ on $\mathbf{G}$.

The set $\mathfrak{M}(\mathbf{G})$ is endowed with the cone topology and the set $\mathfrak{C}(\mathbf{G})$ is endowed with the Michael topology. For more details of hypergroups, one can see [6], [9], [12], [24].

A closed non-empty subset $\mathcal{H}$ of a hypergroup $\mathbf{G}$ is called a subhypergroup of $\mathbf{G}$ if $t^- \in \mathcal{H}$ and $\text{supp}(\delta_t * \delta_s) \subset \mathcal{H}$ for all $t \in \mathcal{H}$.

For a hypergroup $\mathbf{G}$ and a compact subhypergroup $\mathbf{K}$ of $\mathbf{G}$, the double coset of t with respect to $\mathbf{K}$ is defined as

$$\mathbf{K}t\mathbf{K} = \{k_1 * t * k_2 : k_1, k_2 \in \mathbf{K}\} = \bigcup_{k_1, k_2 \in \mathbf{K}} \text{supp}\,(\delta_{k_1} * \delta_t * \delta_{k_2}),$$

where $t * s = \text{supp}(\delta_t * \delta_s)$ for all $t, s \in \mathbf{G}$.

Let $\mathtt{f}$ be a function in the space $C(\mathbf{G})$. We can express $\mathtt{f}(t * s)$ as $(\delta_t * \delta_s)(\mathtt{f})$ or as the integral of $\mathtt{f}(t)$ with respect to the measure $d(\delta_x * \delta_y)(t)$ over the space $\mathbf{G}$. The operation $*$ is defined for μ and ν in $\mathfrak{M}(\mathbf{G})$ as follows:

$$\mu * \nu(\mathtt{f}) = \int_{\mathbf{G}} \int_{\mathbf{G}} \mathtt{f}(t * s) d\mu(t) d\mu(s), \ \mathtt{f} \in C(\mathbf{G}).$$

If the equation $\mathtt{f}(k_1 * t * k_2) = \mathtt{f}(t)$ holds for every $t \in \mathbf{G}$ and $k_1, k_2 \in \mathbf{K}$, then the function $\mathtt{f} \in C(\mathbf{G})$ is $\mathbf{K}$-bi-invariant. Let $C_c(\mathbf{G})$ denote the collection of all continuous functions on $\mathbf{G}$ that have compact support. On the other hand, $C_c^{\natural}(\mathbf{G})$ refers to the subset of $C_c(\mathbf{G})$ that consists of $\mathbf{K}$-bi-invariant functions.

Let us assume that the hypergroup $\mathbf{G}$ possesses a left Haar measure, whereas $\mathbf{K}$ possesses a normalized Haar measure. Let $\mathtt{f} \in \mathbf{G}$. We may define $\mathtt{f}^{\natural}(t)$ as the integral of $\mathtt{f}(k_1 * t * k_2)$ over $\mathbf{K} \times \mathbf{K}$, where k_1 and k_2 are integration variables, i. e.,

$$\mathtt{f}^{\natural}(t) = \int_{\mathbf{K}} \int_{\mathbf{K}} \mathtt{f}(k_1 * t * k_2) dk_1 dk_2.$$

A measure $\mu \in \mathfrak{M}(\mathbf{G})$ is called $\mathbf{K}$-bi-invariant if $\mu^{\natural} = \mu$, where $\mu^{\natural}(\mathtt{f}) = \mu(\mathtt{f}^{\natural}), \mathtt{f} \in C_c(\mathbf{G})$. Let us define

$$\mathfrak{M}_c^{\natural}(\mathbf{G}) = \{\mu \in \mathfrak{M}(\mathbf{G}) : \mu \text{ is } \mathbf{K}\text{-bi-invariant and } \mathrm{supp}\mu \text{ is compact}\}.$$

Consider the hypergroup $\mathbf{G}$ and the compact subhypergroup $\mathbf{K}$. A hypergroup Gelfand pair is defined as the pair $(\mathbf{G}, \mathbf{K})$ for which the space $(\mathfrak{M}_c^{\natural}(\mathbf{G}), *)$ exhibits commutativity. Let $\widehat{\mathbf{G}}^{\natural}$ denote the collection of all bounded continuous functions $\vartheta : \mathbf{G} \to \mathbb{C}$ that satisfies following conditions:

(i) ϑ is $\mathbf{K}$-bi-invariant;

(ii) $\vartheta(e) = 1$ and $\vartheta(t^-) = \overline{\vartheta(t)}$, where the later states that the complex conjugate of $\vartheta(t)$ is equal to $\vartheta(t^-)$;

(iii) $\displaystyle\int_{\mathbf{K}} \vartheta(t * k * s) dk = \vartheta(t)\vartheta(s)$ holds for all t and s in $\mathbf{G}$.

The set $\widehat{\mathbf{G}}^{\natural}$ is defined as the dual of the hypergroup $\mathbf{G}$ [9]. The space $\widehat{\mathbf{G}}^{\natural}$ is a locally compact Hausdorff space equipped with the topology of uniform convergence on compact sets. If the topological space $\mathbf{G}$ satisfies the second axiom of countability, then the dual space $\widehat{\mathbf{G}^\natural}$ is likewise second countable (see [7]).

Let $(\mathbf{G}, \mathbf{K})$ be a hypergroup Gelfand pair. The Fourier transform of $\mathtt{f} \in C_c^\natural(\mathbf{G})$ is defined by

$$\widehat{\mathtt{f}}(\vartheta) = \int\limits_{\mathbf{G}} \vartheta(t^-)\mathtt{f}(t)dt.$$

The following gives the inverse Fourier transform:

$$\check{\widehat{\mathtt{f}}}(t) = \mathtt{f}(t) = \int\limits_{\widehat{\mathbf{G}^\natural}} \vartheta(t)\widehat{\mathtt{f}}(\vartheta)d\pi(\vartheta),$$

where π is the Plancherel measure on $\widehat{\mathbf{G}^\natural}$ (see [11]).

If $\mathbf{G}$ is a locally compact group and $\mathbf{K}$ is a compact subhypergroup of $\mathbf{G}$, such that $(\mathbf{G}, \mathbf{K})$ is a Gelfand pair, the double coset space $\mathbf{G}//\mathbf{K}$ is an example of a commutative hypergroup. Every double coset forms a partition of $\mathbf{G}$. The double coset space $\mathbf{G}//\mathbf{K}$ is equipped with a local topology via the quotient topology with respect to the associated equivalence relation (see [6]). Defined by $p_K(t) = KxK$, $t \in \mathbf{G}$, the natural mapping $p_K : \mathbf{G} \to \mathbf{G}//\mathbf{K}$ is an open surjective continuous mapping. For $\mathtt{f} \in C^\natural(\mathbf{G})$, one can define $\tilde{\mathtt{f}}$ on $\mathbf{G}//\mathbf{K}$ by $\tilde{\mathtt{f}}(\mathbf{K}t\mathbf{K}) = \mathtt{f}(t)$ for all $t \in \mathbf{G}$. $\widehat{\widehat{\mathtt{f}}}$ and $\widehat{\tilde{\mathtt{f}}}$ are in $C_b(\widehat{\mathbf{G}//\mathbf{K}})$ and $\widehat{\tilde{\mathtt{f}}} = \widehat{\widehat{\mathtt{f}}}$, as the authors of [9] have demonstrated. Additionally, $\widehat{\mathtt{f}} = \widehat{\mathtt{f}^\natural}$ for $\mathtt{f} \in L^2(\mathbf{G})$.

Recall that a family F of complex-valued functions on a set S is considered uniformly bounded if, for every $t \in S$ and every $\mathtt{f} \in$ F, there exists a real number M, such that $|\mathtt{f}(t)| \leqslant M$. Further, a collection F of continuous functions from a topological space $\mathtt{X}$ to a metric space (Y, d) is said to be equicontinuous at t_0 in $\mathtt{X}$, if for every positive number ϵ, there is a neighborhood $\mathfrak{U}$ of t_0, such that

$$d(\vartheta(t), \vartheta(t_0)) < \epsilon$$

for all $t \in \mathfrak{U}$ and for all $\vartheta \in$ F.

3. KS^p space over hypergroup. It is known that the space of Henstock-Kurzweil integrable functions, $HK([a, b])$, is a barreled space (see [36, Chapter 7]). The major drawback of the set of Henstock-Kurzweil integrable functions is that it is not a Banach space by nature. Gill and Zachary [13] introduced a Henstock-Kurzweil integrable function space of Banach space-type called Kuelbs-Steadman space. We denote Kuelbs-Steadman space as KS^p. Importantly, KS^p contains L^p space as a continuously dense subset, also contains Henstock-Kurzweil integrable functions.

One can see Banach algebra on HK integrable function spaces $HK([a,b])$ is not possible. To see this, let $M([a,b])$ denote the corresponding measure algebra of all complex Borel measures on $[a,b]$. Convolution of two measures μ and ν on $[a,b] \subset \mathbb{R}$ is given by:

$$(\mu * \mu)(A) = \int\limits_{[a,b]} \mu(dx) \int\limits_{[a,b]} \mu(dy) 1_A(x+y),$$

where A is a subset of $[a,b]$ and 1_A is the indicator function of A. The involution $t \to t^-$ on $[a,b]$ is defined by $f^-(t) = \overline{f(b-t)}$, such that $(f^-)^-(t) = f(t)$ for all $t \in [a,b]$.

Let $\mathcal{P}_t$ be the unit point mass at t. If $\mathtt{f}$ is a Borel function on $[a,b]$ and $t, s \in [a,b]$, then the translation of $\mathtt{f}$ is defined as:

$$\mathtt{f}(t * s) = \mathtt{f}_t(s) = \mathtt{f}^s(t) = \int\limits_a^b \mathtt{f}\, d(\mathcal{P}_t * \mathcal{P}_s),$$

where $\mathcal{P}_t$, $\mathcal{P}_s$ denotes the unit point mass at t, s. We denote $BV([a,b])$ to be the set of all functions of bounded variation on $[a,b]$, with convolution operators and bounded variation norm. It is well known that $BV([a,b]) \subseteq L([a,b])$ [30]. Moreover, $HK([a,b]) \cap BV([a,b]) \subseteq L([a,b]) \subseteq$ $\subseteq HK([a,b])$. The multiplication on $HK([a,b])$ can be defined as the following convolution.

For any pair of Borel functions $\mathtt{f}, \mathtt{g} \in HK([a,b])$, the convolution $\mathtt{f} * \mathtt{g}$ defined as

$$(\mathtt{f} * \mathtt{g})(t) = \int\limits_a^b \mathtt{f}(t * s)\mathtt{g}(s^-) dm(s), \tag{1}$$

for which the function $s \to \mathtt{f}(s)\mathtt{g}(s^{-1}t)$ is Haar integrable. Here m is the Haar measure on $HK([a,b])$.

Theorem 1. [37, Prop 11.(a)] *If $\mathtt{f} \in HK([a,b])$ and $g \in BV([a,b])$, then $\mathtt{f} * \mathtt{g}$ exists on $\mathbb{R}$ and*

$$m(\mathtt{f} * \mathtt{g}(t)) \leqslant ||\mathtt{f}|| \left[\inf |\mathtt{g}| + V\mathtt{g}\right],$$

where $||\cdot||$ *is Alexiewicz norm defined by* $\|f\| = \sup\left\{\left|\int_a^t f\right| : a \leqslant t \leqslant b\right\}$.

If $\mathtt{f}(t) = \dfrac{sin(t)}{|t|^{\frac{1}{2}}}$, $\mathtt{g}(t) = \dfrac{(sin(t) + cos(t))}{|t|^{\frac{1}{2}}}$ are two functions in $\mathbb{R}$, then $\mathtt{f} \in HK([a,b])$ and $\mathtt{g} \in HK([a,b])$ but $\mathtt{f} * \mathtt{g} \notin HK([a,b])$. From this example we conclude the following remark:

Remark 1. *If* $\mathtt{f}, \mathtt{g} \in HK([a,b])$, *then it is possible that* $f * g \notin HK([a,b])$.

Proposition. [37, Prop. 13] *Let* $\mathtt{f} \in HK([a,b])$ *and* $\mathtt{g} \in BV([a,b])$. *Then* $\mathtt{f}*\mathtt{g}$ *exists on* $[a,b]$ *and* $\|\mathtt{f}*\mathtt{g}\| \leqslant \|\mathtt{f}\| \cdot \|\mathtt{g}\|_L$, *where* $\|\cdot\|$ *and* $\|\cdot\|_L$ *represent Alexiewicz norm and Lebesgue norm, respectively.* The equation (1) can be written as $\int_a^b \mathtt{f}_s(t) g(s^-) dm(s)$, so that $\mathtt{f} * g$ may be regarded as a limit of linear combinations of translates of $\mathtt{f}$, but $\|\mathtt{f}*\mathtt{g}\| \nleqslant \|\mathtt{f}\| \cdot \|\mathtt{g}\|$ $\forall\, \mathtt{f}, \mathtt{g} \in HK(\mathbb{R})$. So, $HK([a,b])$ is not a normed algebra with respect to convolution $*$ defined by (1).

Remark 2. *Banach algebra for Henstock-Kurzweil integrable function space is not possible.*

The completeness properties of KS^p motivated us to extent KS^p on a hypergroup. Let $\mathbf{G}$ be a hypergroup, which is second countable, and μ be the unique Haar measure on $\mathbf{G}$. Since $\mathbf{G}$ is second countable, it has a countable basis, say $\mathfrak{B} = \{B_i\}$. Assume that χ_i is the characteristic function on B_i. Note that $\chi_i \in L^p(\mathbf{G}) \cap L^\infty(\mathbf{G})$ for $1 \leqslant p < \infty$. Let $F_i(\cdot)$ on $L^1(\mathbf{G})$ be defined by $F_i(\mathtt{f}) = \int_{\mathbf{G}} \chi_i(t)\mathtt{f}(t) d\mu(t)$. It is easy to see that $F_i(\cdot)$ is bounded on $L^p(\mathbf{G})$, $\|F_i\| \leqslant 1$, and if $F_i(\mathtt{f}) = 0$, then $\mathtt{f} = 0$. So, $\{F_i\}$ is fundamental on $L^p(\mathbf{G})$ whenever $k = 1, 2, \ldots$ and $1 \leqslant p \leqslant \infty$. Let $\{t_i\}$ be a non negative real sequence, such that $\sum_{i=1}^{\infty} t_i = 1$. Define a Haar measure $d\mu = \left[\sum_{i=1}^{\infty} t_i \chi_i(t) \chi_i(s)\right] d\mu(t) d\mu(s)$ on $\mathbf{G} \times \mathbf{G}$. With the help of this, we can construct a Hilbert-type space of Kuelbs-Steadman spaces as follows.

Let us define an inner product $(\cdot)$ on $L^1(\mathbf{G})$ by

$$
\begin{aligned}
(\mathtt{f},\mathtt{g}) &= \int\limits_{\mathbf{G}\times\mathbf{G}} \mathtt{f}(t)\mathtt{g}(s)^c d\mu \\
&= \sum_{i=1}^{\infty} t_i \Big[\int\limits_{\mathbf{G}} \chi_i(t)\mathtt{f}(t)d\mu(t)\Big]\Big[\int\limits_{\mathbf{G}} \chi_i(t)g(s)d\mu(s)\Big]^c.
\end{aligned}
$$

This completion of $L^1(\mathbf{G})$ with the inner product above is called $KS^2(\mathbf{G})$. A relationship of $KS^2(\mathbf{G})$ and $L^p(\mathbf{G})$ can be seen in the following theorem, whose proof is analogous to [13, Theorem 3.25]:

Theorem 2. *The space $L^p(\mathbf{G})$, $1 \leqslant p < \infty$, is dense in the space $KS^2(\mathbf{G})$.*

We will construct the norm of $KS^p(\mathbf{G})$ with the help of $L^p(\mathbf{G})$ as follows. Let $\mathtt{f} \in L^p(\mathbf{G})$ and define

$$
\|\mathtt{f}\|_{KS^p(\mathbf{G})} = \begin{cases} \left(\sum\limits_{i=1}^{\infty} t_i \left|\int\limits_{\mathbf{G}} \chi_i(t)\mathtt{f}(t)\, d\mu(t)\right|^p\right)^{\frac{1}{p}}, & \text{when } 1 \leqslant p < \infty, \\ \sup\limits_{i\geqslant 1} \left|\int\limits_{\mathbf{G}} \chi_i(t)\mathtt{f}(t)\, d\mu(t)\right|, & \text{when } p = \infty. \end{cases}
$$

It is easy to check that $\|\cdot\|_{KS^p(\mathbf{G})}$ is a norm in $L^p(\mathbf{G})$. Kuelbs-Steadman space on $\mathbf{G}$, represented by $KS^p(\mathbf{G})$, is the completion of $L^p(\mathbf{G})$ with respect to the aforementioned norm. One can see [13], [25], [27] and references therein for details of Kuelbs-Steadman spaces.

Theorem 3. **(Hölder-type inequality for KS^p space).**
Let $1 \leqslant p, q < \infty$, such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $\mathtt{f} \in KS^p(\mathbf{G})$ and $\mathtt{g} \in KS^q(\mathbf{G})$, then $\mathtt{f}g \in KS^1(\mathbf{G})$ and $\|\mathtt{fg}\|_{KS^1} \leqslant \|\mathtt{f}\|_{KS^p} \cdot \|\mathtt{g}\|_{KS^q}$.

Proof. To prove the inequality, we use the generalized form of arithmetic-geometric mean inequality: if $A, B \geqslant 0$, and $0 \leqslant \theta \leqslant 1$, then

$$
A^\theta B^\theta \leqslant \theta A + (1-\theta)B. \tag{2}
$$

If $\|\mathtt{f}\|_{KS^p(\mathbf{G})} = 0$ or $\|\mathtt{g}\|_{KS^q(\mathbf{G})} = 0$, then $\mathtt{fg} = 0$ a.e. and the inequality is obvious. So, we consider neither $\|\mathtt{f}\|_{KS^p(\mathbf{G})} = 0$ nor $\|g\|_{KS^q(\mathbf{G})} = 0$. Now, if we replace $\mathtt{f}$ by $\mathtt{f}/\|\mathtt{f}\|_{KS^p}$ and $\mathtt{g}$ by $\mathtt{g}/\|\mathtt{g}\|_{KS^q}$ and assume $\|\mathtt{f}\|_{KS^p(\mathbf{G})} = 1$ and $\|g\|_{KS^q(\mathbf{G})} = 1$, we need to show that $\|\mathtt{f}g\|_{KS^1} \leqslant 1$.

Setting $A = |\mathtt{f}(t)|^p$, $B = |\mathtt{g}(t)|^q$ and $\theta = 1/p$, so that $1-\theta = 1/q$, we get

$$|\mathtt{f}(t)g(t)| \leqslant \frac{1}{p}|\mathtt{f}(t)|^p + \frac{1}{q}|\mathtt{g}(t)|^q. \tag{3}$$

Now, using inequality (3),

$$\begin{aligned}\|\mathtt{fg}\|_{KS^1} &= \sum_{r=1}^{\infty} \tau_r \left| \int_{\mathbf{G}} \chi_r(t)\mathtt{f}(t)\mathtt{g}(t) d\mu(t) \right| \leqslant \sum_{r=1}^{\infty} \tau_r \int_{\mathbf{G}} \chi_r(t) \, |\mathtt{f}(t)\mathtt{g}(t)| \, d\mu(t) \\ &\leqslant \sum_{r=1}^{\infty} \tau_r \int_{\mathbf{G}} \chi_r(t) \Big(\frac{1}{p}|\mathtt{f}(t)|^p + \frac{1}{q}|\mathtt{g}(t)|^q \Big) d\mu(t).\end{aligned}$$

From this conclude that $\|\mathtt{fg}\|_{KS^1} \leqslant 1$, and this completes the proof. □

Theorem 4. *If $\mu(\mathbf{G}) < \infty$ and $1 \leqslant p < q$, then $KS^q(\mathbf{G}) \hookrightarrow KS^p(\mathbf{G})$, where $\hookrightarrow$ represents continuous embedding.*

Proof. Since $\dfrac{p}{q} + \dfrac{q-p}{q} = 1$, using Theorem 3 for $|\mathtt{f}|^p$ and identity function 1, we have

$$\begin{aligned}\|\mathtt{f}\|^p_{KS^p(\mathbf{G})} &= \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(t)\mathtt{f}(t) \, d\mu \right|^p \leqslant \sum_{i=1}^{\infty} t_i \int_{\mathbf{G}} \chi_i |\mathtt{f}(t)|^p \, d\mu \\ &\leqslant \Big(\sum_{i=1}^{\infty} t_i \left| \int_{G} \chi_i |\mathtt{f}(t)|^p \, d\mu \right|^{\frac{q}{p}} \Big)^{\frac{p}{q}} \cdot \Big(\sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i 1 \, d\mu \right|^{\frac{q}{q-p}} \Big)^{\frac{q-p}{q}}.\end{aligned}$$

This implies

$$\begin{aligned}\|\mathtt{f}\|_{KS^p(\mathbf{G})} &\leqslant \Big(\sum_{i=1}^{\infty} t_i \left| \int_{G} \chi_i \mathtt{f}(t) \, d\mu \right|^q \Big)^{\frac{1}{q}} \cdot (\mu(\mathbf{G}))^{\frac{q-p}{pq}} \\ &= \|\mathtt{f}\|_{KS^q(\mathbf{G})} \cdot (\mu(\mathbf{G}))^{\frac{q-p}{pq}}.\end{aligned}$$

Thus, $\|\mathtt{f}\|_{KS^p(\mathbf{G})} \leqslant C \, \|\mathtt{f}\|_{KS^q(\mathbf{G})}$, where $C = (\mu(\mathbf{G}))^{\frac{q-p}{pq}}$ is constant. This completes the proof. □

Next, to prove that KS^p is a Banach algebra, we define the convolution of two function $\mathtt{f}, \mathtt{g} \in KS^p(\mathbf{G})$ as

$$\mathtt{f} * \mathtt{g}(t) = \sum_{i=1}^{\infty} t_i \int_{\mathbf{G}} \chi_i \mathtt{f}(s)\mathtt{g}(s^- * t) \, d\mu(s), \tag{4}$$

where χ_i denotes the characteristic function on the countable basis B_i of $\mathbf{G}$ and t_i is sequence of real numbers, such that $\sum_{i=1}^{\infty} t_i = 1$.

Theorem 5. *$KS^p(\mathbf{G})$ is a Banach algebra with respect to the convolution (4).*

Proof. For $\mathtt{f}, \mathtt{g} \in KS^p(\mathbf{G})$, using the Fubini theorem [36, p. 87], we have

$$\begin{aligned}
\|\mathtt{f} * \mathtt{g}\|^p_{KS^p(\mathbf{G})} &= \sum_{i=1}^{\infty} \left| \int_{\mathbf{G}} \chi_i (\mathtt{f} * \mathtt{g})(t)\, d\mu(t) \right|^p \\
&= \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i \left(\sum_{i=1}^{\infty} t_i \int_{\mathbf{G}} \mathtt{f}(s) \mathtt{g}(s^- * t)\, d\mu(s) \right) d\mu(t) \right|^p \\
&\leqslant \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i \mathtt{f}(s)\, d\mu(s) \right|^p \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i\, \mathtt{g}(s^- * t)\, d\mu(s) \right|^p \\
&= \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i\, \mathtt{f}(s)\, d\mu(s) \right|^p \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i\, \mathtt{g}(z)\, d(\delta_s * \delta_t)(z) \right|^p.
\end{aligned}$$

This implies $\|\mathtt{f} * \mathtt{g}\|^p_{KS^p(\mathbf{G})} \leqslant \|\mathtt{f}\|^p_{KS^p(\mathbf{G})} \|g\|^p_{KS^p(\mathbf{G})}$, i. e., $\|\mathtt{f} * \mathtt{g}\|_{KS^p(\mathbf{G})} \leqslant \|\mathtt{f}\|_{KS^p(\mathbf{G})} \|\mathtt{g}\|_{KS^p(\mathbf{G})}$. Hence, $KS^p(\mathbf{G})$ is Banach algebra. $\square$

Theorem 6. *For a Gelfand pair $(\mathbf{G}, \mathbf{K})$ in the context of hypergroups, there exists a single unique positive measure π on $\widehat{\mathbf{G}^\natural}$, such that*

$$\sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(t) \mathtt{f}(t) d\mu(t) \right|^2 = \sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}^\natural}} \xi_i(\vartheta) \widehat{\mathtt{f}}(\vartheta) d\pi(\vartheta) \right|^2$$

for any $\mathtt{f} \in KS^1(\mathbf{G}) \cap KS^2(\mathbf{G})$.

Proof. The proof is analogous to that of [9, Theorem 3.11]. $\square$

4. HK-Sobolev spaces on Gelfand pair over Hypergroup. In this Section, we define HK-Sobolev space on Gelfand pair over a Hypergroup. Let us consider a second countable hypergroup $\mathbf{G}$ and Gelfand pair $(\mathbf{G}, \mathbf{K})$. We define a collection $KS^{2,\natural}(\mathbf{G})$ with respect to the Haar measure on $\mathbf{G}$ as follows:

$$KS^{2,\natural}(\mathbf{G}) = \{\mathtt{f} \colon \mathbf{G} \to \mathbb{C} \colon \mathtt{f} \in KS^2(\mathbf{G}), \mathtt{f} \text{ is } \mathbf{K}\text{-bi-invariant}\}.$$

Now, let us define

$$KS^2(\widehat{\mathbf{G}^\natural}) = \{\mathtt{f}\colon \widehat{\mathbf{G}^\natural} \to \mathbb{C}\colon \mathtt{f} \in KS^2(\widehat{\mathbf{G}^\natural})(\text{w.r.t the measure } \pi \text{ on } \widehat{\mathbf{G}^\natural})\}.$$

Note that Theorem 6 claims that the Fourier transform can be used to create an isometric isomorphism from $KS^{2,\natural}(\mathbf{G})$ to $KS^2(\widehat{\mathbf{G}^\natural})$.

Definition 1. *Consider a hypergroup Gelfand pair $(\mathbf{G}, \mathbf{K})$ and let $\zeta : \widehat{\mathbf{G}^\natural} \to \mathbb{R}^+$ be a measurable function, and $\alpha > 0$. The HK-Sobolev space over hypergroup Gelfand pair, denoted by $HK_\zeta^{\alpha,\natural}(\mathbf{G})$, is defined as the set of all functions $\mathtt{f} \in KS^{2,\natural}(\mathbf{G})$, such that*

$$\sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}^\natural}} \left(1 + \zeta(\vartheta)^2\right)^\alpha \xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta)d\pi(\vartheta)\right|^2 < \infty,$$

where ξ_i is the characteristic function on the countable basis $\{\widehat{B_i^\natural}\}$ of $\widehat{\mathbf{G}^\natural}$. We define the norm in $HK_\zeta^{\alpha,\natural}(\mathbf{G})$ as follows:

$$\|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})} = \left(\sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}^\natural}} \left(1 + \zeta(\vartheta)^2\right)^\alpha \xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta)d\pi(\vartheta)\right|^2\right)^{\frac{1}{2}}. \tag{5}$$

Theorem 7. *For a hypergroup Gelfand pair $(\mathbf{G}, \mathbf{K})$, the space $\left(HK_\zeta^{\alpha,\natural}(\mathbf{G}), \|\cdot\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})}\right)$ is a complete normed space.*

Proof. Let us define the mapping $\mathtt{f} \longmapsto (1+\zeta(\cdot)^2)^\alpha\widehat{\mathtt{f}}(\cdot)$ from $HK_\zeta^{\alpha,\natural}(\mathbf{G})$ to $KS^2(\widehat{\mathbf{G}^\natural})$. Then one can easily check that it is an isometric isomorphism. Since $KS^2(\widehat{\mathbf{G}^\natural})$ is complete, therefore, $HK_\zeta^{\alpha,\natural}(\mathbf{G})$ is a Banach space. □

Next, we investigate several embedding results on $HK_\zeta^{\alpha,\natural}(\mathbf{G})$.

Theorem 8. *Assume that $(\mathbf{G}, \mathbf{K})$ is a Gelfand pair hypergroup. So, $HK_\zeta^{\alpha,\natural}(\mathbf{G})$ is embedded in $KS^{2,\natural}(\mathbf{G})$ in the continuous sense.*

Proof. If $\mathtt{f} \in HK_\zeta^{\alpha,\natural}$, then using the Theorem 6 we have

$$\|\mathtt{f}\|_{KS^{2,\natural}(\mathbf{G})} = \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(t)\mathtt{f}(t)d\mu(t)\right|^2 = \sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}^\natural}} \xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta)d\pi(\vartheta)\right|^2$$

$$\leqslant \sum_{i=1}^{\infty} t_i \Big| \int_{\widehat{\mathbf{G}}^{\natural}} \left(1+\zeta(\vartheta)^2\right)^{\alpha} \xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta)d\pi(\vartheta)\Big|^2 = \|\mathtt{f}\|_{HK_{\zeta}^{\alpha,\natural}(\mathbf{G})}.$$

Hence, $HK_{\zeta}^{\alpha,\natural}(\mathbf{G})$ is continuously embedded in $KS^{2,\natural}(\mathbf{G})$. □

Theorem 9. *For a hyrpergroup Gelfand pair* $(\mathbf{G},\mathbf{K})$*,* $HK_{\zeta}^{\alpha,\natural}(\mathbf{G})$ *is continuously embedded into* $HK_{\zeta}^{\beta,\natural}(\mathbf{G})$ *whenever* $\alpha>\beta>0$*.*

Proof. Clearly, $(1+\zeta(\vartheta)^2)\geqslant 1$, so, if $\alpha>\beta$, then

$$(1+\zeta(\vartheta)^2)^{\alpha}\geqslant(1+\zeta(\vartheta)^2)^{\beta}$$

and, consequently, $\|\mathtt{f}\|_{HK_{\zeta}^{\alpha,\natural}}\geqslant\|\mathtt{f}\|_{HK_{\zeta}^{\beta,\natural}}$. □

Lemma 1. *Let* $1\leqslant p\leqslant 2$ *and* $\frac{1}{p}+\frac{1}{q}=1$*; then*

(i) If $\mathtt{f}\in KS^p(\mathbf{G})$*, then* $\widehat{\mathtt{f}}\in KS^q(\widehat{\mathbf{G}})$ *and* $\|\widehat{\mathtt{f}}\|_{KS^q(\widehat{\mathbf{G}})}\leqslant\|\mathtt{f}\|_{KS^p(\mathbf{G})}$*.*

(ii) If $\mathtt{f}\in KS^q(\mathbf{G})$*, then* $\widehat{\mathtt{f}}\in KS^p(\widehat{\mathbf{G}})$*,* $\check{\widehat{\mathtt{f}}}=\mathtt{f}\in KS^q(\mathbf{G})$*, and* $\|\mathtt{f}\|_{KS^q(\mathbf{G})}\leqslant\|\widehat{\mathtt{f}}\|_{KS^p(\widehat{\mathbf{G}})}$*.*

Proof.

(i) Suppose $\mathtt{f}\in C_c(\mathbf{G})$. We have, by using the similar approach as in [11, Proposition 3.3], this:

$$\left(\sum_{i=1}^{\infty}\Big|\int_{\widehat{\mathbf{G}}}\xi_i(\vartheta)\widehat{\mathtt{f}^{\natural}}(\vartheta)d\mu(\vartheta)\Big|^q\right)^{\frac{1}{q}}=\left(\sum_{i=1}^{\infty}t_i\Big|\int_{\widehat{\mathbf{G}//\mathbf{K}}}\xi_{i,\mathbf{K}}^{*}(\vartheta)\widehat{\tilde{\mathtt{f}}^{\natural}}(\tilde{\vartheta})d\tilde{\pi}(\tilde{\vartheta})\Big|^q\right)^{\frac{1}{q}}$$

$$=\left(\sum_{i=1}^{\infty}t_i\Big|\int_{\widehat{\mathbf{G}//\mathbf{K}}}\xi_{i,\mathbf{K}}^{*}(\vartheta)\widehat{\tilde{\mathtt{f}}^{\natural}}(\tilde{\vartheta})d\tilde{\pi}(\tilde{\vartheta})\Big|^q\right)^{\frac{1}{q}}$$

$$\leqslant\left(\sum_{i=1}^{\infty}t_i\Big|\int_{\mathbf{G}//\mathbf{K}}\xi_{i,\mathbf{K}}\tilde{\mathtt{f}}^{\natural}(\mathbf{K}t\mathbf{K})dm(\mathbf{K}t\mathbf{K}))\Big|^p\right)^{\frac{1}{p}}$$

$$=\left(\sum_{i=1}^{\infty}t_i\Big|\int_{\mathbf{G}}\xi_i(t)\mathtt{f}^{\natural}(t)d\mu(t)\Big|^p\right)^{\frac{1}{p}}\leqslant\|\mathtt{f}\|_{KS^p(\mathbf{G})},$$

where $\xi_{i,\mathbf{K}}^{*}$ and $\xi_{i,\mathbf{K}}$ represent the characteristic function on the countable basis $\{\widehat{B_i//\mathbf{K}}\}$ and $\{B_i//\mathbf{K}\}$ of $\widehat{\mathbf{G}//\mathbf{K}}$ and $\mathbf{G}//\mathbf{K}$, respectively. Hence, $\|\widehat{\mathtt{f}}\|_{KS^q(\widehat{\mathbf{G}})}=\|\widehat{\mathtt{f}^{\natural}}\|_{KS^q(\widehat{\mathbf{G}})}\leqslant\|\mathtt{f}\|_{KS^p(\mathbf{G})}$.

(ii) If $\mathtt{f} \in C_c(\mathbf{G})$, then $\widehat{\mathtt{f}} \in C_c(\widehat{G})$, $\check{\widehat{\mathtt{f}}} \in C_b^{\natural}(\mathbf{G})$ and $\tilde{\widehat{\mathtt{f}}} \in C_c(\widehat{\mathbf{G}//\mathbf{K}})$. So, $\check{\tilde{\widehat{\mathtt{f}}}}$ and $\tilde{\check{\widehat{\mathtt{f}}}}$ are in $C_b(\mathbf{G}//\mathbf{K})$. It is shown in [11] that $\check{\tilde{\widehat{\mathtt{f}}}} = \tilde{\check{\widehat{\mathtt{f}}}}$. Now, we have

$$\begin{aligned}
\|\check{\widehat{\mathtt{f}}}\|_{KS^q(\mathbf{G})} &= \left\{ \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(t) \check{\widehat{\mathtt{f}}}(t) d\mu(t) \right|^q \right\}^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{\infty} \left| \int_{\mathbf{G}//\mathbf{K}} \xi_{i,\mathbf{K}} \tilde{\check{\widehat{\mathtt{f}}}}(\mathbf{K}t\mathbf{K}) dm(\mathbf{K}t\mathbf{K}) \right|^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{\infty} \left| \int_{\mathbf{G}//\mathbf{K}} \xi_{i,\mathbf{K}} \check{\tilde{\widehat{\mathtt{f}}}}(\mathbf{K}t\mathbf{K}) dm(\mathbf{K}t\mathbf{K}) \right|^q \right)^{\frac{1}{q}} \\
&= \|\check{\tilde{\widehat{\mathtt{f}}}}\|_{KS^q(\mathbf{G}//\mathbf{K})} \leqslant \|\tilde{\widehat{\mathtt{f}}}\|_{KS^p(\widehat{\mathbf{G}//\mathbf{K}})} \\
&= \left(\sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}//\mathbf{K}}} \xi_{i,\mathbf{K}}^* \tilde{\widehat{\mathtt{f}}}(\tilde{\vartheta}) d\tilde{\pi}(\tilde{\vartheta}) \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}}} \xi_i^* \widehat{\mathtt{f}}(\vartheta) d\pi(\vartheta) \right|^p \right)^{\frac{1}{p}} = \|\widehat{\mathtt{f}}\|_{KS^p(\widehat{\mathbf{G}})} \text{ (see [9]).}
\end{aligned}$$

Thus, $\|\mathtt{f}\|_{KS^q(\mathbf{G})} \leqslant \|\widehat{\mathtt{f}}\|_{KS^p(\widehat{\mathbf{G}})}$.

The proof is completed. □

Theorem 10. *Consider a hypergroup Gelfand pair* $(\mathbf{G}, \mathbf{K})$. *Let* $\beta > > \alpha > 0$ *and* $p = \dfrac{2\beta}{\beta + \alpha}$ *and let* q *be such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *If* $(1 + \zeta^2)^{-1} \in \in KS^{\beta}(\widehat{\mathcal{G}^{\natural}})$, *then* $HK_{\zeta}^{\alpha,\natural}(\mathbf{G})$ *is continuously embedded in* $KS^{q,\natural}(\mathbf{G})$.

Proof. Since $\beta > \alpha > 0$ and $p = \dfrac{2\beta}{\beta + \alpha}$, so it implies that $1 < p < 2$ and $\beta = \dfrac{\alpha p}{2 - p}$. Let $\mathtt{f} \in HK_{\zeta}^{\alpha,\natural}$. Then

$$\|\widehat{\mathtt{f}}\|^p_{KS^p(\widehat{G^{\natural}})} = \sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}^{\natural}}} \xi_i \widehat{\mathtt{f}}(\vartheta) d\pi(\vartheta) \right|^p \leqslant \sum_{i=1}^{\infty} t_i \int_{\widehat{\mathbf{G}^{\natural}}} \xi_i |\widehat{\mathtt{f}}(\vartheta)|^p \, d\pi(\vartheta)$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} t_i \int\limits_{\widehat{\mathbf{G}^\natural}} \xi_i |\widehat{\mathtt{f}}(\vartheta)|^p (1+\zeta(\vartheta)^2)^{\alpha p} (1+\zeta(\vartheta)^2)^{-\alpha p}\, d\pi(\vartheta) \\
&\leqslant \Bigg\{ \Bigg(\sum_{i=1}^{\infty} t_i \left| \xi_i |\widehat{\mathtt{f}}(\vartheta)|^p (1+\zeta(\vartheta)^2)^{\alpha p}\, d\pi(\vartheta) \right|^{\frac{2}{p}} \Bigg)^{\frac{p}{2}} \\
&\qquad \times \Bigg(\sum_{i=1}^{\infty} t_i \left| \xi_i (1+\zeta(\vartheta)^2)^{-\alpha p}\, d\pi(\vartheta) \right|^{\frac{2}{2-p}} \Bigg)^{\frac{2-p}{2}} \Bigg\}.
\end{aligned}
$$

As $\dfrac{p}{2} + \dfrac{2-p}{2} = 1$, we use Theorem 3 and we get

$$
\begin{aligned}
\|\widehat{\mathtt{f}}\|_{KS^p(\widehat{G^\natural})} &\leqslant \Bigg\{ \Bigg(\sum_{i=1}^{\infty} t_i \left| \xi_i \widehat{\mathtt{f}}(\vartheta) (1+\zeta(\vartheta)^2)^{\alpha}\, d\pi(\vartheta) \right|^{2} \Bigg)^{\frac{p}{2}} \\
&\qquad \times \Bigg(\sum_{i=1}^{\infty} t_i \left| \xi_i (1+\zeta(\vartheta)^2)^{-2}\, d\pi(\vartheta) \right|^{\frac{\alpha p}{2-p}} \Bigg)^{\frac{2-p}{2p}} \Bigg\}.
\end{aligned}
$$

That implies

$$
\|\widehat{\mathtt{f}}\|_{KS^p(\widehat{G^\natural})} \leqslant \|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}} \cdot \|(1+\zeta^2)\|_{KS^\beta}^{\frac{\alpha}{2}}, \quad \text{where } \beta = \frac{\alpha p}{2-p}.
$$

Now, by Lemma 1 we have $\|\mathtt{f}\|_{KS^q} \leqslant \|\widehat{\mathtt{f}}\|_{KS^p}$. Therefore, we get

$$
\|\mathtt{f}\|_{KS^q} \leqslant \|\widehat{\mathtt{f}}\|_{KS^p} \leqslant \|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}} \cdot \|(1+\zeta^2)\|_{KS^\beta}^{\frac{\alpha}{2}}.
$$

This implies that $HK_\zeta^{\alpha,\natural}(\mathbf{G})$ is continuously embedded in $KS^{q,\natural}(\mathbf{G})$. □

Theorem 11. *Consider a hypergroup Gelfand pair $(\mathbf{G},\mathbf{K})$. If $\widehat{\mathbf{G}^\natural}$ is uniformly bounded, $(1+\zeta(\cdot)^2)^{-\alpha} \in KS^2(\widehat{\mathbf{G}^\natural})$ for some $\alpha > 0$, and $\mathtt{f} \in HK_\zeta^{\alpha,\natural}(\mathbf{G})$, then $\mathtt{f}$ is bounded and there exists a constant $C > 0$ that depends on α and ζ, such that*

$$
\|\mathtt{f}\|_{KS^\infty(\mathbf{G})} \leqslant C \|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})}.
$$

Proof. Using the definition of inverse Fourier transformation, we have

$$
|\mathtt{f}(t)| = \left| \int\limits_{\widehat{G^\natural}} \vartheta(t)\widehat{\mathtt{f}}(\vartheta)\, d\pi(\vartheta) \right| \leqslant \sup_{\vartheta \in \widehat{G^\natural}} |\vartheta(t)| \cdot \left| \int\limits_{\widehat{G^\natural}} \widehat{\mathtt{f}}(\vartheta)\, d\pi(\vartheta) \right|.
$$

This implies that

$$\left|\int\limits_{\mathbf{G}} \chi_i \mathtt{f}(t) d\mu(t)\right| \leqslant \sup_{\vartheta \in \widehat{G^\natural}} |\vartheta(t)| \left\{ \sum_{i=1}^{\infty} t_i \left| \int\limits_{\widehat{\mathbf{G}^\natural}} \xi_i \widehat{\mathtt{f}}(\vartheta) \frac{(1+\zeta(\vartheta)^2)^\alpha}{(1+\zeta(\vartheta)^2)^\alpha} d\pi(\vartheta) \right| \right\}$$

$$\leqslant \sup_{\vartheta \in \widehat{G^\natural}} |\vartheta(t)| \left\{ \left(\sum_{i=1}^{\infty} t_i \left| \int\limits_{\widehat{\mathbf{G}^\natural}} \xi_i \widehat{\mathtt{f}}(\vartheta) \left(1+\zeta(\vartheta)^2\right)^\alpha d\pi(\vartheta) \right|^2 \right)^{\frac{1}{2}} \right.$$

$$\left. \times \left(\sum_{i=1}^{\infty} t_i \left| \int\limits_{\widehat{\mathbf{G}^\natural}} \xi_i \left(1+\zeta(\vartheta)^2\right)^{-\alpha} d\pi(\vartheta) \right|^2 \right)^{\frac{1}{2}} \right\}$$

$$= \sup_{\vartheta \in \widehat{G^\natural}} |\vartheta(t)| \|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})} \|(1+\zeta(\vartheta)^2)^{-\alpha}\|_{KS^2(\hat{\mathbf{G}})},$$

due to Hölder-type inequality for KS^p space, Theorem 3. As $\widehat{\mathbf{G}^\natural}$ is uniformly bounded, we get

$$\sup_{t \in \mathbf{G}} \sup_{\vartheta \in \widehat{\mathbf{G}^\natural}} |\vartheta(t)| < \infty.$$

Hence,

$$\sup_{i \geqslant 1} \left| \int\limits_{\mathbf{G}} \chi_i \mathtt{f}(t)\, d\mu(t) \right| \leqslant \sup_{t \in \mathbf{G}} \sup_{\vartheta \in \widehat{G^\natural}} |\vartheta(t)| \|(1+\zeta(\vartheta)^2)^{-\alpha}\|_{KS^2(\hat{\mathbf{G}})} \|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})} < \infty.$$

Taking $C = \sup_{t \in \mathbf{G}} \sup_{\vartheta \in \widehat{G^\natural}} |\vartheta(t)| \|(1+\zeta(\vartheta)^2)^{-\alpha}\|_{KS^2(\hat{\mathbf{G}})}$, we obtain the required result. □

Theorem 12. *Assume that* $(\mathbf{G}, \mathbf{K})$ *is a hypergroup Gelfand pair and* $\widehat{\mathbf{G}^\natural}$ *is equicontinuous. Let* $(1+\zeta(\cdot)^2)^{-\alpha} \in KS^2(\widehat{\mathbf{G}^\natural})$; *then* $\mathtt{f} \in HK_\zeta^{\alpha,\natural}(\mathbf{G}) \Rightarrow \mathtt{f}$ *is continuous.*

Proof. Let us consider an arbitrary element $t_0 \in \mathbf{G}$. Since $\widehat{\mathbf{G}^\natural}$ is equicontinuous, so for any $\epsilon > 0$ there exist a neighbourhood $\mathtt{U}$ of t_0, such that $|\vartheta(t) - \vartheta(t_0)| < \epsilon$ for all $\vartheta \in \widehat{\mathbf{G}^\natural}$ and all $t \in \mathbf{G}$. Let $\mathtt{f} \in HK_\zeta^{\alpha,\natural}$, then, using Theorem 3,

$$|\mathtt{f}(t) - \mathtt{f}(t_0)| = \left| \int\limits_{\mathbf{G}^\natural} \vartheta(t) \widehat{\mathtt{f}}(\vartheta)\, d\pi(\vartheta) - \int\limits_{\mathbf{G}^\natural} \vartheta(t_0) \widehat{\mathtt{f}}(\vartheta)\, d\pi(\vartheta) \right|$$

$$
\begin{aligned}
&\leqslant \int\limits_{\widehat{\mathbf{G}^\natural}} |\vartheta(t)-\vartheta(x_0)|\cdot\left|\widehat{\mathtt{f}}(\vartheta)\right|\,d\pi(\vartheta) \leqslant \epsilon\sum_{i=1}^{\infty} t_i\left|\int\limits_{\widehat{\mathbf{G}^\natural}} \xi_i\widehat{\mathtt{f}}(\vartheta)\,d\pi(\vartheta)\right| \\
&= \epsilon\sum_{i=1}^{\infty} t_i\left|\int\limits_{\widehat{\mathbf{G}^\natural}} (1+\zeta(\vartheta)^2)^{\alpha}\xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta)(1+\zeta(\vartheta)^2)^{-\alpha}\,d\pi(\vartheta)\right| \\
&\leqslant \epsilon\Bigg\{\left(\sum_{i=1}^{\infty} t_i\left|\int\limits_{\widehat{\mathbf{G}^\natural}} (1+\zeta(\vartheta)^2)^{\alpha}\xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta)\,d\pi(\vartheta)\right|^2\right)^{\frac{1}{2}}\times \\
&\qquad\qquad \left(\sum_{i=1}^{\infty} t_i\left|\int\limits_{\widehat{\mathbf{G}^\natural}} \xi_i(\vartheta)(1+\zeta(\vartheta)^2)^{-\alpha}\,d\pi(\vartheta)\right|^2\right)^{\frac{1}{2}}\Bigg\}.
\end{aligned}
$$

So, $|\mathtt{f}(t)-\mathtt{f}(t_0)| \leqslant \epsilon\cdot\|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})}\|(1+\zeta(\vartheta)^2)^{-\alpha}\|_{KS^2(\widehat{\mathbf{G}^\natural})}$. This implies that $\mathtt{f}$ is continuous. □

5. Rellich-Kondrashov-type theorem on $HK_\zeta^{\alpha,\natural}(\mathbf{G})$. In the context of Sobolev spaces, the Rellich-Kondrashov theorem is a compact embedding theorem. It is named after two mathematicians: Vladimir Iosifovich Kondrashov, a Russian mathematician, and Franz Rellich, an Austrian-German mathematician. Kondrashov expanded the theorem to include L^p spaces, whereas Rellich was the first to establish the theorem for L^2 (see [28], [32]). In this section, we establish Rellich-Kondrashov-type theorem on the HK-Sobolev space over hypergroup $HK_\zeta^{\alpha,\natural}(\mathbf{G})$. To prove the main theorem (Theorem 18) of this section, we begin the section with the following theorem.

Theorem 13. *For a Hypergroup Gelfand pair $(\mathbf{G},\mathbf{K})$, if $\mathtt{f}\in HK_\zeta^{\alpha,\natural}(\mathbf{G})$ and $s\in\mathbf{G}$, then*

$$
\begin{aligned}
&\sum_{i=1}^{\infty} t_i\left|\int\limits_{\mathbf{G}} \chi_i(t)\left(\mathtt{f}(t*s^-)-\mathtt{f}(t)\right)\,dx\right|^2 \leqslant \\
&\qquad\qquad \leqslant \left(\left|\sup_{\vartheta\in\widehat{G^\natural}} \frac{(\vartheta(s^-)-1)}{(1+\zeta(\vartheta)^2)^{\alpha}}\right|^2\right)\cdot\|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})},
\end{aligned}
$$

where χ_i denotes the characteristic function on the countable basis $\{B_i\}$ of $\mathbf{G}$.

Proof. Let us define $f_s(t) = \mathtt{f}(t * s^-)$ for a fixed $s \in \mathbf{G}$. Then

$$\begin{aligned}\widehat{\mathtt{f}}_s(\vartheta) &= \int_{\mathbf{G}} \vartheta(t^-)\mathtt{f}(t * s^-)\, dt = \\ &= \int_{\mathbf{G}} \vartheta(s^- * t^-)\mathtt{f}(t)\, dt = \quad (\text{changing variable from } t \to t * s^-) \\ &= \int_{\mathbf{G}} \vartheta(s^- * t)\mathtt{f}(t^-)\, dt = \quad (\text{changing variable from } t \to t^-) \\ &= (\vartheta * f)(s^-) = (f * \vartheta)(s^-) = \widehat{\mathtt{f}}(\vartheta)\vartheta(s^-).\end{aligned}$$

Next, using Theorem 6, we have

$$\begin{aligned}\sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(t) \left(\mathtt{f}(t * s^-) - \mathtt{f}(t)\right) dt \right|^2 &= \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(t) \left\{ f_y(t) - \mathtt{f}(t) \right\} dt \right|^2 \\ &= \sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}}^\natural} \xi_i(\vartheta) \left\{ \widehat{f}_y(\vartheta) - \widehat{\mathtt{f}}(\vartheta) \right\} d\pi(\vartheta) \right|^2 \\ &= \sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}}^\natural} \xi_i(\vartheta) \left\{ \widehat{\mathtt{f}}(\vartheta)\vartheta(s^-) - \widehat{\mathtt{f}}(\vartheta) \right\} d\pi(\vartheta) \right|^2 \\ &= \sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}}^\natural} \xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta) \left(\vartheta(s^-) - 1\right) d\pi(\vartheta) \right|^2 \\ &= \sum_{i=1}^{\infty} t_i \left| \int_{\widehat{\mathbf{G}}^\natural} \xi_i(\vartheta)\widehat{\mathtt{f}}(\vartheta)\, (1 + \zeta(\vartheta)^2)^\alpha \frac{(\vartheta(s^-) - 1)}{(1 + \zeta(\vartheta)^2)^\alpha}\, d\pi(\vartheta) \right|^2 \\ &\leqslant \left(\left| \sup_{\vartheta \in \widehat{\mathbf{G}}^\natural} \frac{(\vartheta(s^-) - 1)}{(1 + \zeta(\vartheta)^2)^\alpha} \right|^2 \right) \cdot \|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})}.\end{aligned}$$

The proof is completed. □

Theorem 14. *Consider a hypergroup Gelfand pair* $(\mathbf{G}, \mathbf{K})$. *If* $\mathtt{f} \in HK_\zeta^{\alpha,\natural}(\mathbf{G})$, *then there exists* $\theta \in C_c^\natural(\mathbf{G})$, *such that*

$$\|\mathtt{f} * \theta - \mathtt{f}\|_{KS^2(\mathbf{G})} \leqslant \sup_{s \in supp(\theta)} \left(\sup_{\vartheta \in \widehat{\mathbf{G}}^\natural} \frac{(\vartheta(s^-) - 1)}{(1 + \zeta(\vartheta)^2)^\alpha} \right) \cdot \|\mathtt{f}\|_{HK_\zeta^{\alpha,\natural}(\mathbf{G})}.$$

Proof. Since $\mathbf{G}$ is a locally compact Hausdorff space, it is also a Tychonoff space. This means that there exists a function θ in $C_c^{\natural}(\mathbf{G})$, such that $\theta(e) \neq 0$, $\theta \geqslant 0$, and $\int\limits_{\mathbf{G}} \theta(t) d\mu t = 1$. We have

$$\begin{aligned}
\|\mathtt{f} * \theta - \mathtt{f}\|_{KS^2(\mathbf{G})} &= \left(\sum_{i=1}^{\infty} t_i \left| \int\limits_{\mathbf{G}} \chi_i(t) \left(\mathtt{f} * \theta(t) - \mathtt{f}(t)\right) d\mu(t) \right|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{\infty} t_i \left| \int\limits_{\mathbf{G}} \chi_i(t) \left(\int\limits_{\mathbf{G}} \mathtt{f}(t * s^-)\theta(s)\, d\mu(s) - \mathtt{f}(t) \right) d\mu(t) \right|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{\infty} t_i \left| \int\limits_{\mathbf{G}} \chi_i(t) \left(\int\limits_{\mathbf{G}} \{\mathtt{f}(t * s^-) - \mathtt{f}(t)\}\theta(s)\, d\mu(s) \right) d\mu(t) \right|^2 \right)^{\frac{1}{2}} \\
&\leqslant \int\limits_{\mathbf{G}} \left(\sum_{i=1}^{\infty} t_i \left| \int\limits_{\mathbf{G}} \chi_i(t) \{\mathtt{f}(t * s^-) - \mathtt{f}(t)\}\theta(s)\, d\mu(t) \right|^2 \right)^{\frac{1}{2}} d\mu(s) \\
&\leqslant \int\limits_{\mathbf{G}} |\theta(s)| \left(\sum_{i=1}^{\infty} t_i \left| \int\limits_{\mathbf{G}} \chi_i(t) \{\mathtt{f}(t * s^-) - \mathtt{f}(t)\}\, d\mu(t) \right|^2 \right)^{\frac{1}{2}} d\mu(s) \\
&\leqslant \sup_{s \in supp(\theta)} \left(\sup_{\vartheta \in \widehat{\mathbf{G}}^{\natural}} \frac{(\vartheta(s^-) - 1)}{(1 + \zeta(\vartheta)^2)^{\alpha}} \right) \cdot \|\mathtt{f}\|_{HK_{\zeta}^{\alpha,\natural}(\mathbf{G})} \text{ (using Theorem 13).}
\end{aligned}$$

The proof is completed. □

Theorem 15. (Reverse Fatou's lemma for HK integrable function). Let $\mathbf{G}$ be a hypergroup with a Haar measure μ and (f_n) be a sequence of $\mu-$measurable complex-valued functions on $\mathbf{G}$. If there exists $g \in KS^p(\mathbf{G})$, such that $|f_n| \leqslant |g|$ $\mu-$almost everywhere for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \int\limits_{\mathbf{G}} f_n \, d\mu \leqslant \int\limits_{\mathbf{G}} \limsup_{n\to\infty} f_n \, d\mu.$$

Proof. To prove the theorem, apply the linearity of HK-integral and Fatou's lemma (see [13, Therorem 3.13]) to the sequence $g - f_n$ of HK integrable functions. □

Now, to prove the Theorem 17, we need Dominated Convergence Theorem for Kuelbs-Steadman spaces. Next theorem is about DCT on $KS^p(\mathbf{G})$.

Theorem 16. *Let* $\mathbf{G}$ *be a hypergroup with a Haar measure* μ, $1 \leqslant p < \infty$ *be a real number, and* (f_n) *be a sequence of* $\mu-$*measurable complex-valued functions that converges to a* $\mu-$*measurable function* $\mathtt{f}$. *Suppose that there exist a function* $g \in KS^p(\mathbf{G})$, *such that* $|f_n| \leqslant |g|$ $\mu-$*almost everywhere for all* $n \in \mathbb{N}$. *Then all* f_n *as well as* $\mathtt{f}$ *are in* $KS^p(\mathbf{G})$ *and the sequence* (f_n) *converges to* $\mathtt{f}$ *and* $\lim\limits_{n\to\infty} \int\limits_{\mathbf{G}} f_n \, d\mu = \int\limits_{\mathbf{G}} \mathtt{f} \, d\mu$.

Also,

$$\lim_{n\to\infty} \|\mathtt{f}_n - \mathtt{f}\|_{KS^p(\mathbf{G})} = 0.$$

Proof. Since $|\mathtt{f}_n| \leqslant |\mathtt{g}|$, $\mathtt{f}$ is the pointwise limit of the sequence $(\mathtt{f}_n)$, by linearity and monotonicity of HK integral, $\mathtt{f}$ is also dominated by g. Therefore, f_n as well as $\mathtt{f}$ are in $KS^p(\mathbf{G})$.

Now, $|\mathtt{f} - \mathtt{f}_n| \leqslant |\mathtt{f}| + |\mathtt{f}_n| \leqslant 2|\mathtt{g}|$ for all n, and since $\mathtt{f}_n$ converges to $\mathtt{f}$, we have

$$\limsup_{n\to\infty} |\mathtt{f} - \mathtt{f}_n| = 0.$$

By the linearity and monotonicity of HK integral, we have

$$\left| \int\limits_{\mathbf{G}} \mathtt{f} \, d\mu - \int\limits_{\mathbf{G}} f_n \, d\mu \right| = \left| \int\limits_{\mathbf{G}} (\mathtt{f} - \mathtt{f}_n) \, d\mu \right| \leqslant \int\limits_{\mathbf{G}} |\mathtt{f} - f_n| \, d\mu.$$

By reverse Fatou's lemma for HK integrable function, we have

$$\limsup_{n\to\infty} \int\limits_{\mathbf{G}} |f - \mathtt{f}_n| \, d\mu \leqslant \int\limits_{\mathbf{G}} \limsup_{n\to\infty} |\mathtt{f} - \mathtt{f}_n| \, d\mu = 0. \tag{6}$$

This it implies that

$$\lim_{n\to\infty} \int\limits_{\mathbf{G}} |\mathtt{f} - \mathtt{f}_n| \, d\mu = 0. \tag{7}$$

Therefore,

$$\lim_{n\to\infty} \left| \int\limits_{\mathbf{G}} \mathtt{f} \, d\mu - \int\limits_{\mathbf{G}} \mathtt{f}_n \, d\mu \right| \leqslant \lim_{n\to\infty} \int\limits_{\mathbf{G}} |\mathtt{f} - \mathtt{f}_n| \, d\mu = 0.$$

$$\implies \lim_{n\to\infty} \int\limits_{\mathbf{G}} \mathtt{f}_n \, d\mu = \int\limits_{\mathbf{G}} \mathtt{f} \, d\mu.$$

Moreover,

$$\lim_{n\to\infty}\|\mathtt{f}_n-\mathtt{f}\|^p_{KS^p(\mathbf{G})}=\lim_{n\to\infty}\sum_{i=1}^{\infty}t_i\left|\int_{\mathbf{G}}\chi_i(\mathtt{f}_n-\mathtt{f})\,d\mu\right|^p$$
$$\leqslant\sum_{i=1}^{\infty}t_i\int_{\mathbf{G}}\chi_i\lim_{n\to\infty}|\mathtt{f}_n-\mathtt{f}|\,d\mu=0\quad[\text{by }(7)].$$

So, $\lim_{n\to\infty}\|\mathtt{f}_n-\mathtt{f}\|_{KS^p(\mathbf{G})}=0.$ □

Theorem 17. *Assume that* $\mathbf{G}$ *is a compact hypergroup and* $(\mathbf{G},\mathbf{K})$ *is a Gelfand pair. Let* $1<p,p'<\infty$*, such that* $\frac{1}{p}+\frac{1}{p'}=1$*, if* (f_n) *is a sequence in* $KS^{p,\natural}(\mathbf{G})$ *that converges weakly to a function* $\mathtt{f}$*. Then the sequence* $(\mathtt{f}_n*\theta)$ *converges strongly to* $\mathtt{f}*\theta\in KS^{p',\natural}(\mathbf{G})$ *for every* $\theta\in C_c^\natural(\mathbf{G})$*.*

Proof. Since $(\mathtt{f}_n)$ converges weakly to $\mathtt{f}$, by [8, Proposition 3.5], $\exists$ a positive $M\in\mathbb{R}$, so that $\|\mathtt{f}\|_{KS^p(\mathbf{G})}<M$, $\|\mathtt{f}\|_{KS^p(\mathbf{G})}\leqslant M$. Now

$$|f_n*\theta(t)|=\left|\sum_{i=1}^{\infty}t_i\int_{\mathbf{G}}\chi_i f_n(s)\theta(s^-*t)\,ds\right|\leqslant\sum_{i=1}^{\infty}t_i\left|\int_{\mathbf{G}}\chi_i(s)f_n(s)\theta(s^-*t)\,ds\right|\leqslant$$
$$\leqslant\left(\sum_{i=}^{\infty}\left|\int_{\mathbf{G}}\chi_i f_n(s)\,ds\right|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{i=1}^{\infty}t_i\left|\chi_i\theta(s^-*t)\,ds\right|^{p'}\right)^{\frac{1}{p'}}=$$
$$=\|f_n\|_{KS^p(\mathbf{G})}.\|\theta\|_{KS^{p'}(\mathbf{G})}\leqslant M\|\theta\|_{KS^{p'}(\mathbf{G})}.$$

As $\mathbf{G}$ is compact, the constant function $t\mapsto M\|\theta\|_{KS^{p'}(\mathbf{G})}$ is HK-integrable and then from Theorem 16 we have

$$f_n*\theta(t)=\int_{\mathbf{G}}f_n(s)\theta(s^-*t)\,ds\xrightarrow{n\to\infty}\int_{\mathbf{G}}\mathtt{f}(s)\theta(s^-*t)\,ds=\mathtt{f}*\theta(t).$$

Again,

$$|\mathtt{f}_n*\theta(t)-\mathtt{f}*\theta(t)|$$
$$=\left|\sum_{i=1}^{\infty}t_i\int_{\mathbf{G}}\chi_i\mathtt{f}_n(s)\theta(s^-*t)\,ds-\sum_{i=1}^{\infty}t_i\int_{\mathbf{G}}\chi_i\mathtt{f}(s)\theta(s^-*t)\,ds\right|$$
$$=\left|\sum_{i=1}^{\infty}t_i\int_{\mathbf{G}}\chi_i(\mathtt{f}_n(s)-\mathtt{f}(s))\theta(s^-*t)\,ds\right|$$

$$\leqslant \sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(\mathtt{f}_n - \mathtt{f})(s)\theta(s^- * t)\, ds \right|$$

$$\leqslant \left(\sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i(\mathtt{f}_n - \mathtt{f})(s)\, ds \right|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i \theta(s^- * t)\, ds \right|^{p'} \right)^{\frac{1}{p'}}$$

(Using Holder inequality, Theorem 3))

$$\leqslant \|\mathtt{f}_n - \mathtt{f}\|_{KS^p(\mathbf{G})} \left(\sum_{i=1}^{\infty} t_i \left| \int_{\mathbf{G}} \chi_i \theta(z)\, d(\delta_{s^-} * \delta_x)(z) \right|^{p'} \right)^{\frac{1}{p'}} \leqslant 2M\|\theta\|_{KS^{p'}(\mathbf{G})}.$$

Again, by Theorem 16, we have

$$\lim_{n\to\infty} \|f_n * \theta - \mathtt{f} * \theta\|_{KS^{p'}(\mathbf{G})} = 0.$$

Thus, the proof is concluded. □

The following is the Rellich-Kondrashov-type theorem associated with the function space $HK_\zeta^{\alpha,\natural}(\mathbf{G})$.

Theorem 18. *Let* $\mathbf{G}$ *be a compact hypergroup and* $(\mathbf{G}, \mathbf{K})$ *be a hypergroup Gelfand pair. Assume that* $\beta > \alpha > 0$, $p = \dfrac{2\beta}{\beta + \alpha}$, *and* p' *be such that* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. *If* $(1 + \zeta^2)^{-1} \in KS^\beta(\widehat{\mathcal{G}}^\natural)$ *and*

$$\lim_{s\to e} \left(\sup_{\vartheta \in \widehat{\mathcal{G}}^\natural} \frac{|\vartheta(s) - 1}{(1 + \zeta(\vartheta)^2)^{\frac{\alpha}{2}}} \right) = 0,$$

then, for every $1 \leqslant q \leqslant p'$,

$$HK_\zeta^{\alpha,\natural}(\mathbf{G}) \hookrightarrow KS^{q,\natural}(\mathbf{G}),$$

where $\hookrightarrow$ *represents compact embedding.*

Proof. We have already proved above in the Theorem 10 that $HK_\zeta^{\alpha,\natural}(\mathbf{G})$ is continuously embedded in $KS^{p',\natural}(\mathbf{G})$. Since $\mathbf{G}$ is compact and $q < p'$, by Theorem 4 we have $KS^{p',\natural}(\mathbf{G})$ is continuously embedded in $KS^{q,\natural}(\mathbf{G})$. All this indicates that $HK_\zeta^{\alpha,\natural}$ is continuously embedded in $KS^{p',\natural}(\mathbf{G})$.

Now, suppose that $\{\mathtt{f}_n\}$ is a bounded sequence in $HK^{\alpha,\natural}(\mathbf{G})$; then $\{\mathtt{f}_n\}$ is a bounded sequence in $KS^{p',\natural}(\mathbf{G})$. So, $\exists$ a positive real number k, such that $\|\mathtt{f}_n\|_{KS^{p'}(\mathbf{G})} \leqslant k$.

Since for all $h \in KS^{p\natural}(\mathbf{G})$, we have

$$|\langle \mathtt{f}_n\, , \, h\rangle| \leqslant \|\mathtt{f}_n\|_{KS^{p'}} \, \|h\|_{KS^p} \leqslant k\|h\|_{KS^p}.$$

Thus, $\{\mathtt{f}_n\}$ is weakly bounded and this implies that there exist a subsequence $\{a_n\}$ that converges weakly to $a \in KS^{p'}(\mathbf{G})$. Let $\epsilon > 0$ be arbitrary and $\eta \in C_c^{\natural}(\mathbf{G})$ be such that $\|a * \eta - a\|_{KS^2} < \epsilon$.

Now, using Theorem 14 and Theorem 17, we get

$$\begin{aligned}\|a_n - a\|_{KS^2} &\leqslant \|a_n - a_n * \eta\|_{KS^2} + \|a_n * \eta - a * \eta\|_{KS^2} + \|a * \eta - a\|_{KS^2} \\ &\leqslant \sup_{s \in supp(\eta)} \left(\sup_{\vartheta \in \widehat{\mathcal{G}}^{\natural}} \frac{|\vartheta(s) - 1}{(1 + \zeta(\vartheta)^2)^{\frac{\alpha}{2}}} \right) \cdot \|a_n\|_{HK_{\zeta}^{\alpha,\natural}} + \|a_n * \eta - a * \eta\|_{KS^2} + \epsilon \\ &\leqslant 2\epsilon + \|a_n * \eta - a * \eta\|_{KS^2}.\end{aligned}$$

Since ϵ is arbitrary, $\|a_n - a\|_{KS^2} \leqslant \|a_n * \eta - a * \eta\|_{KS^2}$. Thus,

$$\lim_{n\to\infty} \|a_n - a\|_{KS^2} = \lim_{n\to\infty} \|a_n * \eta - a * \eta\|_{KS^2} = 0.$$

Therefore, $\{a_n\}$ converges to a in $KS^{2,\natural}(\mathbf{G})$. Since $q \geqslant 2$ $KS^{q,\natural}(\mathbf{G})$ embedded in $KS^{2,\natural}(\mathbf{G})$; hence $\{a_n\}$ converges to a in $KS^{q,\natural}(\mathbf{G})$. This leads to the conclusion. $\square$

Acknowledgment. The authors would like to express their sincere gratitude to the anonymous reviewers for their valuable comments and constructive suggestions, which have significantly improved the quality and clarity of this work.

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Received December 2, 2024.
In revised form, May 18, 2025.
Accepted May 23, 2025.
Published online June 15, 2025.

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