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S. I. Kalinin, L. V. Pankratova

A SIMPLE PROOF OF THE DAMASCUS INEQUALITY

Abstract. The article is devoted to the presentation of a new proof of the so-called Damascus inequality, formulated in 2016 by Fozi M. Dannan, Professor of the Department of Fundamental Sciences of the Arab International University (Damascus, Syria). The presented proof is based on the apparatus of applying the derivative of a function of one real variable, as well as elementary methods of estimating quantities, including the use of classical inequalities. An important element of the proof is the appeal to the properties of a strictly GA-convex (-concave) function on an interval. The implementation of this method of proof also made it possible to describe the conditions for achieving equality in the inequality under consideration.

Key words: *Damascus inequality, geometric mean, logarithmic mean, GA-convex (-concave) function*

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Brief introduction. In 2016, Syrian professor Fozi M. Dannan gave the following inequality:

$$\frac{x-1}{x^2-x+1}+\frac{y-1}{y^2-y+1}+\frac{z-1}{z^2-z+1}\leqslant 0, \tag{1}$$

where x, y, z are positive numbers satisfying the condition $xyz=1$.

There are several known ways to justify this inequality. In particular, in the work [2], the authors considered proofs using symmetric functions, geometric representations, and the Lagrange multiplier method. In this note, as part of the magazine's feedback with its readership, we consider another way to prove inequality (1); but first, for the reader's convenience, we provide the necessary auxiliary information.

Definition 1. *The logarithmic mean $L(a,b)$ of two positive numbers a and b is* [1]

$$L = L\left(a,b\right) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b. \end{cases} \qquad (2)$$

Its main property is expressed by the inequality

$$G < L < A, \quad a \neq b, \qquad (3)$$

where $A = A(a,b) = \frac{a+b}{2}$ and $G = G(a,b) = \sqrt{ab}$ are the arithmetic mean and the geometric mean of the numbers a and b, respectively. Inequality (3) was established in [4] (problem 3.6.15, p. 272; problem 3.6.17, p. 273). Moreover, it follows from the result of Tang-Po Lin [3]: $L < M_p$ for all $p \geqslant 1/3$ and $L > M_p$ for all $p \leqslant 0$, where $M_p = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$ In the cited article it is shown that the given result is exact in the sense that for other p the indicated inequalities, generally speaking, are not true for all distinct positive numbers a and b.

***GA*-convex functions.** Based on the works [5], we introduce the concepts of GA-convex and GA-concave functions.

Let $l \subseteq (0, +\infty)$ be an arbitrary interval on the real axis Ox and $f\colon l \to \mathbf{R}$ be a function defined on this interval.

A function f is called GA-convex on l if for any $a, b \in l$ and any $\lambda \in [0;1]$ the inequality

$$f\left(a^{\lambda} b^{1-\lambda}\right) \leqslant \lambda f(a) + (1-\lambda) f(b) \qquad (4)$$

is satisfied.

If under the conditions of the given definition the inequality

$$f\left(a^{\lambda} b^{1-\lambda}\right) \geqslant \lambda f(a) + (1-\lambda) f(b) \qquad (5)$$

is satisfied, then the function f is called GA-concave on l.

Moreover, if for any distinct a,$b \in l$ and any $\lambda \in (0;1)$ inequality (4) (respectively inequality (5)) is strict, then we call the function f strictly GA-convex (strictly GA-concave) on the interval under consideration.

It is easy to see that the function $f(x)$ is GA-convex on the interval l if and only if the function $F(t) = f(e^t)$ is convex on the interval l.

This fact allows us to simply justify the following statement about sufficient conditions for strict GA-convexity (-concavity) of a function on an interval.

Theorem 1. *Let $f(x)$ be continuous on the interval l, $l \subseteq (0, +\infty)$, and twice differentiable inside it. Further, let $\Delta_f(x) = f'(x) + xf''(x)$, $x \in l^0$, where l^0 is the interior of l.*

If the condition $\Delta_f(x) > 0$ is satisfied, then the function $f(x)$ is strictly GA-convex on the interval l. If $\Delta_f(x) < 0$, then the function $f(x)$ is strictly GA-concave on the interval under consideration.

Remark. *In the above-cited article [5], GA-convexity of a twice-differentiable function $f(x)$ is characterized in terms of the non-negativity of the quantity $x^2 f''(x) + xf'(x)$; the author emphasizes that all twice-differentiable non-decreasing convex functions are also GA-convex.*

Let us proceed to the proof of inequality (1).

Proof. Let us introduce the function

$$f(x) = \frac{x-1}{x^2-x+1} - \ln x, \quad x > 0.$$

Since

$$f'(x) = \frac{(x-1)(1-x-x^3)}{x(x^2-x+1)^2},$$

it has (due to the decrease of the function $h(x) = 1 - x - x^3$, $x > 0$) no more than two critical points. Let us show that there are exactly two of them.

The first of them $x_1 = 1$ is easy to find. Denote the second $x_2 = a$ and determine its approximate position on the interval $(0; +\infty)$.

Since $h(0{,}6) = 0{,}4 - (0{,}6)^3 > 0$, $h(0{,}7) = 0{,}3 - (0{,}7)^3 < 0$, then $a \in (0{,}6; 0{,}7)$.

Next, from the form of $f'(x)$ we conclude that $f_{max} = f(1) = 0$, $f_{min} = f(a) < 0$.

We will now show that $f(x) \leqslant 0$ if $x \in [0{,}6; +\infty)$. Indeed, we have:

$$f(0{,}6) = \ln\frac{5}{3} - \frac{10}{19} = 2\frac{\ln 5 - \ln 3}{5-3} - 2\cdot\frac{5}{19} < 2\left(\frac{1}{\sqrt{3\cdot 5}} - \frac{5}{19}\right) < 0. \quad (6)$$

Here, when obtaining the intermediate estimate (6), we used the relation (3) between the logarithmic mean (2) and the geometric mean of numbers 3 and 5: $\frac{5-3}{\ln 5 - \ln 3} > \sqrt{5\cdot 3}$.

From (6), as well as from the condition $f_{max} = 0$, the inequality

$$f(x) \leqslant 0, \quad x \in [0{,}6; +\infty)$$

follows.

Let us now estimate the value $f(x) + f(y) + f(z)$ from above, provided that $x, y, z \in [0{,}6; +\infty)$ and $xyz = 1$. We have: $f(x) + f(y) + f(z) \leqslant 0$, which is equivalent to (1), and equality in this ratio will be achieved only when $f(x) = f(y) = f(z) = 0$, i.e., provided that $x = y = z = 1$.

Now, let not all values of x,y,z belong to the interval $[0{,}6; +\infty)$. Then, due to the condition $xyz = 1$, without loss of generality we can assume that either

$$x, y \in (0; 0{,}6); z \in [0{,}6; +\infty), \tag{7}$$

or

$$x \in (0; 0{,}6); y, z \in [0{,}6; +\infty), y \leqslant z. \tag{8}$$

Let us consider each of these cases separately, referring to the function $g(x) = \frac{x-1}{x^2-x+1}$.

From the form of its derivative $g'(x) = \frac{x(2-x)}{(x^2-x+1)^2}$ it follows that it increases on the interval $(0; 0{,}6]$; therefore, $g(x) < g(0.6) = -\frac{10}{19}$, $x \in (0; 0{,}6)$. In addition,

$$\max_{0 \leqslant x \leqslant +\infty} g(x) = g(2) = \frac{1}{3}.$$

Hence, under conditions (7), we have:

$$g(x) + g(y) + g(z) \leqslant -\frac{10}{19} - \frac{10}{19} + \frac{1}{3} < 0,$$

i.e., inequality (1) is satisfied.

Now let (8) be satisfied. If $x \in (0; 0{,}5]$, then $g(x) \leqslant -\frac{2}{3}$ and $g(x) + g(y) + g(z) < -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0$. Note that the obtained estimate is strict due to the condition $xyz = 1$. Inequality (1) is satisfied again. Similarly, if in conditions (8) $y \leqslant 1$, then $g(x)+g(y)+g(z) < -\frac{10}{19} + 0 + \frac{1}{3} < 0$. In this case, inequality (1) is satisfied.

Thus, to achieve finally the goal, we only have to consider one version of conditions (8): $x \in (0{,}5; 0{,}6)$; $y, z \in (1; +\infty)$, $y \leqslant z$. But then, due to the constraint $xyz = 1$, the clarifying condition $1 < y \leqslant z < 2$ must be satisfied. Let us study the function $g(y)$ for GA-convexity (-concavity), resorting to the above-formulated Theorem 1. Since

$$\Delta_g(y) = g'(y) + yg''(y) = \frac{y(y^3 - 3y^2 - 3y + 4)}{(y^2 - y + 1)^3} = \\ = \frac{y(4 - 5y + y(y-1)(y-2))}{(y^2 - y + 1)^3},$$

then, for $y \in (1;2)$, the inequality $\Delta_g(y) < 0$ holds, which allows us to conclude that $g(y)$ is a strictly GA-concave function on the interval $(1;2)$. But then

$$g(y) + g(z) < 2g\left(\sqrt{yz}\right) = 2g\left(\frac{1}{\sqrt{x}}\right),$$

or

$$g(y) + g(z) < \frac{2\left(\sqrt{x} - x\right)}{1 - \sqrt{x} + x}.$$

Therefore, if the conditions $x \in (0{,}5; 0{,}6)$, $1 < y \leqslant z < 2$ are met, we have:

$$g(x) + g(y) + g(z) < \frac{x - 1}{x^2 - x + 1} - \frac{2\left(x - \sqrt{x}\right)}{x - \sqrt{x} + 1} = \\ = \frac{(x-1)\left(x - \sqrt{x} + 1\right) - 2\left(x - \sqrt{x}\right)\left(x^2 - x + 1\right)}{\left(x^2 - x + 1\right)\left(x - \sqrt{x} + 1\right)} = \\ = \frac{(x-1)(x - \sqrt{x})(1 - 2x) - \left(\sqrt{x} - 1\right)^2}{(x^2 - x + 1)(x - \sqrt{x} + 1)} < 0,$$

which proves inequality (1) completely. □

Conclusion. Thus, we have considered yet another proof of the Damascus inequality (1): a proof, we note, done by elementary means. In addition, the conditions for achieving equality in it are additionally described: (1) can turn into equality only when $x = y = z = 1$.

The implemented approach to proving inequality (1) prompted the authors to formulate the following problem.

For what natural values of n for positive numbers x, y, z satisfying the condition $xyz = 1$, is the inequality

$$\frac{x^n - 1}{x^{n+1} + 1} + \frac{y^n - 1}{y^{n+1} + 1} + \frac{z^n - 1}{z^{n+1} + 1} \leqslant 0 \tag{9}$$

true? Formulate the conditions for achieving equality in it.

For $n = 1$, the authors were able to establish the validity of inequality (9) using the technique presented in this article when substantiating (1). For $n = 2$, this inequality is inequality (1), i.e., it is also true. In both cases, equality is achieved only if $x = y = z = 1$.

Thus, it is fundamentally important to consider the given problem for $n \geqslant 3$. We address this question to the interested reader.

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S. I. Kalinin
Department of Fundamental Mathematics, Vyatka State University
Kirov, Russia
E-mail: kalinin_gu@mail.ru

L. V. Pankratova
Department of Fundamental Mathematics, Vyatka State University
Kirov, Russia
E-mail: pankratovalarisa19@rambler.ru