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INEQUALITIES OF THE 3/8-SIMPSON TYPE FOR DIFFERENTIABLE FUNCTIONS VIA GENERALIZED FRACTIONAL OPERATORS

Abstract. Simpson-type inequalities are an important tool in mathematical analysis, particularly in the study of integrals. In this paper, we present new generalized 3/8-Simpson-type inequalities for functions whose first derivative modulus is (h, m)-convex and satisfies the Lipschitz condition via weight integral operators. To obtain these results, we use a new integral identity established in our study. This research generalizes, extends, and complements the existing results in the literature.

Key words: *convex function, $(h,m)-$convex function, Simpson-type inequality, weighted integral operator, Hölder inequality, Power mean inequality, Young inequality, Lipschitz function.*

2020 Mathematical Subject Classification: *26A51, 6A33, 226D10*

1. Introduction. Convexity is a fundamental concept in several applied disciplines, including computational mathematics, optimization theory, inequality theory, and others. It is unique in that it not only pertains to assessing the mean value of a function defined on an interval, but also to determining upper bounds for various well-known quadrature formulas. To enhance and broaden this evaluation, numerous convexity classes have been introduced in the literature. The study in [32] provides a comprehensive overview of these convexity classes.

The concept of convexity also enables us to derive upper bounds for Simpson's inequalities. Here is the first $\frac{1}{3}$-Simpson formula.

If $f \in C^4((\varrho_1, \varrho_2))$ and $\left\|f^{(4)}\right\|_\infty := \sup\limits_{x\in(\varrho_1,\varrho_2)} \left|f^{(4)}(x)\right| < \infty$, then

$$\left|\frac{1}{3}\left[\frac{f(\varrho_1)+f(\varrho_2)}{2}+2f\left(\frac{\varrho_1+\varrho_2}{2}\right)\right]-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2} f(t)dt\right| \leqslant \frac{(\varrho_2-\varrho_1)^4}{2880}\left\|f^{(4)}\right\|_\infty$$

and the second formula, which is called $\frac{3}{8}-$Simpson formula:

$$\left|\frac{1}{8}\Big[f(\varrho_1)+3f\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)+3f\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)+f(\varrho_2)\Big]-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f(t)dt\right|\leqslant$$
$$\leqslant\frac{(\varrho_2-\varrho_1)^4}{6480}\left\|f^{(4)}\right\|_\infty. \quad (1)$$

Many researchers have investigated various $\frac{1}{3}$-Simpson-type inequalities (see, for example [2], [4], [15], [16], [22], [26], [28], [31], [33] and the references therein). In contrast, relatively less attention has been given to $\frac{3}{8}$-Simpson-type inequalities (see, for instance [25], [14], [34], [35], [12], [27]). The primary focus of these studies has been on classes of convex functions, which play a critical role in various applied fields, including finance, economics, and optimization [11], [29].

Definition 1. [9] *Let $h\colon[0,1]\to(0,1]$ and $f\colon I_1=[0,+\infty)\to[0,+\infty)$. If $\forall\varrho_1,\varrho_2\in I_1$ and $\theta\in[0,1]$ the inequality*

$$f(\theta\varrho_1+m(1-\theta)\varrho_2)\leqslant h^s(\theta)f(\varrho_1)+m(1-h(\theta))^sf(\varrho_2)$$

is valid for $s\in[-1,1]$ and $m\in[0,1]$, then on I_1 the function f is called the $(h,m)-$convex modified of the second type.

The use of fractional operators is well established among researchers in various fields of applied science. Commonly employed fractional operators in the literature include the Riemann-Liouville, Caputo, Katugampola, conformable, non–conformable, and weighted integral operators (see, for example [20], [21], [23], [3], [1], [8], [6], [7], [5], [10], [13], [24], [30], [17], [18], [19], among others). Nevertheless, researchers are not restricted to classical fractional operators; they actively define and explore new generalized operators.

Definition 2. *Let $f\in L_1[\varrho_1,\varrho_2]$ with $\varrho_1,\varrho_2\in\mathbb{R}$ and $\varrho_1\geqslant0$. Then the Riemann-Liouville fractional integrals of order $\alpha\in\mathbb{C}$, $Re(\alpha)>0$ are defined by (right and left, respectively):*

$$I^{\alpha}_{\varrho_1+}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{\varrho_1}^{x}(x-t)^{\alpha-1}f(t)\,dt,\quad x>\varrho_1,$$

$$I_{\varrho_2-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\varrho_2} (t-x)^{\alpha-1} f(t)\, dt, \quad x < \varrho_2.$$

Definition 3. [33] *Let $f \in L_1[\varrho_1, \varrho_2]$ with $0 \leqslant \varrho_1 < \varrho_2$ and let w be a continuous and positive function, $w\colon I_2 = [0,1] \to [0,+\infty)$, with first derivative integrable on $[\varrho_1, \varrho_2]$. Then the weighted fractional integrals are defined by (right and left, respectively):*

$$J_{\varrho_1+}^{w} f(x) = \int_{\varrho_1}^{x} w'\left(\frac{x-t}{\varrho_2-\varrho_1}\right) f(t) dt, \quad x > \varrho_1,$$

$$J_{\varrho_2-}^{w} f(x) = \int_{x}^{\varrho_2} w'\left(\frac{t-x}{\varrho_2-\varrho_1}\right) f(t) dt, \quad x < \varrho_2.$$

2. Some notation used for mathematical expressions. To compactly write large mathematical expressions, we will use the following notation:

$$(A) := \int_0^1 (w(t)+r_1) f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1+\varrho_2}{3}\Big)\, dt,$$

$$(B) := \int_0^1 (w(t)+r_2) f'\Big((1-t)\frac{2\varrho_1+\varrho_2}{3} + t\frac{\varrho_1+2\varrho_2}{3}\Big)\, dt,$$

$$(C) := \int_0^1 (w(t)+r_3) f'\Big((1-t)\frac{\varrho_1+2\varrho_2}{3} + t\varrho_2\Big)\, dt,$$

$$(D) := \int_0^1 |w(t)+r_1| \cdot \Big|f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1+\varrho_2}{3}\Big)\Big|\, dt,$$

$$(E) := \int_0^1 |w(t)+r_2| \cdot \Big|f'\Big((1-t)\frac{2\varrho_1+\varrho_2}{3} + t\frac{\varrho_1+2\varrho_2}{3}\Big)\Big|\, dt,$$

$$(F) := \int_0^1 |w(t)+r_3| \cdot \Big|f'\Big((1-t)\frac{\varrho_1+2\varrho_2}{3} + t\varrho_2\Big)\Big|\, dt,$$

$$W_i := \int_0^1 |w(t) + r_i|^p \, dt, \quad i = 1, 2, 3,$$

$$H_1 := \int_0^1 h^s(t) \, dt, \quad H_2 := \int_0^1 (1 - h(t))^s \, dt$$

and

$$\begin{aligned}\mathbf{\Phi}(f, w, r_1, r_2, r_3) :=& \frac{1}{3}\Big\{\Big[(w(1) + r_1) f\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big) - (w(0) + r_1) f(\varrho_1)\Big] + \\ &+ \Big[(w(1) + r_2) f\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big) - (w(0) + r_2) f\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big)\Big] + \\ &+ \Big[(w(1) + r_3) f(\varrho_2) - (w(0) + r_3) f\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big)\Big]\Big\} - \\ &- \frac{1}{\varrho_2 - \varrho_1}\Big[J^w_{(\frac{2\varrho_1+\varrho_2}{3})-} f(\varrho_1) + J^w_{(\frac{\varrho_1+2\varrho_2}{3})-} f\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big) + J^w_{(\varrho_2)-} f\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big)\Big].\end{aligned}$$

3. Main Results. Let us now present the lemma on the basis of which the results were obtained.

Lemma 1. *Let $f\colon I_1 = [0, +\infty) \to \mathbb{R}$ be a differentiable function, such that $\varrho_1, \varrho_2 \in I_1$ and $0 \leqslant \varrho_1 < \varrho_2$. Let $w\colon I_2 = [0, 1] \to \mathbb{R}$ be a positive differentiable function. If $f', w' \in L_1[\varrho_1, \varrho_2]$, then it is true that*

$$\mathbf{\Phi}(f, w, r_1, r_2, r_3) = \frac{\varrho_2 - \varrho_1}{9}\Big[(A) + (B) + (C)\Big], \tag{2}$$

where r_1, r_2, and r_3 are constants.

Proof. Integrating (A) by parts, we have

$$\begin{aligned}(A) = \frac{3}{\varrho_2 - \varrho_1}\Big[(w(1) + r_1) f\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big) - (w(0) + r_1) f(\varrho_1)\Big] - \\ - \int_0^1 w'(t) f\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big) dt.\end{aligned} \tag{3}$$

Making the change of variable $x = (1-t)\varrho_1 + t\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big)$ in (3), we get

$$(A) = \frac{3}{\varrho_2 - \varrho_1}\left[(w(1)+r_1)f\left(\frac{2\varrho_1+\varrho_2}{3}\right) - (w(0)+r_1)f(\varrho_1)\right] - \\ - \left(\frac{3}{\varrho_2-\varrho_1}\right)^2 \int_{\varrho_1}^{\frac{2\varrho_1+\varrho_2}{3}} w'\left(\frac{x-\varrho_1}{\frac{\varrho_2-\varrho_1}{3}}\right) f(x)\,dx = \\ = \frac{3}{\varrho_2 - \varrho_1}\left[(w(1)+r_1)f\left(\frac{2\varrho_1+\varrho_2}{3}\right) - (w(0)+r_1)f(\varrho_1)\right] - \\ - \left(\frac{3}{\varrho_2-\varrho_1}\right)^2 J^w_{\left(\frac{2\varrho_1+\varrho_2}{3}\right)-} f(\varrho_1). \quad (4)$$

Similarly, considering $y = (1-t)\left(\frac{2\varrho_1+\varrho_2}{3}\right) + t\left(\frac{\varrho_1+2\varrho_2}{3}\right)$ and $z = (1-t)\left(\frac{\varrho_1+2\varrho_2}{3}\right) + t\varrho_2$ and using the Definition 3, we obtain

$$(B) = \frac{3}{\varrho_2 - \varrho_1}\left[(w(1)+r_2)f\left(\frac{\varrho_1+2\varrho_2}{3}\right) - (w(0)+r_2)f\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right] - \\ - \left(\frac{3}{\varrho_2-\varrho_1}\right)^2 \int_{\frac{2\varrho_1+\varrho_2}{3}}^{\frac{\varrho_1+2\varrho_2}{3}} w'\left(\frac{y-\frac{2\varrho_1+\varrho_2}{3}}{\frac{\varrho_2-\varrho_1}{3}}\right) f(y)\,dy = \\ = \frac{3}{\varrho_2 - \varrho_1}\left[(w(1)+r_2)f\left(\frac{\varrho_1+2\varrho_2}{3}\right) - (w(0)+r_2)f\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right] - \\ - \left(\frac{3}{\varrho_2-\varrho_1}\right)^2 J^w_{\left(\frac{\varrho_1+2\varrho_2}{3}\right)-} f\left(\frac{2\varrho_1+\varrho_2}{3}\right); \quad (5)$$

$$(C) = \frac{3}{\varrho_2 - \varrho_1}\left[(w(1)+r_3)f(\varrho_2) - (w(0)+r_3)f\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right] - \\ - \left(\frac{3}{\varrho_2-\varrho_1}\right)^2 \int_{\frac{\varrho_1+2\varrho_2}{3}}^{\varrho_2} w'\left(\frac{z-\frac{\varrho_1+2\varrho_2}{3}}{\frac{\varrho_2-\varrho_1}{3}}\right) f(z)\,dz = \\ = \frac{3}{\varrho_2 - \varrho_1}\left[(w(1)+r_3)f(\varrho_2) - (w(0)+r_3)f\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right] - \\ - \left(\frac{3}{\varrho_2-\varrho_1}\right)^2 J^w_{(\varrho_2)-} f\left(\frac{\varrho_1+2\varrho_2}{3}\right). \quad (6)$$

Summing (4)-(6) and then multiplying the resulting equality by $\frac{\varrho_2-\varrho_1}{9}$, we get the desired result. □

Remark 1. *Lemma* 2.1 *from* [25] *and Lemma* 1.2 *from* [12, for $\eta = 3$] *is derived by setting* $w(t) = t - \frac{1}{2}$ *and choosing* $r_1 = \frac{1}{8}$, $r_2 = 0$ *and* $r_3 = -\frac{1}{8}$.

Corollary 1. *Under the conditions of Lemma 1, if we take* $w(t) = t^\alpha$ *and* $r_1 = r_2 = r_3 = 0$, *we have the following identity via Riemann-Liouville operators:*

$$\begin{aligned}\mathbf{\Phi}(f, t^\alpha, 0, 0, 0) = \frac{\varrho_2 - \varrho_1}{9} \int_0^1 t^\alpha f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big) dt + \\ + \frac{\varrho_2 - \varrho_1}{9} \int_0^1 t^\alpha f'\Big((1-t)\frac{2\varrho_1 + \varrho_2}{3} + t\frac{\varrho_1 + 2\varrho_2}{3}\Big) dt + \\ + \frac{\varrho_2 - \varrho_1}{9} \int_0^1 t^\alpha f'\Big((1-t)\frac{\varrho_1 + 2\varrho_2}{3} + t\varrho_2\Big)\, dt. \quad (7)\end{aligned}$$

Proof. Indeed, for $\mathbf{\Phi}$, we get

$$\begin{aligned}\mathbf{\Phi}(f, t^\alpha, 0, 0, 0) = \frac{1}{3}\Big[f\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big) + f\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big) + f(\varrho_2)\Big] - \\ - \frac{3^{\alpha-1}\alpha\Gamma(\alpha)}{(\varrho_2 - \varrho_1)^\alpha}\Big[\frac{1}{\Gamma(\alpha)} \int_{\varrho_1}^{\frac{2\varrho_1+\varrho_2}{3}} (u - \varrho_1)^{\alpha-1} f(u)\, du + \\ + \frac{1}{\Gamma(\alpha)} \int_{\frac{2\varrho_1+\varrho_2}{3}}^{\frac{\varrho_1+2\varrho_2}{3}} \Big(u - \frac{2\varrho_1 + \varrho_2}{3}\Big)^{\alpha-1} f(u)\, du + \frac{1}{\Gamma(\alpha)} \int_{\frac{\varrho_1+2\varrho_2}{3}}^{\varrho_2} \Big(u - \frac{\varrho_1 + 2\varrho_2}{3}\Big)^{\alpha-1} f(u)\, du\Big] = \\ = \frac{1}{3}\Big[f\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big) + f\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big) + f(\varrho_2)\Big] - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^\alpha}\Big[I^\alpha_{(\frac{2\varrho_1+\varrho_2}{3})-} f(\varrho_1) + \\ + I^\alpha_{(\frac{\varrho_1+2\varrho_2}{3})-} f\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big) + I^\alpha_{(\varrho_2)-} f\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big)\Big].\end{aligned}$$

and for (A), (B), and (C), we get:

$$(A) = \int_0^1 t^\alpha f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big)\, dt,$$

$$(B) = \int_0^1 t^\alpha f'\Big((1-t)\frac{2\varrho_1+\varrho_2}{3} + t\frac{\varrho_1+2\varrho_2}{3}\Big)\,dt,$$

$$(C) = \int_0^1 t^\alpha f'\Big((1-t)\frac{\varrho_1+2\varrho_2}{3} + t\varrho_2\Big)dt.$$

Thus, from (2) we obtain (7). □

Remark 2. *It is not difficult to see that for $\alpha = 1$ from (7) it follows:*

$$\mathbf{\Phi}(f,t,0,0,0) = \frac{1}{3}\Big[f\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)+f\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)+f(\varrho_2)\Big] - \frac{1}{\varrho_2-\varrho_1}\int_0^1 f(t)dt =$$

$$= \frac{\varrho_2-\varrho_1}{9}\int_0^1 tf'\Big((1-t)\varrho_1 + t\frac{2\varrho_1+\varrho_2}{3}\Big)dt +$$

$$+ \frac{\varrho_2-\varrho_1}{9}\int_0^1 tf'\Big((1-t)\frac{2\varrho_1+\varrho_2}{3} + t\frac{\varrho_1+2\varrho_2}{3}\Big)dt +$$

$$+ \frac{\varrho_2-\varrho_1}{9}\int_0^1 tf'\Big((1-t)\frac{\varrho_1+2\varrho_2}{3} + t\varrho_2\Big)dt. \quad (8)$$

Theorem 1. *Let $f\colon I_1 = [0,+\infty) \to \mathbb{R}$ be a differentiable function, such that $f' \in L_1[\varrho_1,\varrho_2]$ for $\varrho_1,\varrho_2 \in I_1$ and $0 \leqslant \varrho_1 < \varrho_2$ and let $w\colon I_2 \subset \mathbb{R} \to \mathbb{R}$ be a positive differentiable function. If $|f'|$ is $(h,m)-$convex modified of the second type for some fixed $m \in [0,1]$ and $s \in [-1,1]$, then it is true that*

$$|\mathbf{\Phi}(f,w,r_1,r_2,r_3)| \leqslant \frac{(\varrho_2-\varrho_1)\,m}{9}\int_0^1 (1-h(t))^s\Big[|w(t)+r_1|\cdot\Big|f'\Big(\frac{\varrho_1}{m}\Big)\Big| +$$

$$+ |w(t)+r_2|\cdot\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3m}\Big)\Big| + |w(t)+r_3|\cdot\Big|f'\Big(\frac{\varrho_1+2\varrho_2}{3m}\Big)\Big|\Big]dt +$$

$$+ \frac{\varrho_2-\varrho_1}{9}\int_0^1 h^s(t)\Big[|w(t)+r_1|\cdot\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\Big| +$$

$$+\left|w(t)+r_2\right|\cdot\left|f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\right|+\left|w(t)+r_3\right|\left|f'(\varrho_2)\right|\Big]dt, \quad (9)$$

where r_1, r_2, and r_3 are constants.

Proof. Applying Lemma 1 and the triangle inequality, we obtain:

$$\begin{aligned}\left|\mathbf{\Phi}\left(f,w,r_1,r_2,r_3\right)\right| &\leqslant \frac{\varrho_2-\varrho_1}{9}\Big[|(A)|+|(B)|+|(C)|\Big]\leqslant \\ &\leqslant \frac{\varrho_2-\varrho_1}{9}\Big[(D)+(E)+(F)\Big].\end{aligned} \quad (10)$$

Applying the (h,m) convexity of $|f'|$ in (10), we get the following. For (D):

$$(D)=\int_0^1 |w(t)+r_1|\cdot\left|f'\Big((1-t)\varrho_1+t\frac{2\varrho_1+\varrho_2}{3}\Big)\right|dt\leqslant$$

$$\leqslant\int_0^1 |w(t)+r_1|\cdot\Big[m\left(1-h\left(t\right)\right)^s\left|f'\Big(\frac{\varrho_1}{m}\Big)\right|+h^s(t)\left|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\right|\Big]dt. \quad (11)$$

For (E):

$$(E)=\int_0^1 |w(t)+r_2|\cdot\left|f'\Big((1-t)\frac{2\varrho_1+\varrho_2}{3}+t\frac{\varrho_1+2\varrho_2}{3}\Big)\right|dt\leqslant$$

$$\leqslant\int_0^1 |w(t)+r_2|\cdot\Big[m(1-h(t))^s\left|f'\Big(\frac{2\varrho_1+\varrho_2}{3m}\Big)\right|+h^s(t)\left|f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\right|\Big]dt. \quad (12)$$

For (F):

$$(F)\leqslant\int_0^1 |w(t)+r_3|\cdot\Big[m(1-h(t))^s\left|f'\Big(\frac{\varrho_1+2\varrho_2}{3m}\Big)\right|+h^s(t)\left|f'(\varrho_2)\right|\Big]dt. \quad (13)$$

From (10), summing (11)–(13) and then multiplying the resulting equality by $\frac{\varrho_2-\varrho_1}{9}$, we have

$$
\begin{aligned}
&|\mathbf{\Phi}(f,w,r_1,r_2,r_3)| \leqslant \\
&\leqslant \frac{\varrho_2-\varrho_1}{9}\int_0^1 |w(t)+r_1|\cdot\Big[m(1-h(t))^s\Big|f'\Big(\frac{\varrho_1}{m}\Big)\Big|+h^s(t)\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\Big|\Big]dt+ \\
&+\frac{\varrho_2-\varrho_1}{9}\int_0^1 |w(t)+r_2|\cdot\Big[m(1-h(t))^s\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3m}\Big)\Big|+h^s(t)\Big|f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\Big|\Big]dt+ \\
&+\frac{\varrho_2-\varrho_1}{9}\int_0^1 |w(t)+r_3|\cdot\Big[m(1-h(t))^s\Big|f'\Big(\frac{\varrho_1+2\varrho_2}{3m}\Big)\Big|+h^s(t)\,|f'(\varrho_2)|\Big]dt. \qquad (14)
\end{aligned}
$$

Simplifying (14), we get (9). Thus, the proof is completed. □

Remark 3. *If we take* $w(t)=t-\frac{1}{2}$, $h(t)=t$ *and choose* $r_1=\frac{1}{8}$, $r_2=0$, *and* $r_3=-\frac{1}{8}$, *and* $m=1$, *then from* (9) *we get Theorem* 2.1 *from* [25]. *And if we additionally take* $s=1$, *then we obtain Corollary* 2.1 *and Corollary* 2.2.

Theorem 2. *Let* $f: I_1\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function, such that* $f'\in L_1[\varrho_1,\varrho_2]$ *for* $\varrho_1,\varrho_2\in I_1$ *with* $0\leqslant\varrho_1<\varrho_2$ *and let* $w: I_2\subset\mathbb{R}\to\mathbb{R}$ *be a positive differentiable function. If* $|f'|^q$ *is* $(h,m)-$*convex modified of the second type for some fixed* $m\in[0,1]$ *and* $s\in[-1,1]$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *with* $q>1$, *then it is true that*

$$
\begin{aligned}
&|\mathbf{\Phi}(f,w,r_1,r_2,r_3)| \leqslant \\
&\leqslant \frac{\varrho_2-\varrho_1}{9}\Big\{W_1^{\frac{1}{p}}\Big[m\Big|f'\Big(\frac{\varrho_1}{m}\Big)\Big|^q H_2+\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\Big|^q H_1\Big]^{\frac{1}{q}}+ \\
&+W_2^{\frac{1}{p}}\Big[m\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3m}\Big)\Big|^q H_2+\Big|f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\Big|^q H_1\Big]^{\frac{1}{q}}+ \\
&+W_3^{\frac{1}{p}}\Big[m\Big|f'\Big(\frac{\varrho_1+2\varrho_2}{3m}\Big)\Big|^q H_2+|f'(\varrho_2)|^q H_1\Big]^{\frac{1}{q}}\Big\}. \qquad (15)
\end{aligned}
$$

Proof. In (10) we have:

$$
|\mathbf{\Phi}(f,w,r_1,r_2,r_3)| \leqslant \frac{\varrho_2-\varrho_1}{9}\Big[(D)+(E)+(F)\Big]. \qquad (16)
$$

Using the Hölder's inequality and the (h,m) convexity of $|f'|^q$, we obtain for (D):

$$(D) = \int_0^1 |w(t) + r_1| \cdot \left| f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1+\varrho_2}{3}\Big)\right| dt \leqslant \tag{17}$$

$$\leqslant W_1^{\frac{1}{p}} \left[\int_0^1 \left| f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1+\varrho_2}{3}\Big)\right|^q dt \right]^{\frac{1}{q}} \leqslant$$

$$\leqslant W_1^{\frac{1}{p}} \left[m\left| f'\Big(\frac{\varrho_1}{m}\Big)\right|^q H_2 + \left| f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\right|^q H_1 \right]^{\frac{1}{q}}.$$

For (E):

$$(E) \leqslant W_2^{\frac{1}{p}} \left[\int_0^1 \left| f'\Big((1-t)\frac{2\varrho_1+\varrho_2}{3} + t\frac{\varrho_1+2\varrho_2}{3}\Big)\right|^q dt \right]^{\frac{1}{q}} \leqslant \tag{18}$$

$$\leqslant W_2^{\frac{1}{p}} \left[m\left| f'\Big(\frac{2\varrho_1+\varrho_2}{3m}\Big)\right|^q H_2 + \left| f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\right|^q H_1 \right]^{\frac{1}{q}}.$$

For (F):

$$(F) \leqslant W_3^{\frac{1}{p}} \left[\int_0^1 \left| f'\Big((1-t)\frac{\varrho_1+2\varrho_2}{3} + t\varrho_2\Big)\right|^q dt \right]^{\frac{1}{q}} \leqslant \tag{19}$$

$$\leqslant W_3^{\frac{1}{p}} \left[m\left| f'\Big(\frac{\varrho_1+2\varrho_2}{3m}\Big)\right|^q H_2 + \left| f'(\varrho_2)\right|^q H_1 \right]^{\frac{1}{q}}.$$

From (16), summing (17)-(19) and then multiplying the resulting equality by $\frac{\varrho_2-\varrho_1}{9}$, we have (15). Thus, the proof is completed. □

Corollary 2. *For $h(t) = t$ and $m = 1$, choosing $w(t) = t - \frac{1}{2}$, $r_1 = \frac{1}{8}$, $r_2 = 0$ and $r_3 = -\frac{1}{8}$, from (15) we obtain an estimate for inequality (1):*

$$\left| \frac{1}{8}\left[f(\varrho_1) + 3f\Big(\frac{2\varrho_1+\varrho_2}{3}\Big) + 3f\Big(\frac{\varrho_1+2\varrho_2}{3}\Big) + f(\varrho_2)\right] - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \leqslant$$

$$\leqslant \frac{\varrho_2-\varrho_1}{72(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}} \left\{ \left(\frac{3^{p+1}+5^{p+1}}{8}\right)^{\frac{1}{p}} \left[\left|f'(\varrho_1)\right|^q + \left| f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\right|^q \right]^{\frac{1}{q}} + \right.$$

$$+4\left[\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|^q+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|^q\right]^{\frac{1}{q}}+$$
$$+\left(\frac{3^{p+1}+5^{p+1}}{8}\right)^{\frac{1}{p}}\left[\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|^q+\left|f'\left(\varrho_2\right)\right|^q\right]^{\frac{1}{q}}\Bigg\}. \quad (20)$$

Proof. Indeed, for the $\mathbf{\Phi}$, H_1,H_2, and W_i, we get

$$\left|\mathbf{\Phi}\left(f,t-\frac{1}{2},\frac{1}{8},0,-\frac{1}{8}\right)\right|=\left|\frac{1}{8}\left[f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+\right.\right.$$
$$\left.\left.+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\right]-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f(u)\,du\right|; \quad (21)$$

$$W_1=\int\limits_0^1\left|t-\frac{3}{8}\right|^p dt=\frac{\left(\frac{3}{8}\right)^{p+1}+\left(\frac{5}{8}\right)^{p+1}}{p+1}=\frac{3^{p+1}+5^{p+1}}{8^{p+1}\left(p+1\right)},$$
$$W_2=\int\limits_0^1\left|t-\frac{1}{2}\right|^p dt=2\frac{\left(\frac{1}{2}\right)^{p+1}}{p+1}=\frac{1}{2^p\left(p+1\right)},$$
$$W_3=\int\limits_0^1\left|t-\frac{5}{8}\right|^p dt=\frac{\left(\frac{5}{8}\right)^{p+1}+\left(\frac{3}{8}\right)^{p+1}}{p+1}=\frac{5^{p+1}+3^{p+1}}{8^{p+1}\left(p+1\right)};$$

$$H_1=H_2=\int\limits_0^1 t^s dt=\int\limits_0^1\left(1-t\right)^s dt=\frac{1}{s+1}. \quad (22)$$

Thus, for the right-hand side (15), after some simplifications, we get

$$\frac{\varrho_2-\varrho_1}{9\left(s+1\right)^{\frac{1}{q}}}\left\{\frac{1}{8}\left(\frac{3^{p+1}+5^{p+1}}{8\left(p+1\right)}\right)^{\frac{1}{p}}\left[\left|f'\left(\varrho_1\right)\right|^q+\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|^q\right]^{\frac{1}{q}}+\right.$$
$$+\frac{1}{2\left(p+1\right)^{\frac{1}{p}}}\left[\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|^q+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|^q\right]^{\frac{1}{q}}+$$
$$\left.+\frac{1}{8}\left(\frac{3^{p+1}+5^{p+1}}{8\left(p+1\right)}\right)^{\frac{1}{p}}\left[\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|^q+\left|f'\left(\varrho_2\right)\right|^q\right]^{\frac{1}{q}}\right\}. \quad (23)$$

From (21) and (23) follows (20). □

Remark 4. *Inequality (20) was obtained in Theorem 2.2 from [25].*

Theorem 3. *Let $f\colon I_1 \to \mathbb{R}$ be a differentiable function, such that $f' \in L_1[\varrho_1, \varrho_2]$ for $\varrho_1, \varrho_2 \in I_1$ and let $w\colon I_2 \subset \mathbb{R} \to \mathbb{R}$ be a positive differentiable function. If $|f'|^q$ with $q > 1$ is $(h, m)-$convex modified of the second type for some fixed $m \in [0, 1]$ and $s \in [-1, 1]$, then it is true that*

$$|\mathbf{\Phi}\,(f, w, r_1, r_2, r_3)| \leqslant \frac{\varrho_2 - \varrho_1}{9}\Big(V_1^{1-\frac{1}{q}} G_1^{\frac{1}{q}} + V_2^{1-\frac{1}{q}} G_2^{\frac{1}{q}} + V_3^{1-\frac{1}{q}} G_3^{\frac{1}{q}}\Big). \tag{24}$$

Here $V_i = \int\limits_0^1 |w(t) + r_i| dt\,, i = 1, 2, 3,$

$$G_1 = \int\limits_0^1 |w(t) + r_1| \cdot \Big[m(1 - h(t))^s \Big|f'\Big(\frac{\varrho_1}{m}\Big)\Big|^q + h^s(t)\Big|f'\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big)\Big|^q\Big]dt,$$

$$G_2 = \int\limits_0^1 |w(t) + r_2| \cdot \Big[m(1 - h(t))^s \Big|f'\Big(\frac{2\varrho_1 + \varrho_2}{3m}\Big)\Big|^q + h^s(t)\Big|f'\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big)\Big|^q\Big]dt,$$

$$G_3 = \int\limits_0^1 |w(t) + r_3| \cdot \Big[m(1 - h(t))^s \Big|f'\Big(\frac{\varrho_1 + 2\varrho_2}{3m}\Big)\Big|^q + h^s(t)\,|f'\,(\varrho_2)|^q\Big]dt.$$

Proof. In (10) we have:

$$|\mathbf{\Phi}\,(f, w, r_1, r_2, r_3)| \leqslant \frac{\varrho_2 - \varrho_1}{9}\Big[(D) + (E) + (F)\Big]. \tag{25}$$

Using the power mean inequality and the (h, m) convexity of $|f'|^q$, we obtain the following.

For (D):

$$(D) \leqslant V_1^{1-\frac{1}{q}} \Big[\int\limits_0^1 |w(t) + r_1| \cdot \Big|f'\Big((1 - t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big)\Big|^q dt\Big]^{\frac{1}{q}} \leqslant$$

$$\leqslant V_1^{1-\frac{1}{q}} \Big\{\int\limits_0^1 |w(t) + r_1| \cdot \Big[m\,(1 - h(t))^s \Big|f'\Big(\frac{\varrho_1}{m}\Big)\Big|^q +$$

$$+\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|^q h^s(t)\Big]dt\Bigg\}^{\frac{1}{q}} = V_1^{1-\frac{1}{q}} G_1^{\frac{1}{q}}. \quad (26)$$

For (E):

$$(E) \leqslant V_2^{1-\frac{1}{q}} \left[\int_0^1 |w(t)+r_2| \cdot \left|f'\left((1-t)\frac{2\varrho_1+\varrho_2}{3} + t\frac{\varrho_1+2\varrho_2}{3}\right)\right|^q dt\right]^{\frac{1}{q}} \leqslant$$

$$\leqslant V_2^{1-\frac{1}{q}} \Bigg\{\int_0^1 |w(t)+r_2| \cdot \Big[(1-h(t))^s m\left|f'\left(\frac{2\varrho_1+\varrho_2}{3m}\right)\right|^q +$$

$$+ h^s(t)\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|^q dt\Big]\Bigg\}^{\frac{1}{q}} = V_2^{1-\frac{1}{q}} G_2^{\frac{1}{q}}. \quad (27)$$

For (F) :

$$(F) \leqslant V_3^{1-\frac{1}{q}} \left[\int_0^1 |w(t)+r_3| \cdot \left|f'\left((1-t)\frac{\varrho_1+2\varrho_2}{3} + t\varrho_2\right)\right|^q dt\right]^{\frac{1}{q}} \leqslant$$

$$\leqslant V_3^{1-\frac{1}{q}} \Bigg\{\int_0^1 |w(t)+r_3| \cdot \Big[m(1-h(t))^s\left|f'\left(\frac{\varrho_1+2\varrho_2}{3m}\right)\right|^q +$$

$$+ h^s(t)\left|f'\left(\varrho_2\right)\right|^q\Big]dt\Bigg\}^{\frac{1}{q}} = V_3^{1-\frac{1}{q}} G_3^{\frac{1}{q}}. \quad (28)$$

From (25), summing (26)–(28) and then multiplying the resulting equality by $\frac{\varrho_2-\varrho_1}{9}$, we have (24). Thus, the proof is complete. □

Corollary 3. *For $h(t)=t$ and $m=1$, if we choose $w(t)=t-\frac{1}{2}$, $r_1=\frac{1}{8}$, $r_2=0$, and $r_3=-\frac{1}{8}$, then we obtain from (24) an estimate for (1):*

$$\left|\frac{1}{8}\Big[f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\Big]-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right| \leqslant$$

$$\leqslant \frac{\varrho_2-\varrho_1}{9}\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\Big[\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(G_{11}^{\frac{1}{q}}+G_{31}^{\frac{1}{q}}\right)+\left(\frac{1}{4}\right)^{1-\frac{1}{q}} G_{21}^{\frac{1}{q}}\Big], \quad (29)$$

where

$$G_{11} = \Big[\frac{3s-2}{16}+\Big(\frac{5}{8}\Big)^{s+2}\Big]\cdot\left|f'\left(\varrho_1\right)\right|^q+\Big[\frac{5s+2}{16}+\Big(\frac{3}{8}\Big)^{s+2}\Big]\cdot\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\Big|^q,$$
$$G_{21} = \frac{1+s2^s}{2^{s+2}}\Big[\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\Big|^q+\Big|f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\Big|^q\Big],$$
$$G_{31}=\Big[\frac{5s+2}{16}+\Big(\frac{3}{8}\Big)^{s+2}\Big]\cdot\Big|f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\Big|^q+\Big[\frac{3s-2}{16}+\Big(\frac{5}{8}\Big)^{s+2}\Big]\cdot\left|f'\left(\varrho_2\right)\right|^q.$$

Proof. Indeed, for $\mathbf{\Phi}$ and for V_i, we have

$$\Big|\mathbf{\Phi}\Big(f,t-\frac{1}{2},\frac{1}{8},0,-\frac{1}{8}\Big)\Big| = \Big|\frac{1}{8}\Big[f(\varrho_1)+3f\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)+ + 3f\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)+f(\varrho_2)\Big]-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)\,du\Big|; \quad (30)$$

$$\begin{cases} V_1 = \int_0^1 |w(t)+r_1|\,dt = \int_0^1 \left|t-\frac{3}{8}\right|dt = \frac{17}{64}, \\ V_2 = \int_0^1 |w(t)+r_2|\,dt = \int_0^1 \left|t-\frac{1}{2}\right|dt = \frac{1}{4}, \\ V_3 = \int_0^1 |w(t)+r_3|\,dt = \int_0^1 \left|t-\frac{3}{8}\right|dt = \frac{17}{64}. \end{cases} \quad (31)$$

For (G_1), we have

$$G_1 = \int_0^1 |w(t)+r_1|\cdot\Big[|m(1-h(t))^s\Big|f'\Big(\frac{\varrho_1}{m}\Big)\Big|^q+h^s(t)\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\Big|^q\Big]dt=$$
$$= \left|f'\left(\varrho_1\right)\right|^q\int_0^1\Big|t-\frac{3}{8}\Big|(1-t)^s dt+\Big|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\Big|^q\int_0^1\Big|t-\frac{3}{8}\Big|t^s dt.$$

Here

$$\int_0^1\Big|t-\frac{3}{8}\Big|(1-t)^s dt = \int_0^1\Big|\frac{5}{8}-z\Big|z^s dz = \frac{2}{(s+1)\,(s+2)}\Big[\frac{3s-2}{16}+\Big(\frac{5}{8}\Big)^{s+2}\Big],$$
$$\int_0^1\Big|t-\frac{3}{8}\Big|t^s dt = \frac{2}{(s+1)\,(s+2)}\Big[\frac{5s+2}{16}+\Big(\frac{3}{8}\Big)^{s+2}\Big].$$

Thus, for (G_1), we get

$$G_1 = \frac{2}{(s+1)(s+2)} \cdot \left\{ \left[\frac{3s-2}{16} + \left(\frac{5}{8} \right)^{s+2} \right] \cdot \left| f'(\varrho_1) \right|^q + \right.$$
$$\left. + \left[\frac{5s+2}{16} + \left(\frac{3}{8} \right)^{s+2} \right] \cdot \left| f'\left(\frac{2\varrho_1 + \varrho_2}{3} \right) \right|^q \right\} = \frac{2G_{11}}{(s+1)(s+2)}. \quad (32)$$

For (G_2), we have

$$G_2 = \int_0^1 |w(t) + r_2| \cdot \left[m(1-h(t))^s \left| f'\left(\frac{2\varrho_1 + \varrho_2}{3m} \right) \right|^q + h^s(t) \left| f'\left(\frac{\varrho_1 + 2\varrho_2}{3} \right) \right|^q \right] dt =$$
$$= \left| f'\left(\frac{2\varrho_1 + \varrho_2}{3} \right) \right|^q \int_0^1 \left| t - \frac{1}{2} \right| (1-t)^s dt + \left| f'\left(\frac{\varrho_1 + 2\varrho_2}{3} \right) \right|^q \int_0^1 \left| t - \frac{1}{2} \right| t^s dt.$$

Here

$$\int_0^1 \left| t - \frac{1}{2} \right| (1-t)^s dt = \int_0^1 \left| \frac{1}{2} - z \right| z^s dz = \frac{1}{2(s+1)(s+2)} \left(\left(\frac{1}{2} \right)^s + s \right).$$

Thus, for (G_2) we get

$$G_{21} = \frac{1 + s2^s}{2^{s+1}(s+1)(s+2)} \left[\left| f'\left(\frac{2\varrho_1 + \varrho_2}{3} \right) \right|^q + \left| f'\left(\frac{\varrho_1 + 2\varrho_2}{3} \right) \right|^q \right] =$$
$$= \frac{2G_{21}}{(s+1)(s+2)}. \quad (33)$$

For (G_3), we have

$$G_3 = \int_0^1 |w(t) + r_3| \cdot \left[m(1-h(t))^s \left| f'\left(\frac{\varrho_1 + 2\varrho_2}{3m} \right) \right|^q + h^s(t) \left| f'(\varrho_2) \right|^q \right] dt =$$
$$= \left| f'\left(\frac{\varrho_1 + 2\varrho_2}{3} \right) \right|^q \int_0^1 \left| t - \frac{5}{8} \right| (1-t)^s dt + \left| f'(\varrho_2) \right|^q \int_0^1 \left| t - \frac{5}{8} \right| t^s dt.$$

Here

$$\int_0^1 \left| t - \frac{5}{8} \right| (1-t)^s dt = \int_0^1 \left| \frac{3}{8} - z \right| z^s dz = \frac{2}{(s+1)(s+2)} \left(\frac{5s+2}{16} + \left(\frac{3}{8} \right)^{s+2} \right),$$

$$\int_0^1 \left| t - \frac{5}{8} \right| t^s dt = \frac{2}{(s+1)(s+2)} \left(\frac{3s-2}{16} + \left(\frac{5}{8} \right)^{s+2} \right).$$

Thus, for G_3 we get

$$G_3 = \frac{2}{(s+1)(s+2)} \left[\left(\frac{5s+2}{16} + \left(\frac{3}{8} \right)^{s+2} \right) \left| f' \left(\frac{\varrho_1 + 2\varrho_2}{3} \right) \cdot \right|^q + \right.$$
$$\left. + \left(\frac{3s-2}{16} + \left(\frac{5}{8} \right)^{s+2} \right) |f'(\varrho_2)|^q \right] = \frac{2G_{31}}{(s+1)(s+2)}. \quad (34)$$

From (30)–(34), (29) follows. □

Remark 5. *Inequality (29) was obtained in Theorem 2.3 from [25].*

Theorem 4. *Let $f : I_1 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function, such that $f' \in L_1[\varrho_1, \varrho_2]$ for $\varrho_1, \varrho_2 \in I_1$ and $0 \leqslant \varrho_1 < \varrho_2$, and let $w : I_2 \subset \mathbb{R} \to \mathbb{R}$ be a positive differentiable function. If $|f'|^q$ is $(h,m)-$convex modified of the second type for some fixed $m \in [0,1]$ and $s \in [-1,1]$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$, then it is true that*

$$|\mathbf{\Phi}(f, w, r_1, r_2, r_3)| \leqslant \frac{\varrho_2 - \varrho_1}{9p} \left(V_1^p + V_2^p + V_3^p \right) +$$
$$+ \frac{\varrho_2 - \varrho_1}{9q} \left[m \left| f' \left(\frac{\varrho_1}{m} \right) \right| H_2 + \left| f' \left(\frac{2\varrho_1 + \varrho_2}{3} \right) \right| H_1 \right]^q +$$
$$+ \frac{\varrho_2 - \varrho_1}{9q} \left[m \left| f' \left(\frac{2\varrho_1 + \varrho_2}{3m} \right) \right| H_2 + \left| f' \left(\frac{\varrho_1 + 2\varrho_2}{3} \right) \right| H_1 \right]^q +$$
$$+ \frac{\varrho_2 - \varrho_1}{9q} \left[m \left| f' \left(\frac{\varrho_1 + 2\varrho_2}{3m} \right) \right| H_2 + |f'(\varrho_2)| H_1 \right]^q, \quad (35)$$

where V_i defined in Theorem 3.

Proof. In (10) we have

$$|\mathbf{\Phi}(f, w, r_1, r_2, r_3)| \leqslant \frac{\varrho_2 - \varrho_1}{9} \Big[(D) + (E) + (F) \Big]. \quad (36)$$

Using Young's inequality $\left(uv = \frac{u^p}{p} + \frac{v^q}{q} \right)$ and the (h,m) convexity of $|f'|^q$, we obtain

For (D):

$$(D)=\int_0^1 |w(t)+r_1|\cdot\left|f'\Big((1-t)\varrho_1+t\frac{2\varrho_1+\varrho_2}{3}\Big)\right|dt\leqslant$$

$$\leqslant\frac{1}{p}V_1^p+\frac{1}{q}\left[\int_0^1\left|f'\Big((1-t)\varrho_1+t\frac{2\varrho_1+\varrho_2}{3}\Big)\right|dt\right]^q\leqslant$$

$$\leqslant\frac{1}{p}V_1^p+\frac{1}{q}\left\{\int_0^1\left[m(1-h(t))^s\left|f'\Big(\frac{\varrho_1}{m}\Big)\right|+h^s(t)\cdot\left|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\right|\right]dt\right\}^q=$$

$$=\frac{1}{p}V_1^p+\frac{1}{q}\left[m\left|f'\left(\frac{\varrho_1}{m}\right)\right|H_2+\left|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\right|H_1\right]^q. \quad (37)$$

For (E):

$$(E)\leqslant\frac{1}{p}V_2^p+\frac{1}{q}\left[\int_0^1\left|f'\Big((1-t)\frac{2\varrho_1+\varrho_2}{3}+t\frac{\varrho_1+2\varrho_2}{3}\Big)\right|dt\right]^q=$$

$$=\frac{1}{p}V_2^p+\frac{1}{q}\left[m\left|f'\Big(\frac{2\varrho_1+\varrho_2}{3m}\Big)\right|H_2+\left|f'\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)\right|H_1\right]^q. \quad (38)$$

For (F):

$$(F)\leqslant\frac{1}{p}V_3^p+\frac{1}{q}\left[\int_0^1\left|f'\Big((1-t)\frac{\varrho_1+2\varrho_2}{3}+t\varrho_2\Big)\right|dt\right]^q=$$

$$=\frac{1}{p}V_3^p+\frac{1}{q}\left[m\left|f'\Big(\frac{\varrho_1+2\varrho_2}{3m}\Big)\right|H_2+\left|f'\left(\varrho_2\right)\right|H_1\right]^q. \quad (39)$$

From (36), summing (37)–(39) and then multiplying the resulting equality by $\frac{\varrho_2-\varrho_1}{9}$, we have (35). Thus, the proof is completed. □

Corollary 4. *For $h(t)=t$ and $m=1$ if we choose $w(t)=t-\frac{1}{2}$, $r_1=\frac{1}{8}$, $r_2=0$, and $r_3=-\frac{1}{8}$, then from (35), we obtain an estimate for (1):*

$$\left|\frac{1}{8}\left[f(\varrho_1)+3f\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)+3f\Big(\frac{\varrho_1+2\varrho_2}{3}\Big)+f(\varrho_2)\right]-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)du\right|\leqslant$$

$$\leqslant\frac{\varrho_2-\varrho_1}{9p}\left[2\Big(\frac{17}{64}\Big)^p+\Big(\frac{1}{4}\Big)^p\right]+\frac{\varrho_2-\varrho_1}{9q\,(s+1)}\left\{\left[\left|f'\left(\varrho_1\right)\right|+\left|f'\Big(\frac{2\varrho_1+\varrho_2}{3}\Big)\right|\right]^q+\right.$$

$$+\left[\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|\right]^q+\left[\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|+\left|f'\left(\varrho_2\right)\right|\right]^q\Bigg\}. \quad (40)$$

Proof. Indeed, for $\mathbf{\Phi}$, we have

$$\left|\mathbf{\Phi}\left(f, t-\frac{1}{2}, \frac{1}{8}, 0, -\frac{1}{8}\right)\right| = \left|\frac{1}{8}\left[f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+\right.\right.$$
$$\left.\left.+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\right]-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|,$$

from (31) and (22) we have:

$$V_1 = V_3 = \frac{17}{64},\ V_2 = \frac{1}{4}, H_1 = H_2 = \frac{1}{s+1}.$$

and the expressions in the right-hand side of (35) will look like:

$$\frac{\varrho_2-\varrho_1}{9p}\left[2\left(\frac{17}{64}\right)^p+\left(\frac{1}{4}\right)^p\right]+\frac{\varrho_2-\varrho_1}{9q\,(s+1)}\left[\left|f'\left(\varrho_1\right)\right|+\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|\right]^q+$$
$$+\frac{\varrho_2-\varrho_1}{9q\,(s+1)}\left[\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|\right]^q+$$
$$+\frac{\varrho_2-\varrho_1}{9q\,(s+1)}\left[\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|+\left|f'\left(\varrho_2\right)\right|\right]^q.$$

Thus, from (35), we obtain

$$\left|\mathbf{\Phi}\left(f, t-\frac{1}{2}, \frac{1}{8}, 0, -\frac{1}{8}\right)\right| \leqslant \frac{\varrho_2-\varrho_1}{9p}\left[2\left(\frac{17}{64}\right)^p+\left(\frac{1}{4}\right)^p\right]+$$
$$+\frac{\varrho_2-\varrho_1}{9q\,(s+1)}\left[\left|f'\left(\varrho_1\right)\right|+\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|\right]^q+$$
$$+\frac{\varrho_2-\varrho_1}{9q\,(s+1)}\left[\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|\right]^q+$$
$$+\frac{\varrho_2-\varrho_1}{9q\,(s+1)}\left[\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|+\left|f'\left(\varrho_2\right)\right|\right]^q.$$

The proof is completed. □

The following theorem, in a special case, provides an estimate for (1) in terms of the smallest and largest values of the derivative of the function on $[\varrho_1, \varrho_2]$.

Theorem 5. *Let $f\colon I_1 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I_1° and $f' \in L_1[\varrho_1, \varrho_2]$. If there exist constants $-\infty < m < M < +\infty$, such that $m \leqslant f' \leqslant M$, $\forall x \in [\varrho_1, \varrho_2]$, then we have*

$$|\mathbf{\Phi}(f, w, r_1, r_2, r_3)| \leqslant \leqslant \frac{(\varrho_2 - \varrho_1)(M - m)}{18} \int_0^1 \Big[|w(t) + r_1| + |w(t) + r_2| + |w(t) + r_3|\Big] dt + + \frac{(\varrho_2 - \varrho_1)(M + m)}{18} \left| \int_0^1 (3w(t) + r_1 + r_2 + r_3)\, dt \right|. \tag{41}$$

Proof. In (10), we have

$$|\mathbf{\Phi}(f, w, r_1, r_2, r_3)| \leqslant \frac{\varrho_2 - \varrho_1}{9} \Big| (A) + (B) + (C) \Big|. \tag{42}$$

For (A), we can write

$$(A) = \int_0^1 (w(t) + r_1) \cdot \Big[f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big) - \frac{m+M}{2} + \frac{m+M}{2} \Big] dt = = \int_0^1 (w(t) + r_1) \cdot \Big[f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big) - \frac{m+M}{2} \Big] dt + + \frac{m+M}{2} \int_0^1 (w(t) + r_1)\, dt. \tag{43}$$

By analogy, for (B) and (C) we can write

$$(B) = \int_0^1 (w(t) + r_2) \cdot \Big[f'\Big((1-t)\frac{2\varrho_1 + \varrho_2}{3} + t\frac{\varrho_1 + 2\varrho_2}{3}\Big) - \frac{m+M}{2} \Big] dt + + \frac{m+M}{2} \int_0^1 (w(t) + r_2)\, dt, \tag{44}$$

$$(C)=\int_0^1 (w(t)+r_3)\left[f'\left((1-t)\frac{\varrho_1+2\varrho_2}{3}+t\varrho_2\right)-\frac{m+M}{2}\right]dt+ \\ +\frac{m+M}{2}\int_0^1 (w(t)+r_3)\,dt. \quad (45)$$

Adding equalities (43)–(45) and using properties of absolute value and inequality $\left|f'(x)-\frac{m+M}{2}\right|\leqslant\frac{M-m}{2}, \forall x\in[\varrho_1,\varrho_2]$, we obtain

$$|(A)+(B)+(C)|\leqslant \\ \leqslant\frac{M-m}{2}\left[\int_0^1|w(t)+r_1|\,dt+\int_0^1|w(t)+r_2|\,dt+\int_0^1|w(t)+r_3|\,dt\right]+ \\ +\frac{M+m}{2}\left|\int_0^1(w(t)+r_1)\,dt+\int_0^1(w(t)+r_2)\,dt+\int_0^1(w(t)+r_3)\,dt\right|= \\ =\frac{M-m}{2}\left[\int_0^1|w(t)+r_1|\,dt+\int_0^1|w(t)+r_2|\,dt+\int_0^1|w(t)+r_3|\,dt\right]+ \\ +\frac{M+m}{2}\left|\int_0^1(3w(t)+r_1+r_2+r_3)\,dt\right|.$$

Then, multiplying both sides of the resulting inequality by $\frac{\varrho_2-\varrho_1}{9}$ from (42) we obtain (41). This completes the proof. □

Corollary 5. *For $h(t)=t$ and $m=1$, if we choose $w(t)=t-\frac{1}{2}$, $r_1=\frac{1}{8}$, $r_2=0$, and $r_3=-\frac{1}{8}$, then from (41) we get:*

$$\left|\frac{1}{8}\left[f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\right]-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)du\right|\leqslant \\ \leqslant\frac{25\,(\varrho_2-\varrho_1)\,(M-m)}{576}. \quad (46)$$

Proof. Indeed, for $\mathbf{\Phi}$, we have

$$\left|\mathbf{\Phi}\left(f, t-\frac{1}{2}, \frac{1}{8}, 0, -\frac{1}{8}\right)\right| =$$
$$= \left|\frac{1}{8}\left[f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\right]-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2} f(u)du\right|.$$

Let us calculate the integrals on the right-hand side (41). From (31), for the first integral we have:

$$\int\limits_0^1 \Big(|w(t)+r_1|+|w(t)+r_2|+|w(t)+r_3|\Big)dt = \frac{25}{32}$$

and for the second integral, we get:

$$\int\limits_0^1 (3w(t)+r_1+r_2+r_3)\,dt = 3\int\limits_0^1 \left(t-\frac{1}{2}\right)dt = 0.$$

Thus, the proof is complete. □

Remark 6. *Inequality (46) was obtained in Theorem* 3.1 *from* [25] *and in Theorem* 2.1 *from* [27, for $\eta = 3$].

The following theorem, in a special case, provides an estimate for (1) in terms of the Lipschitz constant.

Theorem 6. *Let* $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ *be a differentiable function* (ϱ_1, ϱ_2), *such that* $f' \in L_1[\varrho_1, \varrho_2]$ *with* $0 \leqslant \varrho_1 < \varrho_2$. *If* f' *is an L-Lipschitzian function on* $[\varrho_1, \varrho_2]$, *then we have*

$$|\mathbf{\Phi}(f, w, r_1, r_2, r_3)| \leqslant$$
$$\leqslant \frac{(\varrho_2-\varrho_1)^2 L}{27}\left[\int\limits_0^1 |w(t)+r_1|\,tdt+\int\limits_0^1 |w(t)+r_2|\,tdt+\int\limits_0^1 |w(t)+r_3|\,t\,dt\right]+$$
$$+\frac{\varrho_2-\varrho_1}{9}\left|\left[f'(\varrho_1)+f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)+f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right]\int\limits_0^1 w(t)dt+\right.$$
$$\left.+f'(\varrho_1)\,r_1+r_2 f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)+f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)r_3\right|. \quad (47)$$

Proof. In (10), we have

$$|\mathbf{\Phi}\left(f, w, r_1, r_2, r_3\right)| \leqslant \frac{\varrho_2 - \varrho_1}{9}\Big|(A) + (B) + (C)\Big|.$$

For (A), we can write:

$$(A) = \int_0^1 (w(t) + r_1)\left[f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big) - f'(\varrho_1) + f'\left(\varrho_1\right)\right]dt =$$

$$= \int_0^1 (w(t) + r_1) \cdot \left[f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big) - f'\left(\varrho_1\right)\right]dt +$$

$$+ f'\left(\varrho_1\right)\int_0^1 \left(w(t) + r_1\right)dt. \quad (48)$$

By analogy, for (B) and (C) we can write:

$$(B) = \int_0^1 (w(t) + r_2)\left[f'\Big((1-t)\frac{2\varrho_1 + \varrho_2}{3} + t\frac{\varrho_1 + 2\varrho_2}{3}\Big) - f'\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big)\right]dt +$$

$$+ f'\Big(\frac{2\varrho_1 + \varrho_2}{3}\Big)\int_0^1 \left(w(t) + r_2\right)dt, \quad (49)$$

$$(C) = \int_0^1 (w(t) + r_3)\left[f'\Big((1-t)\frac{\varrho_1 + 2\varrho_2}{3} + t\varrho_2\Big) - f'\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big)\right]dt +$$

$$+ f'\Big(\frac{\varrho_1 + 2\varrho_2}{3}\Big)\int_0^1 \left(w(t) + r_3\right)dt, \quad (50)$$

Let us add equalities (48) through (50), take the absolute value of both sides of the resulting equation, and use the properties of absolute values. As a result, we obtain:

$$|(A) + (B) + (C)| \leqslant$$

$$\leqslant \int_0^1 |w(t) + r_1| \cdot \left|f'\Big((1-t)\varrho_1 + t\frac{2\varrho_1 + \varrho_2}{3}\Big) - f'\left(\varrho_1\right)\right|dt +$$

$$+\int_0^1 |w(t)+r_2|\cdot\left|f'\left((1-t)\frac{2\varrho_1+\varrho_2}{3}+t\frac{\varrho_1+2\varrho_2}{3}\right)-f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|dt+$$

$$+\int_0^1 |w(t)+r_3|\cdot\left|f'\left((1-t)\frac{\varrho_1+2\varrho_2}{3}+t\varrho_2\right)-f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|dt+$$

$$+\left|f'(\varrho_1)\int_0^1 (w(t)+r_1)\,dt+f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\int_0^1 (w(t)+r_2)\,dt+\right.$$

$$\left.+f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\int_0^1 (w(t)+r_3)\,dt\right|.$$

Assuming that f' is a Lipschitz function, we have:

$$|(A)+(B)+(C)|\leqslant$$

$$\leqslant\frac{(\varrho_2-\varrho_1)L}{3}\left[\int_0^1|w(t)+r_1|\,tdt+\int_0^1|w(t)+r_2|\,tdt+\int_0^1|w(t)+r_3|\,tdt\right]+$$

$$+\left|\left[f'(\varrho_1)+f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)+f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right]\int_0^1 w(t)dt+\right.$$

$$\left.+f'(\varrho_1)\,r_1+r_2f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)+f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)r_3\right|.$$

Then, multiplying both sides of the resulting inequality by $\frac{\varrho_2-\varrho_1}{9}$, we obtain (47). The proof is completed. □

Corollary 6. *For $h(t)=t$ and $m=1$, if we choose $w(t)=t-\frac{1}{2}$, $r_1=\frac{1}{8}$, $r_2=0$, and $r_3=-\frac{1}{8}$, then we get from (47):*

$$\left|\frac{1}{8}\left[f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\right]-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)du\right|\leqslant$$

$$\leqslant\frac{41\,(\varrho_2-\varrho_1)^2\,L}{1728}. \quad (51)$$

Proof. Indeed, for $\mathbf{\Phi}$, we have:

$$\left|\mathbf{\Phi}\left(f, t-\frac{1}{2}, \frac{1}{8}, 0, -\frac{1}{8}\right)\right| = \left|\frac{1}{8}\left[f(\varrho_1) + 3f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + \right.\right.$$
$$\left.\left. + 3f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + f(\varrho_2)\right] - \frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2} f(u)\,du\right|.$$

Calculate the integrals on the right-hand side of (47). For the integrals of the first group we have from (31):

$$\int\limits_0^1 \left|t-\frac{3}{8}\right| t\,dt + \int\limits_0^1 \left|t-\frac{1}{2}\right| t\,dt + \int\limits_0^1 \left|t-\frac{5}{8}\right| t\,dt = \frac{25}{64}$$

and for the integrals of the second group, we get:

$$\left|\left[f'(\varrho_1) + f'\left(\frac{2\varrho_1+\varrho_2}{3}\right) + f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right]\int\limits_0^1 w(t)dt + \right.$$
$$\left. + f'(\varrho_1)\, r_1 + r_2 f'\left(\frac{2\varrho_1+\varrho_2}{3}\right) + f'\left(\frac{\varrho_1+2\varrho_2}{3}\right) r_3\right| =$$
$$= \left|0 + \frac{f'(\varrho_1)}{8} + 0 - \frac{f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)}{8}\right| \leqslant \frac{L}{8}\left|\varrho_1 - \frac{\varrho_1+2\varrho_2}{3}\right| = \frac{(\varrho_2-\varrho_1)\,L}{12}.$$

Thus, for the right-hand side, we have

$$\frac{(\varrho_2-\varrho_1)^2 L}{27}\frac{25}{64} + \frac{(\varrho_2-\varrho_1)^2 L}{9\cdot 12} = \frac{41\,(\varrho_2-\varrho_1)^2 L}{1728}.$$

The proof is complete. □

Remark 7. *Inequality (51) was obtained in Theorem* 3.2 *from* [25] *and in Theorem* 2.2 *from* [27, for $\eta = 3$].

4. Conclusions. In this paper, we present new 3/8−Simpson integral inequalities via generalized (weighted) integral operators for functions whose first derivatives (and their powers) satisfy (h, m)-convexity of the second type and the Lipschitz condition. By introducing a new integral identity, our method unifies and extends a number of known results on these inequalities. In particular, we obtain estimates for classical convex, (h, m)-modified convex, and Lipschitz functions, as well as refined bounds including Riemann–Liouville. Using certain choices of parameters, we show how these results reduce to traditional inequalities. Given the generality and strength of the idea presented in the work, we believe that one of the future possibilities is the application to new integral inequalities,

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