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EXPANSIVE MAPPINGS ENDOWED WITH A DIRECTED GRAPH IN DISLOCATED METRIC SPACE AND SOME NEW FIXED-POINT RESULTS

Abstract. In this paper, we introduce the concept of Gr-expansive and θ-Gr-expansive mapping and derive some new fixed-point results endowed with a directed graph considering such mappings in a complete dislocated metric space. A coupled fixed-point theorem is also established by applying Gr-expansive condition. Examples are provided in support of the derived results. The solution to a Fredholm integral equation is also formulated to show the applicability of the results.

Key words: *expansive mapping, dislocated metric space, directed graph, fixed point*

2020 Mathematical Subject Classification: *47H10, 54H25, 05C20*

1. Introduction. In recent years, applications of fixed-point theory has gained importance in the study of nonlinear analysis with different remarkable outcomes. After the foundation of the metric fixed-point theory by Banach [1] in 1922, researchers have developed and generalized several fixed-point theorems leading to notable advancements and diverse practical applications in various fields. An interesting extension of Banach contraction principle was given by Edelstein in [5]. In [2], Bhaskar et. al. derived a fixed-point theorem for mixed monotone mapping in a metric space with partial order with an application to periodic boundary-value problem. In 2024, Puvar et. al. [17] used generalized Γ-C_F simulation function to obtain a common fixed-point theorem in G-metric space.

In 1984, Wang et al. [20] studied expansive mappings and presented fixed-point results for such mappings within the framework of complete metric space. After that, many researchers have significantly extended and improved fixed-point theorems concerning expansive mappings. In 2000, Hitzler and Seda [10] introduced the concept of dislocated metric space, where the self-distance need not be zero. Since then, dislocated metric spaces have played a vital role in topology, logic programming, electronics engineering, and different other branches. Considering expansive mapping in dislocated metric space, some fixed-point results were discussed by Rahman et. al. [19]. Fixed-point results for F-expanding mappings in complete G-metric space was depicted in the work of Górnicki [7]. In 2024, Das et. al. [4] established some new fixed-point results for expansive type mappings in dislocated quasi-metric space with application to integral equation.

In 2006, Espinola et al. [6] introduced a graph-theoretical framework for fixed-point theorems in R-trees. In 2007, Jachymski [11] developed the concept of Banach G-contraction mappings and proved some fixed-point results. In [3], Bojor defined the notion of G-Reich type mappings with application in fixed-point theory. In [13], Kamran et. al. derived fixed-point theorems in generalized metric space endowed with graphs. Onsod et. al. introduced θ-G contraction mappings in [16] to derive results in metric space endowed with a graph. In 2022, Mebarki et. al. [14] applied coupled fixed-point result in b-metric space endowed with a directed graph to differential equation with infinite delay. In 2024, Rafique et. al. [18] discussed fuzzy multi-valued F-contractive mappings with a directed graph in a parametric metric space.

Motivated by these works, in this paper we present the idea of Gr-expansive mapping and derive some new fixed-point results endowed with a directed graph in the context of a complete dislocated metric space. A coupled fixed-point theorem is also established using Gr-expansive mapping. We provide suitable examples, along with applications that emphasize the significance and practical impact of the derived results.

2. Preliminaries. Hitzler et al. [10] introduced the idea of dislocated metric space in the following way.

Definition 1. *[10] Let X be a nonempty set and $d\colon X \times X \longrightarrow [0, \infty)$ be a distance function satisfying the following conditions:*

(*i*) $d(x, y) = 0 \implies x = y \quad \forall x, y \in X$,

(*ii*) $d(x, y) = d(y, x) \quad \forall x, y \in X$,

(*iii*) $d(x, y) \leqslant d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

Then d is called a dislocated metric (d-metric) on X and (X, d) is called a dislocated metric space (d-metric space).

Example 1. Let $X = \mathbb{R}$ and $d\colon X \times X \longrightarrow [0, \infty)$ be defined by

$$d(x, y) = \max\{|x|, |y|\} \;\forall\; x,y \in X.$$

Then (X, d) is a d-metric space.

Definition 2. *[10] A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$, such that*

$$d(x_m, x_n) < \epsilon \;\forall\; m, n \geqslant n_0.$$

$\{x_n\}$ is said to be convergent to a limit x if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

(X, d) is called complete if every Cauchy sequence in X is convergent with respect to d.

Lemma 1. *[10] Limits in d-metric spaces are unique.*

Lemma 2. *[15] If x is a limit of some sequence $\{x_n\}$ in a d-metric space $(X,)$, then $d(x,) = 0$.*

Let $(X,)$ be a d-metric space and let Δ represent the diagonal of the cartesian product $X \times X$. We take the directed graph $G = (V(G), E(G))$, where the vertex set $V(G)$ corresponds to X and the edge set $E(G)$ includes all the loops, i.e., $\Delta \subseteq E(G)$. Furthermore, G can be regarded as a weighted graph by giving each edge a weight corresponding to the distance between its vertices. We denote $\Psi = \{G\colon G$ is a directed graph with $V(G) = X$ and $\Delta \subseteq E(G)\}$.

In 2007, Jachymski [11] generalized the Banach contraction principle on a complete metric space endowed with a graph. He introduced the notion of Banach G-contraction as follows:

Definition 3. *[11] Let (X, d) be a metric space endowed with the directed graph $G = (V(G), E(G))$. A mapping $T\colon X \longrightarrow X$ is a Banach G-contraction (or simply G-contraction) if it satisfies the following conditions:*

a) T is edge-preserving, i.e., $(x, y) \in E(G) \implies (Tx, Ty) \in E(G)$ $\forall\; x, y \in X$.

b) T decreases weights of edges of G, i.e., there exists a $\lambda \in (0,1)$ such that

$$d(Tx, Ty) \leqslant \lambda d(x, y) \;\forall\; x, y \in X.$$

Definition 4. *[11] Let (X, d) be a complete metric space and G be a directed graph. A self-mapping $T: X \longrightarrow X$ is said to be G-continuous if for each sequence $\{x_n\}$ in X with $x_n \longrightarrow x$ in X and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, we have $Tx_n \longrightarrow Tx$.*

In 1987, Guo and Lakshmikantham [8] introduced the concept of coupled fixed point in Banach spaces. The same concept holds in case of d-metric space also.

Definition 5. *[8] Let (X, d) be a d-metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $T: X \times X \longrightarrow X$ if $T(x, y) = x$ and $T(y, x) = y$.*

Jleli and Samet [12] introduced a new type of contraction mapping called θ-contraction. They denote by Θ the set of functions $\theta: (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$.

(Θ_3) there exists $r \in (0, 1)$ and $l \in (0, \infty]$, such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l.$$

In 2020, Yesilkaya et al. [21] introduced the concept of θ-expansive mapping in ordered metric space.

Definition 6. *[21] Let $(X, \leqslant, d)$ be an ordered metric space. A mapping $T: X \longrightarrow X$ is said to be surjective θ-expansive if there exists $\theta \in \Theta$ and $\eta > 1$, such that*

$$\theta(d(Tx, Ty)) \geqslant [\theta(d(x, y))]^\eta$$

for all $(x, y) \in M$, where

$$M = \{(x, y) \in X \times X: x \leqslant y, d(Tx, Ty) > 0\}.$$

3. Main results. We define the concept of Gr-expansive mapping and derive some fixed-point results considering such mapping in complete d-metric space endowed with a directed graph. Next we show an application of θ-Gr-expansive type mappings to Fredholm integral equation. Here $F(T)$ denotes the set of fixed point of the mapping T.

Definition 7. *Let (X, d) be a d-metric space endowed with the directed graph $G = (V(G), E(G))$. A mapping $T\colon X \longrightarrow X$ is said to be Gr-expansive mapping if it satisfies the following conditions:*

a) $(Tx, Ty) \in E(G) \implies (x, y) \in E(G)\ \forall\ x, y \in X.$

b) *T increases weights of edges of G, i.e., there exists some $\lambda > 1$, such that*

$$d(Tx, Ty) \geqslant \lambda d(x, y)\ \forall\ (x, y) \in E(G). \tag{1}$$

Example 2. Let $X = \mathbb{N} \cup \{0\}$ and define a d-metric $d\colon X \times X \longrightarrow [0, \infty)$ by $d(x, y)=\max\{x, y\}\ \forall\ x, y \in X$.

Define $T\colon X \longrightarrow X$ by $Tx = 5x\ \forall\ x \in X$.

Consider the graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x)\colon x \in X\} \cup \{(0, 1), (0, 5)\}$.

Take $\lambda = \dfrac{3}{2}$. Then T is a Gr-expansive mapping.

Example 3. Let $X = \{0, 1, 2\}$. Define a d-metric $d\colon X \times X \longrightarrow [0, \infty)$ as follows:

$$d(0,0) = d(1,1) = d(2,2) = 0,\ d(0,1) = d(1,0) = 7,$$
$$d(0,2) = d(2,0) = 3,\ d(1,2) = d(2,1) = 4.$$

Define $T\colon X \longrightarrow X$ by

$$T(0) = 0,\ T(1) = 0,\ T(2) = 1.$$

Consider the graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(0, 0), (1, 1), (2, 2), (2, 1)\}$.

For $\lambda = \dfrac{3}{2}$, T is a Gr-expansive mapping.

Theorem 1. *Let (X, d) be a complete d-metric space endowed with the directed graph $G = (V(G), E(G))$ and T be an onto Gr-expansive self-mapping on X.*

If the following conditions hold:

i) *there exists a point $x_1 \in X$, such that $(Tx_1, x_1) \in E(G)$,*

ii) *T is G-continuous,*

then $F(T) \neq \phi$. Moreover, if for each $x, y \in F(T)$ we have $(x, y) \in E(G)$, then T has a unique fixed point in X.

Proof. By i), there exists a point $x_1 \in X$ with $(Tx_1, x_1) \in E(G)$. Let $Tx_1 = x_0$. Therefore,

$$(x_0, x_1) \in E(G). \tag{2}$$

For $x_1 \in X$, since T is onto, there exists $x_2 \in X$, such that $x_1 = Tx_2$. From (2), we have $(Tx_1, Tx_2) \in E(G)$. By condition a) of Definition 7,

$$(x_1, x_2) \in E(G). \tag{3}$$

Also, there exists $\lambda > 1$, such that

$$d(x_0, x_1) = d(Tx_1, Tx_2) \geqslant \lambda\, d(x_1, x_2).$$

This implies that

$$d(x_1, x_2) \leqslant \frac{1}{\lambda}\, d(x_0, x_1),$$

$$\text{i.e., } d(x_1, x_2) \leqslant k\, d(x_0, x_1), \text{ where } k = \frac{1}{\lambda}.$$

Again for $x_2 \in X$, since T is onto, there exists $x_3 \in X$, such that $x_2 = Tx_3$. From (3), we have $(Tx_2, Tx_3) \in E(G)$ and so, $(x_2, x_3) \in E(G)$.
Now,

$$d(x_1, x_2) = d(Tx_2, Tx_3) \geqslant \lambda\, d(x_2, x_3).$$

This implies that

$$d(x_2, x_3) \leqslant \frac{1}{\lambda}\, d(x_1, x_2),$$

$$\text{i.e., } d(x_2, x_3) \leqslant k\, d(x_1, x_2).$$

Continuing in this way, we get a sequence $\{x_n\} \subseteq X$, such that $(x_n, x_{n+1}) \in E(G)$ with $x_n = Tx_{n+1} \ \ \forall\, n \in \mathbb{N} \cup \{0\}$ and

$$d(x_n, x_{n+1}) \leqslant k\, d(x_{n-1}, x_n).$$

Thus,

$$d(x_n, x_{n+1}) \leqslant k^n\, d(x_0, x_1) \quad \forall\, n \in \mathbb{N}.$$

Now, for each $m > n \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leqslant (k^n + k^{n+1} + \ldots + k^{m-1}) d(x_0, x_1) \leqslant \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

Since $k \in (0, 1)$, so $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, $x_n \longrightarrow u$ in X.

As T is a G-continuous, so $Tx_n \longrightarrow Tu$, i.e., $x_{n-1} \longrightarrow Tu$. Hence, $Tu = u$, and thus u is a fixed point of T.

To show the uniqueness, let u and v be two distinct fixed points of T, such that $(u, v) \in E(G)$. Then

$$d(u, v) = d(Tu, Tv) \geqslant \lambda\, d(u, v).$$

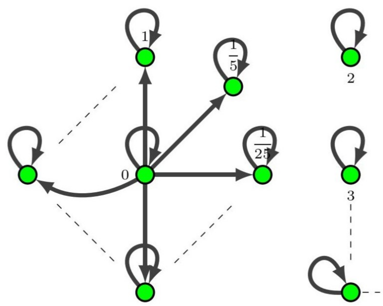
Since $\lambda > 1$, so, $d(u, v) = 0$, which implies $u = v$. □

We illustrate Theorem 1 by the following example.

Example 4. Let $X = [0, \infty)$ and define a complete d-metric $d\colon X \times X \longrightarrow [0, \infty)$ by $d(x, y) = \max\{x, y\}$ $\forall$ $x, y \in X$.
Let $T\colon X \longrightarrow X$ be such that

$$Tx = \begin{cases} 5x, & 0 \leqslant x < 1, \\ 6 - x, & 1 \leqslant x < 2, \\ 2x, & x \geqslant 2. \end{cases}$$

Consider the graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x)\colon x \in X\} \cup \{(0, \frac{1}{5^k})\colon k = 0, 1, 2, \ldots\}$. Then for $\lambda = 1.1$, T is an onto Gr-expansive mapping. Also, all the conditions of Theorem 1 are satisfied. Clearly, 0 is the unique fixed point of T on X.



Remark 1. *It is seen that if the mapping T is not surjective, then Theorem 1 does not hold. For this, we consider the following example.*

Let $X = [0, \infty)$ and define a complete d-metric $d\colon X \times X \longrightarrow [0, \infty)$ by $d(x, y) = |x - y|$ $\forall$ $x, y \in X$. Let $T\colon X \longrightarrow X$ be such that $Tx = 5x + 1$, $x \in X$. Here T is not onto.

Consider the graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x)\colon x \in X\} \cup \{(1, 2), (0, \frac{1}{5}), (1, 0)\}$.

Then for $\lambda = 2$, T is a Gr-expansive mapping. Also, conditions i) and ii) of Theorem 1 are satisfied. But T has no fixed point.

The following is a fixed-point result for another type of Gr-expansive mapping.

Theorem 2. *Let (X, d) be a complete d-metric space endowed with the directed graph $G = (V(G), E(G))$ and T be an onto self-mapping on X satisfying*

$$d(Tx, Ty) + d(Tx, x) \geqslant \lambda\, d(x, y) + d(Ty, y), \tag{4}$$

for all $(x, y) \in E(G)$ with $\lambda > 1$. If the following conditions hold:

i) $(Tx, Ty) \in E(G) \implies (x, y) \in E(G)$,

ii) *there exists $x_1 \in X$, such that $(Tx_1, x_1) \in E(G)$,*

iii) *T is G-continuous,*

then $F(T) \neq \phi$. Moreover, if for each $x, y \in F(T)$ we have $(x, y) \in E(G)$, then T has a unique fixed point in X.

Proof. By ii), there exists $x_1 \in X$ with $(Tx_1, x_1) \in E(G)$. Let $Tx_1 = x_0$. Therefore,

$$(x_0, x_1) \in E(G). \tag{5}$$

For $x_1 \in X$, since T is onto, there exists $x_2 \in X$, such that $x_1 = Tx_2$. From (5), we have $(Tx_1, Tx_2) \in E(G)$; and by i),

$$(x_1, x_2) \in E(G). \tag{6}$$

Using (4), we have,

$$d(Tx_1, Tx_2) + d(Tx_1, x_1) \geqslant \lambda\, d(x_1, x_2) + d(Tx_2, x_2),$$

$$\text{i.e., } 2d(x_0, x_1) \geqslant (\lambda + 1) d(x_1, x_2),$$

$$\text{i.e., } d(x_1, x_2) \leqslant k\, d(x_0, x_1), \quad \text{where } k = \frac{2}{\lambda + 1}.$$

Again for $x_2 \in X$, since T is onto, there exists $x_3 \in X$, such that $x_2 = Tx_3$. From (6), $(Tx_2, Tx_3) \in E(G)$ and by i), $(x_2, x_3) \in E(G)$.

Now,

$$d(Tx_2, Tx_3) + d(Tx_2, x_2) \geqslant \lambda\, d(x_2, x_3) + d(Tx_3, x_3),$$

$$\text{i.e., } 2d(x_1, x_2) \geqslant (\lambda + 1)\, d(x_2, x_3),$$

$$\text{i.e., } d(x_2, x_3) \leqslant k\ d(x_1, x_2).$$

Continuing in this way, we generate a sequence $\{x_n\} \subseteq X$, such that $(x_n, x_{n+1}) \in E(G)$ with $x_n = Tx_{n+1}\ \ \forall\ n \in \mathbb{N} \cup \{0\}$ and

$$d(x_n, x_{n+1}) \leqslant k\ d(x_{n-1}, x_n).$$

Also,

$$d(x_n, x_{n+1}) \leqslant k^n\ d(x_0, x_1)\ \ \forall\ n \in \mathbb{N}.$$

Now, for each $m > n \in \mathbb{N}$,

$$d(x_n, x_m) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \leqslant \frac{k^n}{1-k} d(x_0, x_1).$$

Since $k \in (0, 1)$ for $\lambda > 1$, so, $\{x_n\}$ is a Cauchy sequence in X, which converges to some $u \in X$.

As T is a G-continuous, so, $Tx_n \longrightarrow Tu$ i.e., $x_{n-1} \longrightarrow Tu$. Hence $Tu = u$, and thus u is a fixed point of T.

For the uniqueness, let u and v be two distinct fixed points of X, such that $(u, v) \in E(G)$. Then

$$d(Tu, Tv) + d(Tu, u) \geqslant\ \lambda\ d(u, v) + d(Tv, v).$$

Using Lemma 2, since $d(u, u) = 0$, we get:

$$d(u, v) \geqslant \lambda\ d(u, v) + d(v, v),$$

$$\text{i.e., } d(u, v) \geqslant \lambda\ d(u, v).$$

Since $\lambda > 1$, so, $d(u, v) = 0$, which implies $u = v$. □

Example 5. Let $X = [0, \infty)$ and define a complete d-metric $d\colon X \times X \longrightarrow [0, \infty)$ by $d(x, y) = \max\{x, y\} + |x - y|\ \forall\ x, y \in X$.

Let $T\colon X \longrightarrow X$ be such that

$$Tx = \begin{cases} 4x, & 0 \leqslant x < 1, \\ 5 - x, & 1 \leqslant x < 2, \\ \frac{3x}{2}, & x \geqslant 2. \end{cases}$$

Consider the graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x)\colon x \in X\} \cup \{(\frac{1}{4^k}, 0)\colon k = 0, 1, 2, \ldots\}$.

For $\lambda = 1.1$, it is seen that all the conditions of Theorem 2 are satisfied. Here 0 is the unique fixed point of T.

In 2013, Han et.al. [9] derived some new theorems for expanding mappings in cone metric spaces without using continuity condition. In our next result, we replace the G-continuity of T by another condition.

Theorem 3. *Let (X, d) be a complete d-metric space endowed with the directed graph $G = (V(G), E(G))$ and T be an onto Gr-expansive self-mapping on X.*

If the following conditions hold:

- *i) there exists a point $x_1 \in X$, such that $(Tx_1,x_1) \in E(G)$,*
- *ii) for each sequence $\{x_n\}$, such that $x_n \longrightarrow u$ with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$, if $Tp = u$ then $(p, x_n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$,*

then $F(T) \neq \phi$. Moreover, if for each $x, y \in F(T)$, we have $(x, y) \in E(G)$, then T has a unique fixed point in X.

Proof. As in Theorem 1, $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, $x_n \longrightarrow u$ in X.

Let $p \in X$ be such that $Tp = u$. Then by ii), $(p, x_n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$. Now,

$$d(u, x_n) = d(Tp, Tx_{n+1}) \geqslant \lambda\, d(p, x_{n+1}).$$

Taking limit as $n \to \infty$, and using continuity of $d(u, .)$, $d(p, .) \colon X \longrightarrow \mathbb{R}$, we get,

$$d(u, u) \geqslant \lambda\, d(p, u).$$

Using Lemma 2 and since $\lambda > 1$, we have, $d(p, u) = 0$ and thus $p = u$.

Uniqueness part is similar to Theorem 1. □

4. θ-Gr-expansive type mapping and application to Fredholm integral equation. In this section, we define Reich and Chatterjea-type θ-Gr-expansive mapping and give application to the Fredholm integral equation via the fixed point formulation of such mappings.

Definition 8. *Let (X, d) be a d-metric space. A self-mapping T on X is said to be a Reich-type θ-Gr-expansive mapping if there exists $\theta \in \Theta$ and $G \in \Psi$, such that*

- *i) $(Tx, Ty) \in E(G) \implies (x, y) \in E(G)\ \forall\ x, y \in X$,* *ii) there exsits* $\eta > 1$ *and* $\alpha, \beta \in [0,1]$*, such that*

$$d(Tx,Ty) \neq 0 \implies \theta(d(Tx,Ty)) \geqslant [\theta(\alpha d(Tx,x) + \beta d(Ty,y) + d(x,y))]^{\eta}, \quad (7)$$

for all $x, y \in X$ *with* $(x,y) \in E(G)$*.*

Example 6. Let $X = \{0,1,2\}$. Define a d-metric $d\colon X \times X \longrightarrow [0,\infty)$ as follows:

$$d(0,0) = 0,\ d(1,1) = 3,\ d(2,2) = 1,\ d(0,1) = d(1,0) = 7,$$
$$d(0,2) = d(2,0) = 3,\ d(1,2) = d(2,1) = 4.$$

Define $T\colon X \longrightarrow X$ by $T(0) = 0$, $T(1) = 0$, $T(2) = 1$. Consider the graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(0,0),(1,1),(2,2),(1,0),(2,0)\}$. Take $\theta(t) = e^{\sqrt{t}}, t \in (0,\infty)$, $\alpha = \beta = 0$, $\eta = \frac{3}{2}$.

Clearly, $d(Tx,Ty) \neq 0$ only for the edges $(2,2)$ and $(2,0)$, and T satisfies the condition (7) for these edges. Therefore, T is a Reich-type θ-Gr-expansive mapping.

Definition 9. *Let* (X,d) *be a d-metric space. A self-mapping* T *on* X *is said to be a Chatterjea-type* θ*-*Gr*-expansive mapping if there exists* $\theta \in \Theta$ *and* $G \in \Psi$*, such that*

i) $(Tx,Ty) \in E(G) \implies (x,y) \in E(G)\ \forall\ x,y \in X$*,*

ii) there exists $\eta > 1$ *and* $\alpha \in [0,1]$*, such that*

$$d(Tx,Ty) \neq 0 \implies \theta(d(Tx,Ty)) \geqslant [\theta(\alpha\ d(Tx,y) + d(Ty,x))]^{\eta}, \quad (8)$$

for all $x, y \in X$ *with* $(x,y) \in E(G)$*.*

Example 7. Let $X = \{0,1,2\}$. Define a d-metric $d\colon X \times X \longrightarrow [0,\infty)$ as follows:

$$d(0,0) = 0,\ d(1,1) = 1,\ d(2,2) = \frac{2}{3},\ d(0,1) = d(1,0) = \frac{9}{10},$$
$$d(0,2) = d(2,0) = \frac{7}{10},\ d(1,2) = d(2,1) = \frac{4}{5}.$$

We take T, G and θ as in Example 6. Then for $\alpha = 0$, $\eta = \frac{11}{10}$, T is a Chatterjea-type θ-Gr-expansive mapping.

Theorem 4. *Let* (X,d) *be a complete d-metric space endowed with the directed graph* $G = (V(G), E(G))$ *and* T *be an onto Reich-type* θ*-*Gr*-expansive self-mapping on* X *satisfying the following conditions:*

i) there exists $x_1 \in X$, such that $(Tx_1, x_1) \in E(G)$,

ii) T is G-continuous.

Then $F(T) \neq \phi$. Moreover, if for each $x, y \in F(T)$ we have $(x, y) \in E(G)$, then T has a unique fixed point in X.

Proof. As in Theorem 2, we generate a sequence $\{x_n\} \subseteq X$, such that $(x_n, x_{n+1}) \in E(G)$ with $x_n = Tx_{n+1}$ $\forall\, n \in \mathbb{N} \cup \{0\}$.
If for some n, $x_{n+1} = x_n$, then x_n is a fixed point of T.
Now, assume that $x_n \neq x_{n+1}$ for all n. Without loss the generality, we take $d(Tx_n, Tx_{n+1}) > 0$. Using (7) we have,

$$\begin{aligned}\theta(d(x_{n-1}, x_n)) &= \theta(d(Tx_n, Tx_{n+1})) \\ &\geqslant [\theta(\alpha d(Tx_n, x_n) + \beta d(Tx_{n+1}, x_{n+1}) + d(x_n, x_{n+1}))]^\eta \\ &= [\theta(\alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + d(x_n, x_{n+1}))]^\eta \\ &\geqslant [\theta(d(x_n, x_{n+1}))]^\eta.\end{aligned}$$

Now, proceeding similarly to Theorem 3 in [21], we can show that $\{x_n\}$ is a Cauchy sequence in X, which converges to some $u \in X$.

Since T is a G-continuous, so, $Tx_{n+1} \longrightarrow Tu$, i.e., $x_n \longrightarrow Tu$. Hence, $Tu = u$, and thus u is a fixed point of T.

For the uniqueness, let u and v be two distinct fixed points of X, such that $(u, v) \in E(G)$. Using (7), we have:

$$\begin{aligned}\theta(d(u, v)) &= \theta(d(Tu, Tv)) \\ &\geqslant [\theta(\alpha d(Tu, u) + \beta d(Tv, v) + d(u, v))]^\eta \\ &\geqslant [\theta(d(u, v))]^\eta > [\theta(d(u, v))],\end{aligned}$$

which is a contradiction. Thus, $u = v$. □

Example 8. In Example 4, we take $\theta(t) = e^{\sqrt{t}}$, $\alpha = \beta = 0$, $\eta = 1.1$. Then T is an onto Reich-type θ-Gr-expansive mapping. Also, all the conditions of Theorem 4 are satisfied. Clearly, 0 is the unique fixed point of T on X.

Corollary 1. *Let (X, d) be a complete d-metric space endowed with the directed graph $G = (V(G), E(G))$ and T be an onto self-mapping on X with $\theta \in \Theta$ satisfying*

$$d(Tx, Ty) \neq 0 \implies \theta(d(Tx, Ty)) \geqslant [\theta(d(x, y))]^\eta, \tag{9}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $\eta > 1$. If the following conditions hold:

i) $(Tx, Ty) \in E(G) \implies (x, y) \in E(G)$,

ii) *there exists a point* $x_1 \in X$, *such that* $(Tx_1, x_1) \in E(G)$,

iii) T *is* G*-continuous,*

then $F(T) \neq \phi$. *Moreover, if for each* $x, y \in F(T)$ *we have* $(x, y) \in E(G)$, *then* T *has a unique fixed point on* X.

Remark 2. *Condition (9) is an extension of condition (1).*
Taking $\theta(t) = e^{\sqrt{t}}, t > 0$ *in (9), we obtain*

$$e^{\sqrt{d(Tx,Ty)}} \geqslant \left[e^{\sqrt{d(x,y)}}\right]^{\eta} = e^{\sqrt{\eta^2 d(x,y)}},$$

from which we get $d(Tx, Ty) \geqslant \lambda\, d(x, y)$ *for all* $x, y \in X$ *with* $(x, y) \in E(G)$ *and* $\lambda = \eta^2 > 1$.

Now we consider the following Fredholm integral equation:

$$x(t) = \int_a^b k(t, s, x(s))ds + g(t), \text{ for all } t \in [a, b], \tag{10}$$

where $x, g\colon [a, b] \longrightarrow [a, b]$ and $k\colon [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous mappings. We take x as onto and $|g(t)| \leqslant \mu_1$, $|k(t, s, r)| \leqslant \mu_2$ for all t,$s \in [a, b], r \in \mathbb{R}$, where $0 < \mu_1, \mu_2 < b$ with $\mu_2 \leqslant \frac{b-\mu_1}{b-a}$. Let $X = C([a, b], [a, b])$, the set of all continuous functions from $[a, b]$ to $[a, b]$ and the d-metric $d\colon X \times X \longrightarrow [0, \infty)$ be defined by

$$d(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)| \ \forall\ x, y \in X.$$

Theorem 5. *Suppose that*

i) $k(t, s, .)\colon \mathbb{R} \longrightarrow \mathbb{R}$ *is non-decreasing for each* $t, s \in [a, b]$,

ii) *there exists* $\lambda > 1$, *such that for all* $x, y \in X$,

$$\left|\int_a^b k(t, s, x(s))ds - \int_a^b k(t, s, y(s))ds\right| \geqslant \lambda|x(t) - y(t)| \ \forall\ t \in [a, b].$$

Then the integral equation (10) has a solution.

Proof. Define a mapping $T\colon X \longrightarrow X$ by

$$Tx(t) = \int_a^b k(t,s,x(s))ds + g(t) \;\forall\; t \in [a,b]. \tag{11}$$

Clearly, from i), T is non-decreasing. Also, T is onto, since x is an onto mapping.

Consider a graph G consisting of $V(G) = X$ and $E(G) = \{(x,y) \in X \times X\colon x(t) \leqslant y(t) \;\forall\; t \in [a,b]\}$. Since k and g are continuous, so, T is G-continuous.

Suppose $(Tx, Ty) \in E(G)$. Therefore $Tx(t) \leqslant Ty(t) \;\forall\; t \in [a,b]$. If $x \geqslant y$, since T is non-decreasing, so, $Tx(t) \geqslant Ty(t) \;\forall\; t \in [a,b]$, which is a contradiction. Therefore, $x \leqslant y$, which implies that $(x,y) \in E(G)$.

We define a continuous mapping x_0 on $[a,b]$ by

$$x_0(t) = g(t) + (b - \mu_1),\; t \in [a,b].$$

By (11),

$$\begin{aligned} Tx_0(t) = \int_a^b k(t,s,x_0(s))ds + g(t) &\leqslant \int_a^b |k(t,s,x_0(s))|ds + g(t) \\ &\leqslant \mu_2(b-a) + g(t) \leqslant x_0(t), \end{aligned}$$

which implies that $(Tx_0, x_0) \in E(G)$.

Now, for every $x, y \in X$ with $(x,y) \in E(G)$ and $d(Tx,Ty) > 0$, we have:

$$\begin{aligned} d(Tx,Ty) &= \sup_{t\in[a,b]} \Big|Tx(t) - Ty(t)\Big| \\ &= \sup_{t\in[a,b]} \Big|\int_a^b k(t,s,x(s))ds - \int_a^b k(t,s,y(s))ds\Big| \\ &\geqslant \sup_{t\in[a,b]} \lambda\,|x(t) - y(t)| = \lambda\, d(x,y). \end{aligned}$$

Taking $\theta(t) = e^{\sqrt{t}}, t > 0$, so that $\theta \in \Theta$, we have:

$$\theta(d(Tx,Ty)) = e^{\sqrt{d(Tx,Ty)}} \geqslant e^{\sqrt{\lambda d(x,y)}} = \left[e^{\sqrt{d(x,y)}}\right]^{\eta} = [\theta(d(x,y))]^{\eta}$$

$\forall\ (x, y) \in E(G)$, where $\eta = \sqrt{\lambda} > 1$ as $\lambda > 1$. Thus, the mapping T satisfies the conditions of Corollary 1. So, there exists a fixed point of the mapping T in X, which is a solution for the integral equation (10). □

Similar results as in Theorem 4 and Theorem 5 also hold in the case of Chatterjea-type θ-Gr-expansive mappings.

Theorem 6. *Let (X, d) be a complete d-metric space endowed with the directed graph $G = (V(G), E(G))$ and $T\colon X \longrightarrow X$ be an onto Chatterjea-type θ-Gr-expansive mapping satisfying the following conditions:*

i) there exists $x_1 \in X$, such that $(Tx_1, x_1) \in E(G)$

ii) T is G-continuous.

Then $F(T) \neq \phi$. Moreover, if for each $x, y \in F(T)$ we have $(x, y) \in E(G)$, then T has a unique fixed point on X.

The proof follows as in Theorem 4.

Theorem 7. *Suppose that*

i) $k(t, s, .)\colon \mathbb{R} \longrightarrow \mathbb{R}$ is non-decreasing for each $t, s \in [a, b]$,

ii) there exists $\lambda > 1$, such that for all $x, y \in X$,

$$\Big|\int_a^b k(t,s,x(s))ds - \int_a^b k(t,s,y(s))ds\Big|$$
$$\geqslant \lambda\Big|\int_a^b k(t,s,y(s))ds + g(t) - x(t)\Big|\ \forall\ t \in [a, b].$$

Then the integral equation (10) has a solution.

Proof. Taking $\alpha = 0$ in (8), the proof is analogous to the proof of Theorem 5. □

5. Coupled fixed-point theorem with Gr-expansive-type condition. This section contains a coupled fixed-point theorem for onto Gr-expansive type mapping in a complete d-metric space endowed with graph $G = (V(G),E(G))$, where $V(G) = X \times X$ and $E(G) \subseteq X^2 \times X^2$ includes all the loops.

Theorem 8. *Let (X, d) be a complete d-metric space endowed with the directed graph G and $T\colon X \times X \longrightarrow X$ be a continuous onto mapping satisfying the condition:*

$$d(T(x, y), T(u, v)) \geqslant a_1 d(x, u) + a_2 d(y, v), \tag{12}$$

for all $((x, u), (y, v)) \in E(G)$ with $a_1, a_2 > 1$. If the following conditions hold:

- *i) there exist x_1, y_1 in X, such that $((T(x_1, y_1), T(y_1, x_1)), (x_1, y_1))$ and $((T(y_1, x_1), T(x_1, y_1)), (y_1, x_1)) \in E(G)$,*
- *ii) for each $(x, y) \in X \times X$,*

$$((T(x, y), T(y, x)), (x, y)) \in E(G),$$

then T has a coupled fixed point. Moreover, if for each coupled fixed point (x, y) and (u, v), we have $((x, y), (u, v))$ and $((y, x), (v, u)) \in E(G)$, then T has a unique coupled fixed point.

Proof. We take $T(x_1 y_1) = x_0$ and $T(y_1 x_1) = y_0$. Then, by i),

$$((x_0, y_0),(x_1, y_1)) \text{ and } ((y_0, x_0), (y_1, x_1)) \in E(G). \tag{13}$$

For $x_1 \in X$, since T is onto, there exists $(x_2, y_2) \in X \times X$, such that $T(x_2, y_2) = x_1$. Let $T(y_2, x_2) = y_1$. By ii), we have:

$$((x_1, y_1), (x_2, y_2)) \text{ and } ((y_1, x_1), (y_2 x_2)) \in E(G).$$

Continuing in this way, we generate two sequences $\{x_n\}$, $\{y_n\} \subseteq X$, such that

$$((x_n, y_n), (x_{n+1}, y_{n+1})) \text{ and } ((y_n, x_n), (y_{n+1}, x_{n+1})) \in E(G)$$

with

$$T(x_{n+1}, y_{n+1}) = x_n \text{ and } T(y_{n+1}, x_{n+1}) = y_n \ \forall\ n \in \mathbb{N} \cup \{0\}.$$

Using (12), for $n \in \mathbb{N}$, we have:

$$\begin{aligned} d(x_{n-1}, x_n) &= d(T(x_n, y_n), T(x_{n+1}, y_{n+1})) \\ &\geqslant a_1 d(x_n, x_{n+1}) + a_2 d(y_n, y_{n+1}) \geqslant a_1 d(x_n, x_{n+1}). \end{aligned}$$

This implies

$$d(x_n, x_{n+1}) \leqslant \frac{1}{a_1} d(x_{n-1}, x_n) = k\ d(x_{n-1}, x_n) \leqslant k^n d(x_0, x_1), \text{where } k = \frac{1}{a_1}.$$

Now, for each $m > n \in \mathbb{N}$,

$$d(x_n, x_m) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \leqslant \frac{k^n}{1-k} d(x_0, x_1).$$

Similarly, we can show that

$$d(y_n, y_m) \leqslant \frac{k^n}{1-k} d(y_0, y_1).$$

Since $k \in (0, 1)$, so, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, which converge to some p and q in X, respectively.
Since T is continuous,

$$p = \lim_{n \longrightarrow \infty} x_n = \lim_{n \longrightarrow \infty} T(x_{n+1}, y_{n+1}) = T(p, q).$$

Similarly, $q = T(q, p)$. Thus, (p, q) is a coupled fixed point of T.

For the uniqueness, let (r, s) be another coupled fixed point of T with $((p, q), (r, s))$ and $((q, p), (s, r)) \in E(G)$. Using (12), we have

$$d(p, r) = d(T(p, q), T(r, s)) \geqslant a_1 d(p, r) + a_2 d(q, s),$$

which implies that

$$d(p, r) \geqslant a_1 d(p, r).$$

Since $a_1 > 1$, so, $p = r$ and, similarly, we can show that $q = s$. □

An illustrative example of Theorem 8 is provided below.

Example 9. Let $X = [0, \infty)$ and define a complete d-metric

$$d \colon X \times X \longrightarrow [0, \infty) \text{ by } d(x, y) = \max\{x, y\}\ \forall\ x, y \in X.$$

Let $T \colon X \times X \longrightarrow X$ be such that $T(x, y) = 3x + 4y\ \forall\ (x, y) \in X \times X$. Consider the graph $G = (V(G), E(G))$, where $V(G) = X \times X$ and $E(G) = \{((x, y), (x, y)) \in X^2 \times X^2 : x, y \in X\} \cup \{((3x+4y, 3y+4x), (x, y)) : x, y \in X\}$. Then for $a_1 = a_2 = 2$, all the conditions of Theorem 8 are satisfied. Thus $(0, 0)$ is the unique coupled fixed point of T.

Remark 3. *If the mapping is not surjective, then Theorem 8 does not hold. For example, let* $X = \mathbb{N} \cup \{0\}$ *and define a complete d-metric* $d\colon X \times X \longrightarrow [0, \infty)$ *by* $d(x, y) = x + y \ \forall\ x, y \in X$. *Let* $T\colon X \times X \longrightarrow X$ *be such that* $T(x, y) = 3x + 3y + 1 \ \forall\ (x, y) \in X \times X$. *Consider the graph* $G = (V(G), E(G))$, *where* $V(G) = X \times X$ *and* $E(G) = \{((x, y), (x, y)) \in X^2 \times X^2\colon x, y \in X\} \cup \{((3x + 3y + 1, 3x + 3y + 1), (x, y))\colon x, y \in X\}$.

Then for $a_1 = a_2 = 2$, T *satisfies the conditions (12), i) and ii) of Theorem 8. But* T *has no coupled fixed point.*

It can also be seen that the coupled fixed point in Theorem 8 is not necessarily unique if there is no edge connecting the coupled fixed points.

For example, take $X = \mathbb{N} \cup \{0\}$ *and a complete d-metric* $d\colon X \times X \longrightarrow [0, \infty)$ *defined by* $d(x, y) = |x - y| \ \forall\ x, y \in X$.

Let $T\colon X \times X \longrightarrow X$ *be such that* $T(x, y) = x \ \forall\ x \in X$. *Consider the graph* $G = (V(G), E(G))$ *where* $V(G) = X \times X$ *and* $E(G) = \{((x, y), (x, y)) \in X^2 \times X^2\colon x, y \in X\}$. *Then, for* $a_1 = a_2 = 2$, *all the conditions of Theorem 8 are satisfied. However,* $(0, 0), (1, 1), \ldots$ *are all coupled fixed points of* T *and there is no edge connecting them.*

References

[1] Banach S. *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 1922, vol. 3 no. 1, pp. 133 – 181.

[2] Bhaskar T. G., Lakshmikantham V. *Fixed point theorems in partially ordered metric spaces and applications*. Nonlinear Anal., 2006, vol. 65, no. 7, pp. 1379 – 1393. DOI: https://doi.org/10.1016/j.na.2005.10.017

[3] Bojor F. *Fixed point theorems for Reich type contractions on metric spaces with a graph*. Nonlinear Anal., 2012, vol. 75, no 9, pp. 3895 – 3901. DOI: https://doi.org/10.1016/j.na.2012.02.009

[4] Das H., Goswami N. *Expansive type mappings in dislocated quasi-metric space with some fixed point results and application*. Korean J. Math., 2024, vol. 32, no. 2, pp. 245 – 257. DOI: https://doi.org/10.11568/kjm.2024.32.2.245

[5] Edelstein M. *An extension of Banach contraction principle*. Proc. Amer. Math. Soc., 1961, vol. 12, no. 1, pp. 7 – 10. DOI: https://doi.org/10.2307/2034113

[6] Espinola R., Kirk W. A. *Fixed point theorems in R-trees with applications to graph theory*. Topology Appl., 2006, vol. 153, no. 7, pp. 1046 – 1055. DOI: https://doi.org/10.1016/j.topol.2005.03.001

[7] Górnicki J. *Fixed point theorems for F-expanding mappings.* Fixed Point Theory and Appl., 2017, 9 (2016).
DOI: https://doi.org/10.1186/s13663-017-0602-3

[8] Guo D., Lakshmikantham V. *Coupled fixed points of nonlinear operators with applications.* Nonlinear Anal., 1987, vol. 11, no. 5, pp. 623 – 632.
DOI: https://doi.org/10.1016/0362-546X(87)90077-0

[9] Han Y., Xu S. *Some new theorems of expanding mappings without continuity in cone metric spaces.* Fixed Point Theory and Appl., 2013, vol. 2013, Article 3, 9 p. DOI: https://doi.org/10.1186/1687-1812-2013-3

[10] Hitzler P., Seda A. K. *Dislocated topologies.* J. Electr. Eng., 2000, vol. 51, no. 12, pp. 3 – 7.

[11] Jachymski J. *The contraction principle for mappings on a metric space with a graph.* Proc. Amer. Math. Soc., 2008, vol. 136, no. 4, pp. 1359 – 1373.
DOI: https://doi.org/10.1090/S0002-9939-07-09110-1

[12] Jleli M., Samet B. *A new generalization of the Banach contraction principle.* J. Inequal. Appl., 2014, vol. 2014, p. 38.
DOI: https://doi.org/10.1186/1029-242X-2014-38

[13] Kamran T., Postolache M., Fahimuddin, Ali M. U. *Fixed point theorems on generalized metric space endowed with graph.* J. Nonlinear Sci. Appl., 2016, vol. 9, no. 6, pp. 4277 – 4285.
DOI: https://doi.org/10.22436/jnsa.009.06.69

[14] Mebarki K., Boudaoui A., Shatanawi W., Abodayeh K., Shatnawi T. A. M. *Solution of differential equations with infinite delay via coupled fixed point.* Heliyon, 2022, vol. 8, no. 2, e08849.
DOI: https://doi.org/10.1016/j.heliyon.2022.e08849

[15] Mhanna S., Baiz O., Benaissa H., El Moutawakil D. *Some new results of fixed point in dislocated quasi-metric spaces.* J. Math. Comp. Sci., 2022, vol. 24, pp. 22 – 32. DOI: http://dx.doi.org/10.22436/jmcs.024.01.03

[16] Onsod W., Saleewong T., Ahmad J., Al-Mazrooei A. E., Kumam P. *Fixed points of a Θ-contraction on metric spaces with a graph.* Commun. Nonlinear Anal., 2016, vol. 2, pp. 139 – 149.

[17] Puvar S. V., Vyas R. G. *Common fixed point in G-metric spaces via generalized Γ-C_F-simulation function.* Probl. Anal. Issues Anal., 2024, vol. 13 (31), no. 3, pp. 64 – 78.
DOI: http://dx.doi.org/10.15393/j3.art.2024.15970

[18] Rafique M., Nazir T., Kalita H. *Fixed points of fuzzy multivalued F-contractive mappings with a directed graph in parametric metric spaces.* Probl. Anal. Issues Anal., 2024, vol. 13 (31), no. 3, pp. 79 – 100.
DOI: http://dx.doi.org/10.15393/j3.art.2024.15870

[19] Rahman M., Sarwar M. *Fixed point theorems for expanding mappings in dislocated metric space*. Math. Sci. Lett., 2015, vol. 4, no. 1, pp. 69 – 73. DOI: http://dx.doi.org/10.12785/msl/040114

[20] Wang S. Z., Li B. Y., Gao Z. M., Iseki K. *Some fixed point theorerms on expansion mappings*. Math. Japonica., 1984, vol. 29, pp. 631 – 636.

[21] Yeşilkaya S. S., Aydın C. *Fixed point results of expansive mappings in metric spaces*. Mathematics, 2020, vol. 8, no. 10, p. 1800. DOI: https://doi.org/10.3390/math8101800

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