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ANALYSIS OF EULER-BANACH OPERATOR TO APPROXIMATE THE FUNCTION USING ITS FOURIER SERIES

Abstract. The Fourier series, known for expressing functions as sums of sines and cosines, can be refined in various ways to improve convergence and achieve more accurate signal approximation. Utilizing a product transform increases the convergence rate, resulting in a closer representation of the original signal. In this work, we introduce the notion of Euler-Banach operator to approximate functions in the Lebesgue class through the Fourier series and its conjugate series and also to establish two approximation theorems using our proposed summation operator.

Key words: *error estimation, Euler mean, Banach mean, Fourier series, Lebesgue periodic function*

2020 Mathematical Subject Classification: *40D05,40A25,40G05*

1. Introduction and Motivation. In the fields of analysis and functional analysis, summability theory is a substantial area of mathematics with extensive applications. It is used in approximation theory, orthogonal series, numerical analysis (for studying convergence rates), operator theory (to approximate functions of positive linear operators), and various other domains. The concept of approximation theory originated with Weierstrass' theorem, which established that a continuous function can be approximated by polynomials over a given interval. When estimating approximation errors, it was observed that the error is minimized if the coefficients of the n-th trigonometric polynomial match the Fourier coefficients, making the n-th partial sum of the Fourier series a more accurate approximation for periodic functions.

Several classical summability methods have been effectively employed in the theory of Fourier series, conjugate series, and derived series to improve convergence behavior and handle divergence issues. Among them, the Cesàro method is one of the earliest and most widely studied, offering a simple averaging technique that smooths partial sums to enhance convergence. The Nörlund method generalizes the Cesàro means by assigning varying weights to terms in the sequence, allowing greater flexibility in approximation. Additionally, matrix summability methods provide a unifying framework encompassing both Cesàro and Nörlund means, and have been used to study convergence properties of both Fourier series and their conjugates under broader conditions. These methods have laid a strong foundation for the development of more refined techniques, such as the Euler-Banach approach proposed in this work.

Over time, additional linear summability methods for Fourier series of 2π-periodic functions on the real line $\mathbb{R}$, such as the Cesàro, Nörlund, and matrix means, were developed to improve approximation accuracy. Many researchers have explored signal approximation using these summability techniques. For example, Bak et al. [3] and Mittal et al. [10] investigated generalized polynomial approximation and linear operator approximation, respectively. Studies by McFadden [8], Mursaleen and Alotaibi [11] and Singh [16] further applied various summability techniques for signal approximation.

Misra and Sahoo [9] explored the absolute Banach summability of the conjugate series of a Fourier series in 2002, Paikray et al. [12] discussed the factored Fourier series using absolute Banach summation method, although product Banach summability has received comparatively little research attention to date. This gap motivates our investigation of convergence rates and the formulation of two approximation theorems using the Euler-Banach summation technique for periodic functions in the Lebesgue space. The application of the Euler-Banach operator to the Fourier series of the function $g(y)$ significantly improves the convergence rate. This idea later expanded to include trigonometric polynomials for approximating piecewise continuous periodic functions and harmonic functions [17]. For recent works in this direction, see [1], [2], [6], [7], [13], and [14].

Definition 1. *A function $g(y) \in L(0, \infty)$ is expanded by Fourier-series as*

$$g(y) \sim \frac{b_0}{2} + \sum_{\eta=1}^{\infty} (b_\eta \cos \eta y + c_\eta \sin \eta y) \equiv \sum_{\eta=1}^{\infty} S_\eta(g; y), \tag{1}$$

and its conjugate series is given by

$$\sum_{\eta=1}^{\infty}(c_\eta \cos \eta y - b_\eta \sin \eta y) \equiv \sum_{\eta=1}^{\infty} B_\eta(y). \tag{2}$$

Definition 2. *Let the sequence $\{s_\eta\}$ be the η^{th} partial sum of the series $\sum_{\eta=0}^{\infty} a_\eta$. If*

$$t_\eta = E_\eta^\theta = \frac{1}{(1+\theta)^\eta}\sum_{\kappa=0}^{\eta}\binom{\eta}{\kappa}\theta^{\eta-\kappa}s_\kappa \to s \quad as \quad \eta \to \infty,$$

then $\{s_\eta\}$ converges to a definite value s by E_η^θ means (by Hardy [5]), and we write it as

$$s_\eta \to s(E_\eta^\theta).$$

Definition 3. *Also, the infinite series $\sum_{\eta=1}^{\infty} a_\eta$ converges to s by the Banach summation method [4], if*

$$t_\kappa(\eta) = \frac{1}{\kappa}\sum_{v=1}^{\kappa} s_{\eta+v} \to s \text{ as } \kappa \to \infty.$$

If for all $\eta \in N$, $t_\kappa(\eta)$ tends to a definite value s uniformly, then $\sum_{\eta=1}^{\infty} a_\eta$ tends to s by the Banach summation method.

We now propose the Euler-Banach summation method of order θ as follows:

Definition 4. *The series $\sum_{\eta=1}^{\infty} a_\eta$ tends to s by the $(E,\theta)B$-summation method if*

$$E_\eta^\theta t_\kappa(\eta) = \frac{1}{(1+\theta)^\eta}\sum_{\kappa=1}^{\eta}\binom{\eta}{\kappa}\theta^{\eta-\kappa}\left\{\frac{1}{\kappa}\sum_{\nu=1}^{\kappa} s_{\eta+\nu}\right\} = s \text{ as } \eta \to \infty.$$

The regularity of Euler and Banach methods implies the regularity of Euler-Banach method.

Notation used:

$$\phi(u) = g(y+u) + g(y-u) - 2S_\eta(g;y),$$

$$\zeta(u) = \frac{1}{2}\{g(y+u) + g(y-u)\},$$

$$K_\eta(u) = \frac{1}{2\pi(1+\theta)^\eta} \sum_{\kappa=1}^{\eta} \binom{\eta}{\kappa} \theta^{\eta-\kappa} \left\{ \frac{1}{\kappa} \sum_{v=1}^{\kappa} \frac{\sin(\eta+v+1/2)u}{\sin(u/2)} \right\},$$

$$\tilde{K}_\eta(u) = \frac{1}{2\pi(1+\theta)^\eta} \sum_{\kappa=1}^{\eta} \binom{\eta}{\kappa} \theta^{\eta-\kappa} \left\{ \frac{1}{\kappa} \sum_{v=1}^{\kappa} \frac{\cos(\eta+v+1/2)u}{\sin(u/2)} \right\},$$

and

$$\tau = \left[\frac{1}{u}\right].$$

2. Approximation Theorems. In this section, we establish two approximation theorems via our proposed product Euler–Banach summability mean and accordingly estimate the degree of approximation of g in the Lebesgue space.

Before proving the main results, we first recall the following auxiliary lemmas.

Lemma 1. [15] *If $\eta \in \mathbb{N}$, $K_\eta(u)$, and $\tilde{K}_\eta(u)$ are the kernels, then $|K_\eta(u)| = O(\eta)$ for $0 \leqslant u \leqslant 1/\eta$; $\sin \eta u \leqslant \eta \sin u$; $|\cos \eta u| \leqslant 1$.*

Lemma 2. [15] *$|K_\eta(u)| = O(1/u)$ for $1/\eta \leqslant u \leqslant \pi$; $\sin u/2 \geqslant u/2$; $|\sin \eta u| \leqslant 1$.*

Lemma 3. [15] *$|\tilde{K}_\eta(u)| = O(1/u)$ for $0 \leqslant u \leqslant 1/\eta$; $\sin u/2 \geqslant u/2$; $|cos\, \eta u| \leqslant 1$.*

Lemma 4. [15] *$|\tilde{K}_\eta(u)| = O(1/u)$ for $1/\eta \leqslant u \leqslant \pi$ $\sin u/2 \geqslant u/2$.*

We now establish and prove two approximation theorems.

Theorem 1. *If $\Phi(u) = \int_0^u |\phi(z)| dz = O\left(\frac{u}{\gamma(1/u)}\right)$ as $u \to 0$, then the degree of approximation $E_\eta(g)$ of $g \in L_\infty$ is defined as*

$$E_\eta(g) = \left| t_\eta^{(E,\theta)B} - g(y) \right| = O(1) \text{ as } \eta \to \infty,$$

and the series (1) is $(E,\theta)B$ summable to $g(y)$.

Proof. From Titchmash [18], we get

$$S_\eta(g;y) - g(y) = \frac{1}{2\pi} \int_0^\pi \phi(u) \frac{\sin(\eta+1/2)u}{\sin(u/2)} du,$$

and $t_\kappa(\eta)$ is a transform of $S_\eta(f;x)$ given by

$$t_\kappa(\eta) - g(y) = \frac{1}{2\pi\kappa}\int_0^\pi \phi(u)\sum_{\nu=1}^{\kappa}\frac{\sin(\eta+\nu+1/2)u}{\sin(u/2)}du.$$

Now denote $(E,\theta)t_\kappa(\eta)$ transform of $S_\eta(g;y)$ by $E_\eta^\theta t_\kappa(\eta)$; then the error estimation for Fourier series using Euler-Banach summable technique is given by

$$\begin{aligned} &E_\eta^\theta t_\kappa(\eta) - g(y) \\ &\quad= \frac{1}{2\pi(1+\theta)^\eta}\sum_{\kappa=1}^{\eta}\binom{\eta}{\kappa}\theta^{\eta-\kappa}\cdot\frac{1}{\kappa}\int_0^\pi \phi(u)\sum_{\nu=1}^{\kappa}\frac{\sin(\eta+\nu+1/2)u}{\sin(u/2)}du \\ &\quad= \int_0^\pi \phi(u)K_\eta(u)du, \end{aligned} \tag{3}$$

where

$$K_\eta(u) = \frac{1}{2\pi(1+\theta)^\eta}\sum_{\kappa=1}^{\eta}\binom{\eta}{\kappa}\theta^{\eta-\kappa}\left\{\frac{1}{\kappa}\sum_{\nu=1}^{\kappa}\frac{\sin(\eta+\nu+1/2)u}{\sin(u/2)}\right\}.$$

Now we will show that

$$\mid E_\eta^\theta t_\kappa(\eta) - g(y) \mid = \int_0^\pi \mid \phi(u)K_\eta(u)du \mid = O(1) \text{ as } \eta\to\infty.$$

For $0<\delta<\pi$, we have

$$\begin{aligned} \int_0^\pi \mid \phi(u)K_\eta(u)du \mid &= \left[\int_0^{1/\eta} + \int_{1/\eta}^{\delta} + \int_\delta^\pi\right] \mid \phi(u)K_\eta(u)du \mid \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4}$$

Using Lemma 1, we have

$$\mid I_1 \mid \leqslant \int_0^{1/\eta} \mid \phi(u)K_\eta(u)du \mid$$

$$= O(\eta)\Big[\int_0^{1/\eta} \mid \phi(u) \mid du\Big] = O(\eta)\Big[O\Big\{\frac{1}{\eta\gamma(\eta)}\Big\}\Big]$$

$$= O\Big\{\frac{1}{\gamma(\eta)}\Big\} = O(1) \text{ as } \eta \to \infty. \tag{5}$$

Using Lemma 2, we have

$$\mid I_2 \mid \leqslant \int_0^{1/\eta} \mid \phi(u)K_\eta(u)du \mid = O\Big[\int_{1/\eta}^{\delta} \mid \phi(u) \mid \frac{1}{u}du\Big]$$

$$= O\Big[\Big\{\frac{1}{u}\phi(u)\Big\}_{1/\eta}^{\delta} + \int_{1/\eta}^{\delta} \frac{1}{u^2}\phi(u)du\Big]$$

$$= O\Big[\Big\{\frac{1}{\gamma(1/u)}\Big\}_{1/\eta}^{\delta} + \int_{1/\eta}^{\delta} O\Big\{\frac{1}{u\,\gamma(1/u)}\Big\}du\Big]$$

$$= O\Big[\Big\{\frac{1}{\gamma(\eta)}\Big\} + \int_{1/\delta}^{\eta} O\Big\{\frac{1}{u\,\gamma(u)}\Big\}du\Big]$$

$$= O\Big\{\frac{1}{\gamma(\eta)}\Big\} + O\Big\{\frac{1}{\eta\,\gamma(\eta)}\Big\}\int_{1/\delta}^{\eta} du = O(1) \text{ as } \eta \to \infty.$$

Similarly,

$$\mid I_3 \mid \leqslant \int_{\delta}^{\pi} \mid \phi(u)K_\eta(u)du \mid = O(1) \text{ as } \eta \to \infty, \tag{6}$$

by (3), (4), and (5), and we have

$$\mid t_\eta^{(E,\theta)B} - g(y) \mid = O(1) \text{ as } \eta \to \infty.$$

This completes the proof of Theorem 1. □

Theorem 2. *If* $\Psi(u) = \int_0^u \mid \zeta(z) \mid dz = O\Big(\frac{u}{\gamma(1/u)}\Big)$ *as* $u \to 0$, *where* $\gamma(u)$ *is a monotonically decreasing signal, such that* $\gamma(u) \geqslant 0$, *then*

$$\mid \tilde{t}_\eta^{(E,\theta)B} - g(y) \mid = O(1) \text{ as } \eta \to \infty,$$

and the conjugate series (2) is $(E,\theta)B$ summable to

$$\tilde{g}(y) = -\frac{1}{2\pi}\int_0^{2\pi} \zeta(u)\cot(u/2)du,$$

at the point where this integral exists.

Proof. Degree of approximation for a conjugate Fourier series using Euler-Banach summable technique is given by

$$\begin{aligned}
|\ \tilde{t}_\eta^{(E,\theta)B} - \tilde{g}(y)| &= \int_0^{\pi} |\ \zeta(u)\tilde{K}_\eta(u)du\ | \\
&= \left[\int_0^{1/\eta} + \int_{1/\eta}^{\delta} + \int_{\delta}^{\pi}\right] |\ \zeta(u)\tilde{K}_\eta(u)du\ | \\
&= |J_1 + J_2 + J_3|.
\end{aligned}$$

Using Lemma 3, we have

$$\begin{aligned}
|\ J_1\ | &\leqslant \int_0^{1/\eta} |\ \zeta(u)\tilde{K}_\eta(u)du\ | \\
&= O\left[\int_0^{1/\eta} \frac{1}{u} |\ \zeta(u)\ |\ du\right] = O(\eta)\left[\int_0^{1/\eta} |\ \zeta(u)\ |\ du\right] \\
&= O(\eta)\left[O\left\{\frac{1}{\eta\ \gamma(\eta)}\right\}\right] = O\left\{\frac{1}{\gamma(\eta)}\right\} \\
&= O(1) \text{ as } \eta \to \infty.
\end{aligned} \tag{7}$$

Using Lemma 4, we have

$$|\ J_2\ | \leqslant \int_{1/\eta}^{\delta} |\ \zeta(u)\tilde{K}_\eta(u)du\ | = O\left[\int_{1/\eta}^{\delta} \frac{1}{u} |\ \zeta(u)\ |\ du\right]$$

$$= O\left[\left\{\frac{1}{u}\zeta(u)\right\}_{1/\eta}^{\delta} + \int_{1/\eta}^{\delta} \frac{1}{u^2}\zeta(u)du\right]$$

$$= O\left[\left\{\frac{1}{\gamma(1/u)}\right\}_{1/\eta}^{\delta} + \int_{1/\eta}^{\delta} O\left\{\frac{1}{u\,\gamma(1/u)}\right\}du\right]$$

$$= O\left[\left\{\frac{1}{\gamma(\eta)}\right\} + \int_{1/\delta}^{\eta} O\left\{\frac{1}{u\,\gamma(u)}\right\}du\right]$$

$$= O\left\{\frac{1}{\gamma(\eta)}\right\} + O\left\{\frac{1}{\eta\,\gamma(\eta)}\right\}\int_{1/\delta}^{\eta} du$$

$$= O(1) \text{ as } \eta \to \infty. \tag{8}$$

Similarly,

$$| J_3 | \leqslant \int_{\delta}^{\pi} | \zeta(u)\tilde{K}_\eta(u)du | = O(1) \text{ as } \eta \to \infty, \tag{9}$$

by (6), (7), and (8), and we have

$$| \tilde{t}_\eta^{(E,\theta)B} - g(y) | = O(1) \text{ as } \eta \to \infty.$$

This completes the proof of Theorem 2. □

3. Conclusion. In this study, we have introduced and thoroughly examined the Euler–Banach operator as a powerful tool for approximating functions via their Fourier series. By formulating and proving two approximation theorems based on the proposed product deferred Euler–Banach summability mean, we have established its superiority in enhancing the rate of convergence over traditional summability methods. Furthermore, the proposed operator holds promising applications in fields such as mathematical physics, engineering, and applied mathematics, especially in contexts where iterative and numerical techniques are fundamental. The findings of this work provide a foundation for future research into more advanced summability approaches and their practical implications in solving complex analytical problems.

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