

UDC 517.544

S. S. VOLOSIVETS

## SHARP CONDITIONS FOR WEIGHTED INTEGRABILITY OF $q$ -DUNKL FOURIER TRANSFORMS

**Abstract.** We obtain sufficient conditions for the weighted integrability of the  $q$ -Dunkl Fourier transforms of functions from generalized integral Lipschitz classes. In the  $L^2$  case, we prove the sharpness of these conditions.

**Key words:**  $q$ -Dunkl Fourier transform,  $q$ -Dunkl translation, modulus of smoothness,  $q$ -Dunkl differential operator

**2020 Mathematical Subject Classification:** 44A15, 47A10

**1. Introduction.** Let  $f \in L^1(\mathbb{R})$ , i.e.,  $f: \mathbb{R} \rightarrow \mathbb{C}$  be Lebesgue-integrable function on  $\mathbb{R}$ . Then the Fourier transform of  $f$  is defined by

$$\hat{f}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(t) e^{-itx} dt, \quad x \in \mathbb{R}.$$

If  $f \in L^p(\mathbb{R})$ ,  $1 < p \leq 2$ , then  $\hat{f}(x)$  is defined as a limit of

$$(2\pi)^{-1/2} \int_{-a}^b f(x) e^{-itx} dx$$

in the norm of  $L^{p'}(\mathbb{R})$ ,  $1/p + 1/p' = 1$ , as  $a, b \rightarrow +\infty$ . In particular,  $\hat{f} \in L^{p'}(\mathbb{R})$ . The following Hausdorff-Young inequality proved by Titchmarsh [16, Ch. IV]

$$\|\hat{f}\|_{p'} \leq C \|f\|_p := C \left( \int_{\mathbb{R}} |f(t)|^p dt \right)^{1/p}, \quad f \in L^p(\mathbb{R}), \quad 1 < p \leq 2, \quad (1)$$

is valid. For  $p = p' = 2$ , we have the Plancherel equality  $\|\hat{f}\|_2 = \|f\|_2$ . More about these results see in [16, Ch. III and IV].

Szász [15] obtained the sufficient conditions for convergence of the series  $\sum_{n=1}^{\infty} (|a_n(f)|^\beta + |b_n(f)|^\beta)$ , where  $a_n(f)$  ( $b_n(f)$ ) are cosine (sine) Fourier coefficients of  $2\pi$ -periodic function  $f \in L^p([0, 2\pi])$ ,  $1 < p \leq 2$ . An analogue of Szász's result for Fourier transform was established by Titchmarsh [16, Ch. IV, Theorem 84] and this result is well-known.

Gogoladze and Meskhia [7] proposed a class of weighted sequences satisfying a condition similar to the reverse Hölder inequality and studied convergence of series of  $|a_n(f)|^\beta + |b_n(f)|^\beta$  with such weights. Moricz [11] introduced the continual analogues of Gogoladze-Meskhia classes as follows. Let  $\gamma \geq 1$ ,  $\lambda(t) \in L^1_{loc}(\mathbb{R})$  (i.e.,  $\lambda(t)$  be Lebesgue-integrable on each compact from  $\mathbb{R}$ ),  $\lambda(t)$  be even and nonnegative, such that the inequality

$$\left( \int_{2^i}^{2^{i+1}} \lambda^\gamma(t) dt \right)^{1/\gamma} \leq C(\gamma) 2^{i(1-\gamma)/\gamma} \int_{2^{i-1}}^{2^i} \lambda(t) dt, \quad i \in \mathbb{Z},$$

holds for some  $C(\gamma) \geq 1$ . Then  $\lambda(t)$  belongs to the class  $A_\gamma$ .

Let us define the modulus of continuity of  $f \in L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ , by

$$\omega(f, \delta)_{L^p(\mathbb{R})} = \sup_{0 \leq h \leq \delta} \|f(\cdot + h) - f(\cdot)\|_p.$$

Moricz [11] proved

**Theorem 1.** *Let  $1 < p \leq 2$ ,  $f \in L^p(\mathbb{R})$ ,  $1/p + 1/p' = 1$ ,  $0 < r < p'$  and  $\lambda \in A_{p/(p-rp+r)}$ . Then*

$$\int_{|t| \geq 2} \lambda(t) |\hat{f}(t)|^r dt \leq \int_1^\infty \lambda(t) t^{-r/p'} \omega^r(f, \pi/t)_{L^p(\mathbb{R})} dt.$$

Seen a more general variant of Theorem 1 in [10, Theorem 4.1], while its sharpness was discussed in [10, Theorem 4.2].

The aim of this paper is to obtain an analogue of Theorem 1 for  $q$ -Dunkl Fourier transform and to show its sharpness in the case  $p = 2$ . We note that for  $q$ -Bessel Fourier transform such sufficient conditions are proved by Krotova [9], while for the classical Dunkl transform one can see [19]. Particular results for generalized Lipschitz and Dini-Lipschitz classes of functions and non-weighted case were obtained by Daher and Tyr [5].

**2. Definitions.** Let  $0 < q < 1$ ,  $\alpha > -1/2$  and  $\mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\}$ ,  $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$  and  $\widehat{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$ . For  $a \in \mathbb{C}$  and  $n \in \mathbb{N} = \{1, 2, \dots\}$ , we set

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i).$$

and

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad [n]_q! = \frac{(q; q)_n}{(1 - q)^n}, \quad n \in \mathbb{Z}_+ = \{0, 1, \dots\}.$$

The  $q$ -gamma function is given by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad x \neq 0, -1, -2, \dots$$

Then (see [3])  $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$ ,  $\Gamma_q(1) = 1$  and  $\lim_{q \rightarrow 1-0} \Gamma_q(x) = \Gamma(x)$  for  $\operatorname{Re} x > 0$  ( $\operatorname{Re} z$  is the real part of  $z \in \mathbb{C}$ ). A  $q$ -analogue of exponential function  $e^{ix}$  is introduced by Rubin [13]

$$e(x, q^2) = \cos(-ix, q^2) + i \sin(-ix, q^2),$$

where

$$\cos(x, q^2) = \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)} \frac{x^{2j}}{[2j]_q!}, \quad \sin(x, q^2) = \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)} \frac{x^{2j+1}}{[2j+1]_q!}.$$

The  $q$ -derivative of a function  $f$  defined on  $\mathbb{R}_q$  in  $x \neq 0$  is given by

$$D_q(f)(x) = (f(x) - f(qx)) / ((1 - q)x), \quad x \neq 0,$$

$D_q(f)(0) = f'(0)$ , if  $f'(0)$  exists. Also we consider  $D_q^+(f)(x) = D_q f(q^{-1}x)$ . Then it is easy to see that for  $\nu \in \mathbb{R}$  one has  $D_q x^\nu = [\nu]_q x^{\nu-1}$  and  $D_q^+ x^\nu = q^{-\nu} [\nu]_q x^{\nu-1}$ .

If  $f$  is defined on  $\mathbb{R}_q$ , we can consider the even and odd parts of  $f$ :

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2}, \quad (2)$$

and Rubin's  $q$ -differential operator [13]

$$\partial_q(f)(x) = D_q^+ f_e(x) + D_q f_o(x).$$

Let us introduce the  $q$ -integral of Jackson for  $f$  defined on  $\mathbb{R}_q$  on intervals  $[0, a]$ ,  $[0, +\infty)$ ,  $[a, b]$ ,  $a, b \in \mathbb{R}_q^+$ , and  $(-\infty, \infty)$  by

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n; \quad \int_0^{\infty} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} q^n f(q^n),$$

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

$$\int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} q^n [f(q^n) + f(-q^n)].$$

Definitions of  $\int_a^0 f(x) d_q x$  for  $a < 0$ ,  $a \in \mathbb{R}_q$ , and  $\int_{-\infty}^0 f(x) d_q x$  may be obtained in the same way. Then

$$D_q \left( \int_a^x f(t) d_q(t) \right) = f(x), \quad \int_a^b D_q f(t) d_q t = f(b) - f(a).$$

More about  $q$ -analysis see in [8], e.g., the following simple change of variables formula may be found in (19.14) from this book:

$$\int_a^b f(\lambda x) d_q x = \lambda^{-1} \int_{\lambda a}^{\lambda b} f(x) d_q(x), \quad \lambda \in \mathbb{R}_q^+.$$

For  $\alpha > -1/2$ , we consider  $q$ -Dunkl differential operator

$$\Lambda_{q,\alpha}(f)(x) = \partial_q[f_e + q^{2\alpha+1}f_o](x) + [2\alpha + 1]_q \frac{f(x) - f(-x)}{2x},$$

where  $f_e, f_o$  are defined in (2). Bettaibi and Bettaieb [3] proved that the equation  $\Lambda_{q,\alpha}(f)(x) = i\lambda f(x)$  with the initial condition  $f(0) = 1$  has an unique solution

$$\Psi_{q,\alpha}(\lambda x) = j_\alpha(\lambda x, q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x), \quad x \in \mathbb{R}_q.$$

Here  $j_\alpha(x, q^2)$  is the third Jackson normalized  $q$ -Bessel function of the first kind given by

$$j_\alpha(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha + 1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha + n + 1) \Gamma_{q^2}(n + 1)} \left( \frac{x}{1 + q} \right)^{2n}.$$

It is easy to see that

$$\partial_q j_\alpha(x, q^2) = D_q^+ j_\alpha(x, q^2) = -\frac{x}{[2\alpha + 2]_q j_{\alpha+1}(x, q^2)}.$$

Therefore,  $\Psi_{q,\alpha}(x) = j_\alpha(x, q^2) - i\partial_q j_\alpha(x, q^2)$ .

Further, we consider the measure

$$d\mu_{q,\alpha}(x) = \frac{(1 + q)^{-\alpha}}{2\Gamma_{q^2}(\alpha + 1)} |x|^{2\alpha+1} d_q(x) =: c_{\alpha,q} |x|^{2\alpha+1} d_q(x). \quad (3)$$

For  $1 \leq p < \infty$ , the space  $L_{q,\alpha}^p(\mathbb{R}_q)$  consists of all real functions  $f$  on  $\mathbb{R}_q$  for which

$$\|f\|_{p,q,\alpha}^p = \int_{-\infty}^{+\infty} |f(x)|^p d\mu_{q,\alpha}(x) < \infty.$$

The space  $L_{q,\alpha}^\infty(\mathbb{R}_q)$  is the space of bounded on  $\mathbb{R}_q$  functions with uniform norm  $\|f\|_\infty = \|f\|_{\infty,q,\alpha} = \sup_{x \in \mathbb{R}_q} |f(x)|$ .

A  $q$ -analogue  $S_q(\mathbb{R}_q)$  of the Schwartz space consists of all functions with finite seminorms  $P_{n,m,q}(f) = \sup_{x \in \mathbb{R}_q} |x^m \partial_q^n f(x)|$ ,  $m, n \in \mathbb{Z}_+$ , and such that  $\lim_{x \rightarrow 0, x \in \mathbb{R}_q} \partial_q^n f(x)$  exists for all  $n \in \mathbb{Z}_+$ . It is known that  $\Psi_{q,\alpha} \in S_q(\mathbb{R}_q)$  and

$$|\Psi_{q,\alpha}(x)| \leq \frac{4}{(q, q)_\infty}, \quad x \in \mathbb{R}_q, \quad (4)$$

(see [3]). Let us define the  $q$ -Dunkl Fourier transform of  $f \in L_{q,\alpha}^1(\mathbb{R}_q)$  by

$$\mathcal{F}_{q,\alpha}^d(f)(y) = \int_{-\infty}^{+\infty} f(x) \Psi_{q,\alpha}(-yx) d\mu_{q,\alpha}(x),$$

where the value of  $c_{\alpha,q}$  is given in (3). In [3, Theorem 11] it is proved that  $\mathcal{F}_{q,\alpha}^d(S_q(\mathbb{R}_q)) \subset S_q(\mathbb{R}_q)$  and

$$\|\mathcal{F}_{q,\alpha}^d(f)\|_{2,q,\alpha} = \|f\|_{2,q,\alpha} \quad (5)$$

for  $f \in S_q(\mathbb{R}_q)$ . Therefore,  $\mathcal{F}_{q,\alpha}^d$  can be uniquely extended to an isometry of  $L_{q,\alpha}^2(\mathbb{R}_q)$  and (5) holds for  $f \in L_{q,\alpha}^2(\mathbb{R}_q)$ . On the other hand, from (4) we deduce that

$$\|\mathcal{F}_{q,\alpha}^d(f)\|_{\infty,q,\alpha} \leq \frac{4c_{\alpha,q}}{(q,q)_\infty} \|f\|_{1,q,\alpha}, \quad f \in L_{q,\alpha}^1(\mathbb{R}_q), \quad (6)$$

(see also [3, Proposition 9]). From (5) and (6) by the Riesz-Thorin theorem we obtain that  $\mathcal{F}_{q,\alpha}^d$  can be extended to  $L_{q,\alpha}^p(\mathbb{R})$ ,  $1 < p \leq 2$ , and

$$\|\mathcal{F}_{q,\alpha}^d(f)\|_{p',q,\alpha} \leq C(p,q) \|f\|_{p,q,\alpha}, \quad f \in L_{q,\alpha}^p(\mathbb{R}_q), \quad 1/p + 1/p' = 1. \quad (7)$$

The generalized  $q$ -Dunkl translation is defined for  $f \in S_q(\mathbb{R}_q)$ ,  $h, x \in \mathbb{R}_q$  by the formula (see [4])

$$T_h^{q,\alpha} f(x) = \int_{-\infty}^{+\infty} \mathcal{F}_{q,\alpha}^d(f)(y) \Psi_{q,\alpha}(yx) \Psi_{q,\alpha}(yh) d\mu_{q,\alpha}(y),$$

$T_0^{q,\alpha} f = f$ . Using the density of  $S_q(\mathbb{R}_q)$  in  $L_{q,\alpha}^2(\mathbb{R}_q)$ , Bettaibi et al proved in [4, Proposition 2] that for  $f \in L_{q,\alpha}^2(\mathbb{R}_q)$

$$\|T_h^{q,\alpha}(f)\|_{2,q,\alpha} \leq \frac{4}{(q,q)_\infty} \|f\|_{2,q,\alpha}, \quad h \in \mathbb{R}_q,$$

and

$$\mathcal{F}_{q,\alpha}^d(T_h^{q,\alpha}(f))(y) = \Psi_{q,\alpha}(yh) \mathcal{F}_{q,\alpha}^d(f)(y). \quad (8)$$

In the same way, the boundedness of translation  $T_h^{q,\alpha}$  and (8) can be proved for  $f \in L_{q,\alpha}^p(\mathbb{R}_q)$ ,  $1 \leq p \leq 2$ . For  $f \in L_{q,\alpha}^p(\mathbb{R}_q)$ ,  $1 \leq p \leq 2$ , and  $m \in \mathbb{N}$ , we introduce the difference of order  $m$  with step  $h \in \mathbb{R}_q^+$  by

$$\Delta_{h,q,\alpha}^m(f)(x) = (I - T_h^{q,\alpha})^m f(x)$$

and the  $q$ -modulus of smoothness of order  $m$

$$\omega_m(f, \delta)_{p,q,\alpha} = \sup_{0 \leq h \leq \delta, h \in \mathbb{R}_q^+ \cup \{0\}} \|\Delta_{h,q,\alpha}^m(f)\|_{p,q,\alpha}.$$

For conditions of Gogoladze-Meskhia-Moricz type we consider even functions  $\lambda(t)$  defined on  $\mathbb{R}_q$ , such that for  $i \in \mathbb{Z}$  and  $\gamma \geq 1$

$$\left( \int_{(1/q)^i}^{(1/q)^{i+1}} \lambda^\gamma(t) d\mu_{q,\alpha}(t) \right)^{1/\gamma} \leq C q^{-i(2\alpha+2)(1/\gamma-1)} \int_{(1/q)^{i-1}}^{(1/q)^i} \lambda(t) d\mu_{q,\alpha}(t). \quad (9)$$

If (9) holds, we write  $\lambda(t) \in A_{\gamma,q,\alpha}$ . Note that for  $\gamma = p'/(p'-r)$ ,  $0 < r < p'$ , one has  $1/\gamma - 1 = 1 - r/p' - 1 = -r/p'$ . In [9, Lemma 4], it is proved that  $\lambda(t) = |t|^\delta$ ,  $\delta \in \mathbb{R}$ , belongs to all classes  $A_{\gamma,q,\alpha}$ ,  $\gamma \geq 1$ .

Let  $W_{p,q,\alpha}^m(\mathbb{R}_q)$ , where  $m \in \mathbb{N}$ ,  $1 \leq p \leq 2$ , be the Sobolev spaces of functions  $f \in L_{q,\alpha}^p(\mathbb{R}_q)$ , such that  $\Lambda_{q,\alpha}^j(f) \in L_{q,\alpha}^p(\mathbb{R}_q)$  for  $j = 1, 2, \dots, m$ .

Let  $\Phi$  be the set of nondecreasing continuous on  $\mathbb{R}_+ = [0, \infty)$  functions  $\omega$ , such that  $\omega(0) = 0$ . If  $\omega \in \Phi$  and  $\int_0^\delta t^{-1} \omega(t) dt = O(\omega(\delta))$ , then  $\omega$  belongs to the Bary class  $B$ ; if  $\omega \in \Phi$  and  $\delta^m \int_\delta^\infty t^{-m-1} \omega(t) dt = O(\omega(\delta))$ ,  $m > 0$ , then  $\omega$  belongs to the Bary-Stechkin class  $B_m$  (see [2]).

### 3. Auxiliary propositions

**Lemma 1.** *Let  $\alpha \geq 0$ ,  $0 < q < 1$ . Then there exists  $\gamma > 0$ , such that*

$$|j_\alpha(x, q^2)| \leq 1, \quad x \in \mathbb{R}_q^+, \quad \alpha \geq 0, \quad (10)$$

$$|j_\alpha(x, q^2) - 1| \geq \gamma, \quad x \in \mathbb{R}_q^+ \cap (1, +\infty). \quad (11)$$

**Proof.** The inequality (10) was noted by Dhaouadi [6, Remark 1], the inequality (11) was proved by Achak, Daher, Dhaouadi and Loualid in [1]. Similar estimates for the classical Bessel function can be found in [12].  $\square$

The result of Lemma 2 can be found in [17, Lemma 3.2].

**Lemma 2.** *Let  $\alpha \geq 0$ ,  $0 < q < 1$ ,  $k, m \in \mathbb{N}$ ,  $k > m$ ,  $f \in W_{2,q,\alpha}^m(\mathbb{R}_q)$ . Then for some  $C > 0$  we have*

$$\omega_k(f, \delta)_{2,q,\alpha} \leq C \delta^m \omega_{k-m}(\Lambda_{q,\alpha}^m f, \delta)_{2,q,\alpha}, \quad \delta > 0.$$

Lemma 3 easily follows from (8).

**Lemma 3.** *Let  $m \in \mathbb{N}$ ,  $\alpha \geq 0$ ,  $1 \leq p \leq 2$ . Then*

$$\mathcal{F}_{q,\alpha}^d(\Delta_{h,q,\alpha}^m f)(x) = (1 - \Psi_{q,\alpha}(hx))^m \mathcal{F}_{q,\alpha}^d(f)(x), \quad h, x \in \mathbb{R}_q.$$

Lemma 4 is an analogue of the famous Titchmarsh equivalence theorem. It is proved by Tyr [18].

**Lemma 4.** *Let  $\omega \in B_m$ ,  $m \in \mathbb{N}$  and  $f \in L_{q,\alpha}^2(\mathbb{R}_q)$ . Then the statements  $\omega_m(f, t)_{2,q,\alpha} = O(\omega(t))$ ,  $t \in \mathbb{R}_q^+$ , and*

$$\int_{|y| \geq h^{-1}} |\mathcal{F}_{q,\alpha}^d(f)(y)|^2 d\mu_{q,\alpha}(y) = O(\omega^2(h)), \quad h \in \mathbb{R}_q^+, \quad (12)$$

are equivalent.

**Lemma 5.** *Let  $\alpha \geq 0$  and*

$$D_{y,q,\alpha}(x) = \int_{|t| \leq y} \Psi_{q,\alpha}(-tx) d\mu_{q,\alpha} = \mathcal{F}_{q,\alpha}^d(\chi_{[-y,y]})(x), \quad y \in \mathbb{R}_q^+.$$

where  $\chi_E$  be the indicator of a set  $E$ . Then

$$\|D_{y,q,\alpha}\|_{2,q,\alpha} = Ky^{\alpha+1}, \quad y \in \mathbb{R}_q^+,$$

where  $K$  does not depend on  $y$ .

**Proof.** By the Plancherel equality (5) for  $y = q^s$ ,  $s \in \mathbb{Z}$ , we obtain

$$\begin{aligned} \|D_{y,q,\alpha}\|_{2,q,\alpha}^2 &= \|\chi_{[-y,y]}\|_{2,q,\alpha}^2 = 2 \int_0^{q^s} 1 d\mu_{q,\alpha}(x) = \\ &= C_1(1-q)q^s \sum_{n=0}^{\infty} (q^{n+s})^{2\alpha+1} q^n = C_1((1-q)q^{s(2\alpha+2)} \sum_{n=0}^{\infty} (q^n)^{2\alpha+2} = \\ &= C_1 \frac{(1-q)q^{s(2\alpha+2)}}{1-q^{2\alpha+2}} = C_2 y^{2\alpha+2}. \end{aligned}$$

□

### 3. Main results.

**Theorem 2.** *Let  $1 < p \leq 2$ ,  $1/p + 1/p' = 1$ ,  $\alpha \geq 0$ ,  $f \in L_{q,\alpha}^p(\mathbb{R}_q)$ ,  $m \in \mathbb{N}$ . If  $\lambda \in A_{p/(p-pr+r),q,\alpha} = A_{p'/(p'-r),q,\alpha}$  for some  $r \in (0, p')$ ,  $\lambda(x)\chi_{[-1,1]}(x) \in L_{q,\alpha}^{p'/(p'-r)}(\mathbb{R}_q)$ , and the integral*

$$\int_1^\infty \lambda(t) t^{-r(2\alpha+2)/p'} \omega_m^r(f, t^{-1})_{p,q,\alpha} d\mu_{q,\alpha}(t)$$

converges, then  $\lambda(t)|\mathcal{F}_{q,\alpha}^d(f)(t)|^r \in L_{q,\alpha}^1(\mathbb{R}_q)$ .

**Proof.** Let  $M_i = [(1/q)^i, (1/q)^{i+1}] \cup [-(1/q)^{i+1}, -(1/q)^i]$ ,  $i \in \mathbb{Z}_+$ . By Lemma 3, (11) from Lemma 1, and Hausdorff-Young type inequality (7), we have

$$C_1 \int_{M_i} |\mathcal{F}_{q,\alpha}^d(f)(y)|^{p'} d\mu_{q,\alpha}(y) \leq$$



$$\begin{aligned}
&\leq \int_{M_i} |\mathcal{F}_{q,\alpha}^d(f)(y)|^{p'} |1 - \Psi_{q,\alpha}(q^i y, q^2)|^{mp'} d\mu_{q,\alpha}(y) \leq \\
&\int_{-\infty}^{\infty} |\mathcal{F}_{q,\alpha}^d(f)(y)|^{p'} |1 - \Psi_{q,\alpha}(q^i y, q^2)|^{mp'} d\mu_{q,\alpha}(y) = \|\mathcal{F}_{q,\nu}(\Delta_{h,q,\alpha}^m(f))\|_{p,q,\alpha}^{p'} \leq \\
&\leq C_2 \|\Delta_{q,\nu,h}^m(f)\|_{p,q,\alpha}^{p'} \leq C_2 \omega_m^{p'}(f, q^i)_{p,q,\alpha}.
\end{aligned}$$

By the condition  $\lambda \in A_{p'/(p'-r),q,\alpha}$  and the Hölder inequality with exponents  $p'/r$  and  $p'/(p'-r)$ , we have

$$\begin{aligned}
&\int_{M_i} \lambda(t) |\mathcal{F}_{q,\alpha}(f)(t)|^r d\mu_{q,\alpha}(t) \leq \left( \int_{M_i} |\mathcal{F}_{q,\alpha}^d(f)(t)|^{p'} d\mu_{q,\alpha}(t) \right)^{r/p'} \times \\
&\quad \times \left( \int_{M_i} |\lambda(t)|^{p'/(p'-r)} d\mu_{q,\alpha}(t) \right)^{1-r/p'} \leq \\
&\leq C_3 \omega_m^r(f, q^i)_{p,q,\alpha} q^{ir(2\alpha+2)/p'} \int_{M_{i-1}} \lambda(t) d\mu_{q,\alpha}(t).
\end{aligned}$$

By the monotonicity of the integral, we have

$$\int_{M_i} \lambda(t) |\mathcal{F}_{q,\alpha}^d(f)(t)|^r d\mu_{q,\alpha}(t) \leq C_4 \int_{M_{i-1}} \frac{\lambda(t) \omega_m(f, 1/t)_{p,q,\alpha}}{t^{r(2\alpha+2)/p'}} d\mu_{q,\alpha}(t). \quad (13)$$

Summing up (13) over  $i = 0, 1, 2, \dots$ , we obtain

$$\begin{aligned}
&\int_{t \in \mathbb{R}_q, |t| \geq 1} \lambda(t) |\mathcal{F}_{q,\alpha}(f)(t)|^r d\mu_{q,\alpha}(t) \leq \\
&\leq C_4 \int_{t \in \mathbb{R}_q, |t| \geq 1/q} \frac{\lambda(t) \omega_m(f, |t|^{-1})_{p,q,\alpha}}{|t|^{r(2\nu+2)/p'}} d\mu_{q,\alpha}(t). \quad (14)
\end{aligned}$$

Since  $\lambda \in L_{q,\alpha}^{p'/(p'-r)}[-1, 1]$ ,  $|t|^{-1}$  and  $\omega_m(t, |t|^{-1})_{p,q,\alpha}$  are bounded for  $|t| \in [1/q, 1]$ , the integral in the right-hand side of (14) is finite.

Finally, by the condition  $\lambda \in L_{q,\alpha}^{p'/(p'-r)}[-1, 1]$ , the Hölder inequality with the same exponents, and (7), we obtain that  $\lambda(t) |\mathcal{F}_{q,\alpha}^d(f)(t)|^r \in L_{q,\alpha}^1[-1, 1]$  and the theorem is proved.  $\square$

Let  $0 < r < p'$ . By Lemma 5 in [9] it is easy to see that  $\lambda_\beta(t) = |t|^\beta$  belongs to  $L_{q,\alpha}^{p'/(p'-r)}[-1, 1]$  if and only if  $\beta > -(1 - r/p')(2\alpha + 2)$ . From this note and Theorem 2, we obtain

**Corollary 1.** *Let  $1 < p \leq 2$ ,  $m \in \mathbb{N}$ ,  $1/p + 1/p' = 1$ ,  $\alpha \geq 0$ ,  $\beta \in \mathbb{R}$ ,  $f \in L_{q,\alpha}^p(\mathbb{R}_q)$ , and  $\omega_m(f, t)_{p,q,\alpha} = O(t^\delta)$  for some  $\delta > 0$  and all  $t \in \mathbb{R}_q$ . If  $r \in (0, p')$ ,  $\beta > (r/p' - 1)(2\alpha + 2)$  and*

$$p' > r > \frac{p'(\beta + 2\alpha + 2)}{2\alpha + 2 + p'\delta}, \quad (15)$$

then  $t^\beta |\mathcal{F}_{q,\alpha}^d(f)(t)|^r \in L_{q,\alpha}^1(\mathbb{R}_q)$ .

If  $\beta = 0$ , then in (15) the case  $p' = r$  can be added.

**Corollary 2.** *Let  $\alpha \geq 0$ ,  $k, m \in \mathbb{N}$ ,  $0 < r < 2$ ,  $f \in W_{2,q,\alpha}^m(\mathbb{R}_q)$ . If  $\lambda \in A_{2/(2-r),q,\alpha}$ ,  $\lambda(x)\chi_{[-1,1]}(x) \in L_{q,\alpha}^{2/(2-r)}(\mathbb{R}_q)$  and the integral*

$$\int_1^\infty \lambda(t) t^{-mr-r(\alpha+1)} \omega_k^r(\Lambda_{q,\alpha}^m f, t^{-1})_{2,q,\alpha} d\mu_{q,\alpha}(t) \quad (16)$$

converges, then  $\lambda(t) |\mathcal{F}_{q,\alpha}^d(f)(t)|^r \in L_{q,\alpha}^1(\mathbb{R}_q)$ .

**Proof.** By Lemma 2, from the convergence of integral (16) the convergence of

$$\int_1^\infty \lambda(t) t^{-r(2\alpha+2)/2} \omega_{m+k}^r(f, t^{-1})_{2,q,\alpha} d\mu_{q,\alpha}(t)$$

follows. Applying Theorem 2, we obtain the result of the present Corollary.  $\square$

Theorem 3 can be considered as a counterpart of Theorem 2 from [11].

**Theorem 3.** *Let  $1 < p \leq 2$ ,  $1/p + 1/p' = 1$ ,  $1 \leq s < p$ ,  $\alpha \geq 0$ ,  $f \in L_{q,\alpha}^p(\mathbb{R}_q) \cap L_{q,\alpha}^s(\mathbb{R}_q)$  is bounded on  $\mathbb{R}_q$ . If  $m \in \mathbb{N}$ ,  $\omega_m(f, t)_{s,q,\alpha} = O(t^{1/s})$ ,  $t \in \mathbb{R}_q^+$ ,  $\lambda \in A_{p'/(p'-r),q,\alpha}$  for some  $r \in (0, p')$ ,  $\lambda(x)\chi_{[-1,1]}(x) \in L_{q,\alpha}^{p'/(p'-r)}(\mathbb{R}_q)$  and the integral*

$$\int_1^\infty \lambda(t) t^{-r(2\alpha+2)/p'-r/p} \omega_m^{r(1-s/p)}(f, t^{-1})_{\infty,q,\alpha} d\mu_{q,\alpha}(t)$$

converges, then  $\lambda(t) |\mathcal{F}_{q,\alpha}^d(f)(t)|^r \in L_{q,\alpha}^1(\mathbb{R}_q)$ .

**Proof.** By definition, under conditions of Theorem for  $h \in \mathbb{R}_q^+$ , we have

$$\begin{aligned} \|\Delta_{h,q,\alpha}^m f\|_{p,q,\alpha}^p &= \int_{-\infty}^{\infty} |\Delta_{h,q,\alpha}^m f(t)|^p d\mu_{q,\alpha}(t) \leq \\ &\leq \sup_{t \in \mathbb{R}_q} |\Delta_{h,q,\alpha}^m f(t)|^{p-s} \int_{-\infty}^{\infty} |\Delta_{h,q,\alpha}^m f(t)|^s p d\mu_{q,\alpha}(t) \leq C_1 h \omega_m^{p-s}(f, h)_{\infty,q,\alpha}, \end{aligned}$$

whence

$$\omega_m(f, t)_{p,q,\alpha} \leq C_1 t^{1/p} \omega_m^{1-s/p}(f, t)_{\infty,q,\alpha}, \quad t \in \mathbb{R}_q^+. \quad (17)$$

Substituting (17) into integral from the condition of Theorem 2, we conclude that under conditions of Theorem 3 the function  $\lambda(t)|\mathcal{F}_{q,\alpha}^d(f)(t)|^r$  belongs to  $L_{q,\alpha}^1(\mathbb{R}_q)$ .  $\square$

Theorem 4 shows the sharpness of Theorem 2 in the case  $p = 2$  under some restrictions.

**Theorem 4.** Let  $\alpha \geq 0$ ,  $m \in \mathbb{N}$ ,  $\omega \in B \cap B_m$ ,  $r \in (0, 2)$ . If  $\lambda(t)$  is even positive on  $\mathbb{R}_q$  function, such that the integral

$$\int_1^{\infty} \lambda(t) t^{-r(\alpha+1)} \omega^r(t^{-1}) d\mu_{q,\alpha}(t)$$

diverges, then there exists  $f_0 \in L_{q,\alpha}^2(\mathbb{R}_q)$  with property  $\omega_m(f_0, t)_{2,q,\alpha} = O(\omega(t))$ ,  $t \in \mathbb{R}_q^+$ , but  $\lambda(t)|\mathcal{F}_{q,\alpha}^d(f_0)(t)|^r \notin L_{q,\alpha}^1(\mathbb{R}_q)$ .

**Proof.** Let us consider the function

$$f_0(t) = \sum_{j=1}^{\infty} \omega(q^j) q^{j(\alpha+1)} (D_{q^{-j},q,\alpha}(t) - D_{q^{1-j},q,\alpha}(t)),$$

where  $D_{y,q,\alpha}$  are defined in Lemma 5. If  $g \in L_{q,\alpha}^2(\mathbb{R}_q)$  is even, then the equality  $\mathcal{F}_{q,\alpha}^d(\mathcal{F}_{q,\alpha}^d(g)) = g$  holds. Therefore,

$$\mathcal{F}_{q,\alpha}^d(f_0)(y) = \sum_{j=1}^{\infty} \omega(q^j) q^{j(\alpha+1)} \chi_{\{q^{1-j} \leq |\cdot| \leq q^{-j}\}}(y)$$

and (see the proof of Lemma 5)

$$\int_{|y| \geq q^{-m}, y \in \mathbb{R}_q} |\mathcal{F}_{q,\alpha}^d(f_0)(t)|^2 d\mu_{q,\alpha}(y) =$$

$$\begin{aligned}
&= 2 \sum_{j=m+1}^{\infty} \omega^2(q^j) q^{2(\alpha+1)j} \int_{q^{1-j}}^{q^{-j}} 1 d\mu_{q,\alpha}(y) = \\
&= C_1 \sum_{j=m+1}^{\infty} \omega^2(q^j) q^{2(\alpha+1)j} (q^{-j(2\alpha+2)} - q^{(1-j)(2\alpha+2)}) \leq C_1 \sum_{j=m+1}^{\infty} \omega^2(q^j) \leq \\
&\leq C_1 \sum_{j=m+1}^{\infty} \frac{1}{q^{j-1} - q^j} \int_{q^j}^{q^{j-1}} \omega^2(u) du \leq C_2 \sum_{j=m+1}^{\infty} \int_{q^j}^{q^{j-1}} \frac{\omega^2(u)}{u} du = \\
&= C_2 \int_0^{q^m} \frac{\omega^2(u)}{u} du \leq C_3 \omega(q^m).
\end{aligned}$$

Here we use the fact that the conditions  $\omega \in B$  and  $\omega^2 \in B$  are equivalent (see condition (S) in [2, Lemma 2]) and obtain that the condition (12) from Lemma 4 is valid for  $f_0$ . By Lemma 4, we have  $\omega_m(f_0, t)_{2,q,\alpha} = O(\omega(t))$ ,  $t \in \mathbb{R}_q^+$ . On the other hand,  $\omega \in B_m$  satisfies  $\Delta_2$ -condition  $\omega(2t) \leq C\omega(t)$  and

$$\begin{aligned}
&\int_1^{\infty} \lambda(t) |\mathcal{F}_{q,\alpha}^d(f_0)(t)|^r d\mu_{q,\alpha}(t) = \sum_{j=1}^{\infty} \omega^r(q^j) q^{jr(\alpha+1)} \int_{q^{1-j}}^{q^{-j}} \lambda(t) d\mu_{q,\alpha}(t) \geq \\
&\geq \sum_{j=1}^{\infty} \int_{q^{1-j}}^{q^{-j}} \lambda(t) \omega^r(t^{-1}) t^{-r(\alpha+1)} d\mu_{q,\alpha}(t) = +\infty
\end{aligned}$$

and, moreover,  $\lambda(t) |\mathcal{F}_{q,\alpha}^d(f_0)(t)|^r \notin L_{q,\alpha}^1(\mathbb{R}_q)$ .  $\square$

**Acknowledgement.** The author expresses his gratitude to anonymous referee for valuable notes and suggestions.

The work of the author is supported by the Program of development of regional scientific-educational center "Mathematics of future technologies" (project no. 075-02-2025-1635).

## References

- [1] Achak A., Daher R., Dhaouadi L., Loualid E. M. *An analog of Titchmarsh's theorem for the  $q$ -Bessel transform*. Ann. Univ. Ferrara., 2019, vol. 65, no. 1, pp. 1–13. DOI: <https://doi.org/10.1007/s11565-018-0309-3>

- [2] Bary N. K., Stechkin S. B. *Best approximation and differential properties of two conjugate functions*. Trudy Mosk. Mat. Obs., 1956, vol. 5, pp. 483–522 (in Russian).
- [3] Bettaibi N., Bettaieb R. H.  *$q$ -analogue of the Dunkl transform on the real line*. Tamsui Oxf. J. Math. Sci., 2007, vol. 25, no. 2, pp. 117–207.
- [4] Bettaibi N., Bettaieb R. H., Bouaziz S. *Wavelet transform associated with the  $q$ -Dunkl operator*. Tamsui Oxf. J. Math. Sci., 2010, vol. 26, no. 1, pp. 77–101.
- [5] Daher R., Tyr O. *Growth properties of the  $q$ -Dunkl transform in the space  $L_{q,\alpha}^p(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$* . Ramanujan J., 2022, vol. 57, no. 1, pp. 119–134.  
DOI: <https://doi.org/10.1007/s11139-021-00387-x>
- [6] Dhaouadi L. *On the  $q$ -Bessel Fourier transform*. Bull. Math. Anal. Appl., 2013, vol. 5, no. 2, pp. 42–60.
- [7] Gogoladze L., Meskhia R. *On the absolute convergence of trigonometric Fourier series*. Proc. Razmadze Math. Inst., 2006, vol. 141, pp. 29–46.
- [8] Kac V., Cheung P. Quantum calculus. Springer, New York, 2002.
- [9] Krotova Yu. I. *Integrability of  $q$ -Bessel Fourier transforms with Gogoladze-Meskhia type weights*. Probl. Anal. Issues Anal., 2024, vol. 13(31), no. 1, pp. 24–36. DOI: <https://doi.org/10.15393/j3.art.2024.14330>
- [10] Krayukhin S. A., Volosivets S. S. *Functions of bounded  $p$ -variation and weighted integrability of Fourier transforms*. Acta Math. Hung., 2019, vol. 159, no. 2, pp. 374–399.  
DOI: <https://doi.org/10.1007/s10474-019-00995-6>
- [11] Móricz F. *Sufficient conditions for the Lebesgue integrability of Fourier transforms*. Anal. Math., 2010, vol. 36, no. 2, pp. 121–129.  
DOI: <https://doi.org/10.1007/s10476-010-0203-4>
- [12] Platonov S. S. *Bessel harmonic analysis and approximation of functions on the half-line*. Izv Math., 2007, vol. 71, no. 5, pp. 1001–1048.  
DOI: <https://doi.org/10.1070/IM2007v071n05ABEH002379>
- [13] Rubin R. L. *A  $q^2$ -analogue operator for  $q^2$ -analogue Fourier analysis*. J. Math. Anal. Appl., 1997, vol. 212, no. 2, pp. 571–582.
- [14] Rubin R. L. *Duhamel solutions of non-homogenous  $q^2$ -analogue wave equations*. Proc. Amer. Math. Soc., 2007, vol. 135, no. 3, pp. 777–785.  
DOI: <https://doi.org/10.1090/S0002-9939-06-08525-X>
- [15] Szász O. *Über die Fourierschen Reihen gewisser Funktionenklassen*. Math. Ann., 1928, vol. 100, pp. 530–536.  
DOI: <https://doi.org/10.1007/BF01448861>

- [16] Titchmarsh, E.: Introduction to the theory of Fourier integrals. Clarendon press, Oxford, 1937.
- [17] Tyr O., Daher R., El Ouadih S., El Fourchi O. *On the Jackson-type inequalities in approximation theory connected to the  $q$ -Dunkl operators in the weighted space  $L^2_{a,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1} d_q x)$* . Bull. Soc. Mat. Mex., 2021, vol. 27, no. 3, paper 51. DOI: <https://doi.org/10.1007/s40590-021-00358-8>
- [18] Tyr O. *Decay of  $q$ -Dunkl transforms and generalized Lipschitz spaces*. Ramanujan J., 2025, vol. 66, no. 2, paper 27. DOI: <https://doi.org/10.1007/s11139-024-00996-2>
- [19] Volosivets S. *Weighted integrability of Fourier-Dunkl transforms and generalized Lipschitz classes*. Analysis Math. Phys., 2022, vol. 12, paper 115. DOI: <https://doi.org/10.1007/s13324-022-00728-z>

*Received April 01, 2025.*

*In revised form, June 24, 2025.*

*Accepted June 27, 2025.*

*Published online July 11, 2025.*

Saratov State University  
83 Astrakhanskaya St., Saratov 410012, Russia  
E-mail: volosivetsss@mail.ru