DOI: 10.15393/j3.art.2025.17930

UDC 517.544

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SHARP CONDITIONS FOR WEIGHTED INTEGRABILITY OF q-DUNKL FOURIER TRANSFORMS

Abstract. We obtain sufficient conditions for the weighted integrability of the q-Dunkl Fourier transforms of functions from generalized integral Lipschitz classes. In the L^2 case, we prove the sharpness of these conditions.

Key words: *q-Dunkl Fourier transform, q-Dunkl translation, modulus of smoothness, q-Dunkl differential operator*

2020 Mathematical Subject Classification: *44A15, 47A10*

1. Introduction. Let $f \in L^1(\mathbb{R})$, i.e., $f\colon \mathbb{R} \to \mathbb{C}$ be Lebesgue-integrable function on $\mathbb{R}$. Then the Fourier transform of f is defined by

$$\widehat{f}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(t) e^{-itx}\, dt, \quad x \in \mathbb{R}.$$

If $f \in L^p(\mathbb{R})$, $1 < p \leqslant 2$, then $\widehat{f}(x)$ is defined as a limit of

$$(2\pi)^{-1/2} \int_{-a}^{b} f(x) e^{-itx}\, dx$$

in the norm of $L^{p'}(\mathbb{R})$, $1/p + 1/p' = 1$, as $a, b \to +\infty$. In particular, $\widehat{f} \in L^{p'}(\mathbb{R})$. The following Hausdorff-Young inequality proved by Titchmarsh [16, Ch. IV]

$$\|\widehat{f}\|_{p'} \leqslant C\|f\|_p := C\Big(\int_{\mathbb{R}} |f(t)|^p\, dt \Big)^{1/p}, \quad f \in L^p(\mathbb{R}), \quad 1 < p \leqslant 2, \qquad (1)$$

is valid. For $p = p' = 2$, we have the Plancherel equality $\|\widehat{f}\|_2 = \|f\|_2$. More about these results see in [16, Ch. III and IV].

Szász [15] obtained the sufficient conditions for convergence of the series $\sum_{n=1}^{\infty}(|a_n(f)|^\beta+|b_n(f)|^\beta)$, where $a_n(f)$ $(b_n(f))$ are cosine (sine) Fourier coefficients of 2π-periodic function $f \in L^p([0,2\pi])$, $1 < p \leqslant 2$. An analogue of Szász's result for Fourier transform was established by Titchmarsh [16, Ch. IV, Theorem 84] and this result is well-known.

Gogoladze and Meskhia [7] proposed a class of weighted sequences satisfying a condition similar to the reverse Hölder inequality and studied convergence of series of $|a_n(f)|^\beta + |b_n(f)|^\beta$ with such weights. Moricz [11] introduced the continual analogues of Gogoladze-Meskhia classes as follows. Let $\gamma \geqslant 1$, $\lambda(t) \in L^1_{loc}(\mathbb{R})$ (i.e., $\lambda(t)$ be Lebesgue-integrable on each compact from $\mathbb{R}$), $\lambda(t)$ be even and nonnegative, such that the inequality

$$\Big(\int_{2^i}^{2^{i+1}} \lambda^\gamma(t)dt\Big)^{1/\gamma} \leqslant C(\gamma)2^{i(1-\gamma)/\gamma}\int_{2^{i-1}}^{2^i}\lambda(t)dt, \quad i \in \mathbb{Z},$$

holds for some $C(\gamma) \geqslant 1$. Then $\lambda(t)$ belongs to the class A_γ.

Let us define the modulus of continuity of $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, by

$$\omega(f,\delta)_{L^p(\mathbb{R})} = \sup_{0\leqslant h\leqslant\delta}\|f(\cdot+h)-f(\cdot)\|_p.$$

Moricz [11] proved

Theorem 1. *Let $1 < p \leqslant 2$, $f \in L^p(\mathbb{R})$, $1/p + 1/p' = 1$, $0 < r < p'$ and $\lambda \in A_{p/(p-rp+r)}$. Then*

$$\int_{|t|\geqslant 2}\lambda(t)\big|\widehat{f}(t)\big|^r dt \leqslant \int_1^\infty \lambda(t)t^{-r/p'}\omega^r(f,\pi/t)_{L^p(\mathbb{R})}\,dt.$$

Seen a more general variant of Theorem 1 in [10, Theorem 4.1], while its sharpness was discussed in [10, Theorem 4.2].

The aim of this paper is to obtain an analogue of Theorem 1 for q-Dunkl Fourier transform and to show its sharpness in the case $p = 2$. We note that for q-Bessel Fourier transform such sufficient conditions are proved by Krotova [9], while for the classical Dunkl transform one can see [19]. Particular results for generalized Lipschitz and Dini-Lipschitz classes of functions and non-weighted case were obtained by Daher and Tyr [5].

2. Definitions. Let $0 < q < 1$, $\alpha > -1/2$ and $\mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\}$, $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and $\widehat{\mathbb{R}_q} = \mathbb{R}_q \cup \{0\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, we set

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1}(1 - aq^i), \quad (a; q)_\infty = \prod_{i=0}^{\infty}(1 - aq^i).$$

and

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad [n]_q! = \frac{(q, q)_n}{(1 - q)^n}, \quad n \in \mathbb{Z}_+ = \{0, 1, \dots\}.$$

The q-gamma function is given by

$$\Gamma_q(x) = \frac{(q, q)_\infty}{(q^x, q)_\infty}(1 - q)^{1-x}, \quad x \neq 0, -1, -2, \dots.$$

Then (see [3]) $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$, $\Gamma_q(1) = 1$ and $\lim_{q \to 1-0} \Gamma_q(x) = \Gamma(x)$ for $\operatorname{Re} x > 0$ ($\operatorname{Re} z$ is the real part of $z \in \mathbb{C}$). A q-analogue of exponential function e^{ix} is introduced by Rubin [13]

$$e(x, q^2) = \cos(-ix, q^2) + i \sin(-ix, q^2),$$

where

$$\cos(x, q^2) = \sum_{j=0}^{\infty}(-1)^j q^{j(j+1)} \frac{x^{2j}}{[2j]_q!}, \quad \sin(x, q^2) = \sum_{j=0}^{\infty}(-1)^j q^{j(j+1)} \frac{x^{2j+1}}{[2j + 1]_q!}.$$

The q-derivative of a function f defined on $\mathbb{R}_q$ in $x \neq 0$ is given by

$$D_q(f)(x) = (f(x) - f(qx))/((1 - q)x), \ x \neq 0,$$

$D_q(f)(0) = f'(0)$, if $f'(0)$ exists. Also we consider $D_q^+(f)(x) = D_q f(q^{-1}x)$. Then it is easy to see that for $\nu \in \mathbb{R}$ one has $D_q x^\nu = [\nu]_q x^{\nu-1}$ and $D_q^+ x^\nu = q^{-\nu}[\nu]_q x^{\nu-1}$.

If f is defined on $\mathbb{R}_q$, we can consider the even and odd parts of f:

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2}, \tag{2}$$

and Rubin's q-differential operator [13]

$$\partial_q(f)(x) = D_q^+ f_e(x) + D_q f_o(x).$$

Let us introduce the q-integral of Jackson for f defined on $\mathbb{R}_q$ on intervals $[0,a]$, $[0,+\infty)$, $[a,b]$, $a,b\in\mathbb{R}_q^+$, and $(-\infty,\infty)$ by

$$\int_0^a f(x)\,d_qx=(1-q)a\sum_{n=0}^{\infty}f(aq^n)q^n;\quad \int_0^{\infty}f(x)\,d_qx=(1-q)\sum_{n\in\mathbb{Z}}q^nf(q^n),$$

$$\int_a^b f(x)\,d_qx=\int_0^b f(x)\,d_qx-\int_0^a f(x)\,d_qx,$$

$$\int_{-\infty}^{\infty}f(x)\,d_qx=(1-q)\sum_{n\in\mathbb{Z}}q^n[f(q^n)+f(-q^n)].$$

Definitions of $\int_a^0 f(x)\,d_qx$ for $a<0$, $a\in\mathbb{R}_q$, and $\int_{-\infty}^0 f(x)\,d_qx$ may be obtained in the same way. Then

$$D_q\Big(\int_a^x f(t)\,d_q(t)\Big)=f(x),\quad \int_a^b D_qf(t)\,d_qt=f(b)-f(a).$$

More about q-analysis see in [8], e.g., the following simple change of variables formula may be found in (19.14) from this book:

$$\int_a^b f(\lambda x)d_qx=\lambda^{-1}\int_{\lambda a}^{\lambda b}f(x)\,d_q(x),\quad \lambda\in\mathbb{R}_q^+.$$

For $\alpha>-1/2$, we consider q-Dunkl differential operator

$$\Lambda_{q,\alpha}(f)(x)=\partial_q[f_e+q^{2\alpha+1}f_o](x)+[2\alpha+1]_q\frac{f(x)-f(-x)}{2x},$$

where f_e, f_o are defined in (2). Bettaibi and Bettaieb [3] proved that the equation $\Lambda_{q,\alpha}(f)(x)=i\lambda f(x)$ with the initial condition $f(0)=1$ has an unique solution

$$\Psi_{q,\alpha}(\lambda x)=j_\alpha(\lambda x,q^2)+\frac{i\lambda x}{[2\alpha+2]_q}j_{\alpha+1}(\lambda x),\quad x\in\mathbb{R}_q.$$

Here $j_\alpha(x,q^2)$ is the third Jackson normalized q-Bessel function of the first kind given by

$$j_\alpha(x,q^2)=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}\Big(\frac{x}{1+q}\Big)^{2n}.$$

It is easy to see that

$$\partial_q j_\alpha(x,q^2)=D_q^+ j_\alpha(x,q^2)=-\frac{x}{[2\alpha+2]_q j_{\alpha+1}(x,q^2)}.$$

Therefore, $\Psi_{q,\alpha}(x)=j_\alpha(x,q^2)-i\partial_q j_\alpha(x,q^2)$.

Further, we consider the measure

$$d\mu_{q,\alpha}(x)=\frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}|x|^{2\alpha+1}\,d_q(x)=:c_{\alpha,q}|x|^{2\alpha+1}\,d_q(x). \tag{3}$$

For $1\leqslant p<\infty$, the space $L^p_{q,\alpha}(\mathbb{R}_q)$ consists of all real functions f on $\mathbb{R}_q$ for which

$$\|f\|^p_{p,q,\alpha}=\int_{-\infty}^{+\infty}|f(x)|^p\,d\mu_{q,\alpha}(x)<\infty.$$

The space $L^\infty_{q,\alpha}(\mathbb{R}_q)$ is the space of bounded on $\mathbb{R}_q$ functions with uniform norm $\|f\|_\infty=\|f\|_{\infty,q,\alpha}=\sup_{x\in\mathbb{R}_q}|f(x)|$.

A q-analogue $S_q(\mathbb{R}_q)$ of the Schwartz space consists of all functions with finite seminorms $P_{n,m,q}(f)=\sup_{x\in\mathbb{R}_q}|x^m\partial_q^n f(x)|$, $m,n\in\mathbb{Z}_+$, and such that $\lim\limits_{x\to 0,\,x\in\mathbb{R}_q}\partial_q^n f(x)$ exists for all $n\in\mathbb{Z}_+$. It is known that $\Psi_{q,\alpha}\in S_q(\mathbb{R}_q)$ and

$$|\Psi_{q,\alpha}(x)|\leqslant\frac{4}{(q,q)_\infty},\qquad x\in\mathbb{R}_q, \tag{4}$$

(see [3]). Let us define the q-Dunkl Fourier transform of $f\in L^1_{q,\alpha}(\mathbb{R}_q)$ by

$$\mathcal{F}^d_{q,\alpha}(f)(y)=\int_{-\infty}^{+\infty}f(x)\Psi_{q,\alpha}(-yx)\,d\mu_{q,\alpha}(x),$$

where the value of $c_{\alpha,q}$ is given in (3). In [3, Theorem 11] it is proved that $\mathcal{F}^d_{q,\alpha}(S_q(\mathbb{R}_q))\subset S_q(\mathbb{R}_q)$ and

$$\|\mathcal{F}^d_{q,\alpha}(f)\|_{2,q,\alpha}=\|f\|_{2,q,\alpha} \tag{5}$$

for $f \in S_q(\mathbb{R}_q)$. Therefore, $\mathcal{F}^d_{q,\alpha}$ can be uniquely extended to an isometry of $L^2_{q,\alpha}(\mathbb{R}_q)$ and (5) holds for $f \in L^2_{q,\alpha}(\mathbb{R}_q)$. On the other hand, from (4) we deduce that

$$\|\mathcal{F}^d_{q,\alpha}(f)\|_{\infty,q,\alpha} \leqslant \frac{4c_{\alpha,q}}{(q,q)_\infty}\|f\|_{1,q,\alpha}, \quad f \in L^1_{q,\alpha}(\mathbb{R}_q), \tag{6}$$

(see also [3, Proposition 9]). From (5) and (6) by the Riesz-Thorin theorem we obtain that $\mathcal{F}^d_{q,\alpha}$ can be extended to $L^p_{q,\alpha}(\mathbb{R})$, $1 < p \leqslant 2$, and

$$\|\mathcal{F}^d_{q,\alpha}(f)\|_{p',q,\alpha} \leqslant C(p,q)\|f\|_{p,q,\alpha}, \quad f \in L^p_{q,\alpha}(\mathbb{R}_q), \quad 1/p + 1/p' = 1. \tag{7}$$

The generalized q-Dunkl translation is defined for $f \in S_q(\mathbb{R}_q)$, $h, x \in \mathbb{R}_q$ by the formula (see [4])

$$T^{q,\alpha}_h f(x) = \int_{-\infty}^{+\infty} \mathcal{F}^d_{q,\alpha}(f)(y)\Psi_{q,\alpha}(yx)\Psi_{q,\alpha}(yh)\, d\mu_{q,\alpha}(y),$$

$T^{q,\alpha}_0 f = f$. Using the density of $S_q(\mathbb{R}_q)$ in $L^2_{q,\alpha}(\mathbb{R}_q)$, Bettaibi et al proved in [4, Proposition 2] that for $f \in L^2_{q,\alpha}(\mathbb{R}_q)$

$$\|T^{q,\alpha}_h(f)\|_{2,q,\alpha} \leqslant \frac{4}{(q,q)_\infty}\|f\|_{2,q,\alpha}, \quad h \in \mathbb{R}_q,$$

and

$$\mathcal{F}^d_{q,\alpha}(T^{q,\alpha}_h(f))(y) = \Psi_{q,\alpha}(yh)\mathcal{F}^d_{q,\alpha}(f)(y). \tag{8}$$

In the same way, the boundedness of translation $T^{q,\alpha}_h$ and (8) can be proved for $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, $1 \leqslant p \leqslant 2$. For $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, $1 \leqslant p \leqslant 2$, and $m \in \mathbb{N}$, we introduce the difference of order m with step $h \in \mathbb{R}^+_q$ by

$$\Delta^m_{h,q,\alpha}(f)(x) = (I - T^{q,\alpha}_h)^m f(x)$$

and the q-modulus of smoothness of order m

$$\omega_m(f,\delta)_{p,q,\alpha} = \sup_{0 \leqslant h \leqslant \delta, h \in \mathbb{R}^+_q \cup \{0\}} \|\Delta^m_{h,q,\alpha}(f)\|_{p,q,\alpha}.$$

For conditions of Gogoladze-Meskhia-Moricz type we consider even functions $\lambda(t)$ defined on $\mathbb{R}_q$, such that for $i \in \mathbb{Z}$ and $\gamma \geqslant 1$

$$\Big(\int_{(1/q)^i}^{(1/q)^{i+1}} \lambda^\gamma(t)\, d\mu_{q,\alpha}(t)\Big)^{1/\gamma} \leqslant Cq^{-i(2\alpha+2)(1/\gamma-1)} \int_{(1/q)^{i-1}}^{(1/q)^i} \lambda(t)\, d\mu_{q,\alpha}(t). \tag{9}$$

If (9) holds, we write $\lambda(t) \in A_{\gamma,q,\alpha}$. Note that for $\gamma = p'/(p'-r)$, $0 < r < p'$, one has $1/\gamma - 1 = 1 - r/p' - 1 = -r/p'$. In [9, Lemma 4], it is proved that $\lambda(t) = |t|^\delta$, $\delta \in \mathbb{R}$, belongs to all classes $A_{\gamma,q,\alpha}$, $\gamma \geqslant 1$.

Let $W^m_{p,q,\alpha}(\mathbb{R}_q)$, where $m \in \mathbb{N}$, $1 \leqslant p \leqslant 2$, be the Sobolev spaces of functions $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, such that $\Lambda^j_{q,\alpha}(f) \in L^p_{q,\alpha}(\mathbb{R}_q)$ for $j = 1, 2, \ldots, m$.

Let Φ be the set of nondecreasing continuous on $\mathbb{R}_+ = [0, \infty)$ functions ω, such that $\omega(0) = 0$. If $\omega \in \Phi$ and $\int_0^\delta t^{-1}\omega(t)\,dt = O(\omega(\delta))$, then ω belongs to the Bary class B; if $\omega \in \Phi$ and $\delta^m \int_\delta^\infty t^{-m-1}\omega(t)\,dt = O(\omega(\delta))$, $m > 0$, then ω belongs to the Bary-Stechkin class B_m (see [2]).

3. Auxiliary propositions

Lemma 1. *Let $\alpha \geqslant 0$, $0 < q < 1$. Then there exists $\gamma > 0$, such that*

$$|j_\alpha(x, q^2)| \leqslant 1, \quad x \in \mathbb{R}^+_q, \quad \alpha \geqslant 0, \tag{10}$$

$$|j_\alpha(x, q^2) - 1| \geqslant \gamma, \quad x \in \mathbb{R}^+_q \cap (1, +\infty). \tag{11}$$

Proof. The inequality (10) was noted by Dhaouadi [6, Remark 1], the inequality (11) was proved by Achak, Daher, Dhaouadi and Loualid in [1]. Similar estimates for the classical Bessel function can be found in [12]. □

The result of Lemma 2 can be found in [17, Lemma 3.2].

Lemma 2. *Let $\alpha \geqslant 0$, $0 < q < 1$, $k, m \in \mathbb{N}$, $k > m$, $f \in W^m_{2,q,\alpha}(\mathbb{R}_q)$. Then for some $C > 0$ we have*

$$\omega_k(f, \delta)_{2,q,\alpha} \leqslant C\delta^m \omega_{k-m}(\Lambda^m_{q,\alpha} f, \delta)_{2,q,\alpha}, \quad \delta > 0.$$

Lemma 3 easily follows from (8).

Lemma 3. *Let $m \in \mathbb{N}$, $\alpha \geqslant 0$, $1 \leqslant p \leqslant 2$. Then*

$$\mathcal{F}^d_{q,\alpha}(\Delta^m_{h,q,\alpha} f)(x) = (1 - \Psi_{q,\alpha}(hx))^m \mathcal{F}^d_{q,\alpha}(f)(x), \quad h, x \in \mathbb{R}_q.$$

Lemma 4 is an analogue of the famous Titchmarsh equivalence theorem. It is proved by Tyr [18].

Lemma 4. *Let $\omega \in B_m$, $m \in \mathbb{N}$ and $f \in L^2_{q,\alpha}(\mathbb{R}_q)$. Then the statements $\omega_m(f, t)_{2,q,\alpha} = O(\omega(t))$, $t \in \mathbb{R}^+_q$, and*

$$\int_{|y| \geqslant h^{-1}} |\mathcal{F}^d_{q,\alpha}(f)(y)|^2 \, d\mu_{q,\alpha}(y) = O\left(\omega^2(h)\right), \quad h \in \mathbb{R}^+_q, \tag{12}$$

are equivalent.

Lemma 5. *Let $\alpha \geqslant 0$ and*

$$D_{y,q,\alpha}(x) = \int\limits_{|t|\leqslant y} \Psi_{q,\alpha}(-tx)\, d\mu_{q,\alpha} = \mathcal{F}^d_{q,\alpha}(\chi_{[-y,y]})(x), \quad y \in \mathbb{R}_q^+.$$

where χ_E be the indicator of a set E. Then

$$\|D_{y,q,\alpha}\|_{2,q,\alpha} = K y^{\alpha+1}, \quad y \in \mathbb{R}_q^+,$$

where K does not depend on y.

Proof. By the Plancherel equality (5) for $y = q^s$, $s \in \mathbb{Z}$, we obtain

$$\|D_{y,q,\alpha}\|^2_{2,q,\alpha} = \|\chi_{[-y,y}\|^2_{2,q,\alpha} = 2\int\limits_0^{q^s} 1\, d\mu_{q,\alpha}(x) =$$

$$= C_1(1-q)q^s \sum_{n=0}^{\infty} (q^{n+s})^{2\alpha+1} q^n = C_1((1-q)q^{s(2\alpha+2)} \sum_{n=0}^{\infty} (q^n)^{2\alpha+2} =$$

$$= C_1 \frac{(1-q)q^{s(2\alpha+2)}}{1-q^{2\alpha+2}} = C_2 y^{2\alpha+2}.$$

□

3. Main results.

Theorem 2. *Let $1 < p \leqslant 2$, $1/p + 1/p' = 1$, $\alpha \geqslant 0$, $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, $m \in \mathbb{N}$. If $\lambda \in A_{p/(p-pr+r),q,\alpha} = A_{p'/(p'-r),q,\alpha}$ for some $r \in (0, p')$, $\lambda(x)\chi_{[-1,1]}(x) \in L^{p'/(p'-r)}_{q,\alpha}(\mathbb{R}_q)$, and the integral*

$$\int\limits_1^{\infty} \lambda(t) t^{-r(2\alpha+2)/p'} \omega^r_m(f, t^{-1})_{p,q,\alpha}\, d\mu_{q,\alpha}(t)$$

converges, then $\lambda(t)|\mathcal{F}^d_{q,\alpha}(f)(t)|^r \in L^1_{q,\alpha}(\mathbb{R}_q)$.

Proof. Let $M_i = [(1/q)^i, (1/q)^{i+1}] \cup [-(1/q)^{i+1}, -(1/q)^i]$, $i \in \mathbb{Z}_+$. By Lemma 3, (11) from Lemma 1, and Hausdorff-Young type inequality (7), we have

$$C_1 \int\limits_{M_i} |\mathcal{F}^d_{q,\alpha}(f)(y)|^{p'}\, d\mu_{q,\alpha}(y) \leqslant$$

$$\leqslant \int\limits_{M_i} |\mathcal{F}^d_{q,\alpha}(f)(y)|^{p'} |1-\Psi_{q,\alpha}(q^i y,q^2)|^{mp'}\, d\mu_{q,\alpha}(y) \leqslant$$

$$\int\limits_{-\infty}^{\infty} |\mathcal{F}^d_{q,\alpha}(f)(y)|^{p'} |1-\Psi_{q,\alpha}(q^i y,q^2)|^{mp'}\, d\mu_{q,\alpha}(y) = \|\mathcal{F}_{q,\nu}(\Delta^m_{h,q,\alpha}(f)\|^{p'}_{p,q,\alpha} \leqslant$$

$$\leqslant C_2 \|\Delta^m_{q,\nu,h}(f)\|^{p'}_{p,q,\alpha} \leqslant C_2 \omega^{p'}_m(f,q^i)_{p,q,\alpha}.$$

By the condition $\lambda \in A_{p'/(p'-r),q,\alpha}$ and the Hölder inequality with exponents p'/r and $p'/(p'-r)$, we have

$$\int\limits_{M_i} \lambda(t)|\mathcal{F}_{q,\alpha}(f)(t)|^r\, d\mu_{q,\alpha}(t) \leqslant \Big(\int_{M_i} |\mathcal{F}^d_{q,\alpha}(f)(t)|^{p'}\, d\mu_{q,\alpha}(t)\Big)^{r/p'} \times$$

$$\times \Big(\int\limits_{M_i} |\lambda(t)|^{p'/(p'-r)}\, d\mu_{q,\alpha}(t)\Big)^{1-r/p'} \leqslant$$

$$\leqslant C_3 \omega^r_m(f,q^i)_{p,q,\alpha} q^{ir(2\alpha+2)/p'} \int\limits_{M_{i-1}} \lambda(t)\, d\mu_{q,\alpha}(t).$$

By the monotonicity of the integral, we have

$$\int\limits_{M_i} \lambda(t)|\mathcal{F}^d_{q,\alpha}(f)(t)|^r\, d\mu_{q,\alpha}(t) \leqslant C_4 \int\limits_{M_{i-1}} \frac{\lambda(t)\omega_m(f,1/t)_{p,q,\alpha}}{t^{r(2\alpha+2)/p'}}\, d\mu_{q,\alpha}(t). \quad (13)$$

Summing up (13) over $i = 0,1,2,\ldots,$ we obtain

$$\int\limits_{t\in\mathbb{R}_q, |t|\geqslant 1} \lambda(t)|\mathcal{F}_{q,\alpha}(f)(t)|^r\, d\mu_{q,\alpha}(t) \leqslant$$

$$\leqslant C_4 \int\limits_{t\in\mathbb{R}_q, |t|\geqslant 1/q} \frac{\lambda(t)\omega_m(f,|t|^{-1})_{p,q,\alpha}}{|t|^{r(2\nu+2)/p'}}\, d\mu_{q,\alpha}(t). \quad (14)$$

Since $\lambda \in L^{p'/(p'-r)}_{q,\alpha}[-1,1]$, $|t|^{-1}$ and $\omega_m(t,|t|^{-1})_{p,q,\alpha}$ are bounded for $|t| \in [1/q,1]$, the integral in the right-hand side of (14) is finite.

Finally, by the condition $\lambda \in L^{p'/(p'-r)}_{q,\alpha}[-1,1]$, the Hölder inequality with the same exponents, and (7), we obtain that $\lambda(t)|\mathcal{F}^d_{q,\alpha}(f)(t)|^r \in L^1_{q,\alpha}[-1,1]$ and the theorem is proved. $\square$

Let $0 < r < p'$. By Lemma 5 in [9] it is easy to see that $\lambda_\beta(t) = |t|^\beta$ belongs to $L^{p'/(p'-r)}_{q,\alpha}[-1, 1]$ if and only if $\beta > -(1 - r/p')(2\alpha + 2)$. From this note and Theorem 2, we obtain

Corollary 1. *Let $1 < p \leqslant 2$, $m \in \mathbb{N}$, $1/p + 1/p' = 1$, $\alpha \geqslant 0$, $\beta \in \mathbb{R}$, $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, and $\omega_m(f, t)_{p,q,\alpha} = O(t^\delta)$ for some $\delta > 0$ and all $t \in \mathbb{R}_q$. If $r \in (0, p')$, $\beta > (r/p' - 1)(2\alpha + 2)$ and*

$$p' > r > \frac{p'(\beta + 2\alpha + 2)}{2\alpha + 2 + p'\delta}, \tag{15}$$

then $t^\beta |\mathcal{F}^d_{q,\alpha}(f)(t)|^r \in L^1_{q,\alpha}(\mathbb{R}_q)$.

If $\beta = 0$, then in (15) the case $p' = r$ can be added.

Corollary 2. *Let $\alpha \geqslant 0$, $k, m \in \mathbb{N}$, $0 < r < 2$, $f \in W^m_{2,q,\alpha}(\mathbb{R}_q)$. If $\lambda \in A_{2/(2-r),q,\alpha}$, $\lambda(x)\chi_{[-1,1]}(x) \in L^{2/(2-r)}_{q,\alpha}(\mathbb{R}_q)$ and the integral*

$$\int_1^\infty \lambda(t) t^{-mr-r(\alpha+1)} \omega^r_k(\Lambda^m_{q,\alpha} f, t^{-1})_{2,q,\alpha}\, d\mu_{q,\alpha}(t) \tag{16}$$

converges, then $\lambda(t)|\mathcal{F}^d_{q,\alpha}(f)(t)|^r \in L^1_{q,\alpha}(\mathbb{R}_q)$.

Proof. By Lemma 2, from the convergence of integral (16) the convergence of

$$\int_1^\infty \lambda(t) t^{-r(2\alpha+2)/2} \omega^r_{m+k}(f, t^{-1})_{2,q,\alpha}\, d\mu_{q,\alpha}(t)$$

follows. Applying Theorem 2, we obtain the result of the present Corollary. □

Theorem 3 can be considered as a counterpart of Theorem 2 from [11].

Theorem 3. *Let $1 < p \leqslant 2$, $1/p + 1/p' = 1$, $1 \leqslant s < p$, $\alpha \geqslant 0$, $f \in L^p_{q,\alpha}(\mathbb{R}_q) \cap L^s_{q,\alpha}(\mathbb{R}_q)$ is bounded on $\mathbb{R}_q$. If $m \in \mathbb{N}$, $\omega_m(f, t)_{s,q,\alpha} = O(t^{1/s})$, $t \in \mathbb{R}^+_q$, $\lambda \in A_{p'/(p'-r),q,\alpha}$ for some $r \in (0, p')$, $\lambda(x)\chi_{[-1,1]}(x) \in L^{p'/(p'-r)}_{q,\alpha}(\mathbb{R}_q)$ and the integral*

$$\int_1^\infty \lambda(t) t^{-r(2\alpha+2)/p' - r/p} \omega^{r(1-s/p)}_m(f, t^{-1})_{\infty,q,\alpha}\, d\mu_{q,\alpha}(t)$$

converges, then $\lambda(t)|\mathcal{F}^d_{q,\alpha}(f)(t)|^r \in L^1_{q,\alpha}(\mathbb{R}_q)$.

Proof. By definition, under conditions of Theorem for $h \in \mathbb{R}_q^+$, we have

$$\|\Delta_{h,q,\alpha}^m f\|_{p,q,\alpha}^p = \int_{-\infty}^{\infty} |\Delta_{h,q,\alpha}^m f(t)|^p \, d\mu_{q,\alpha}(t) \leqslant$$

$$\leqslant \sup_{t\in\mathbb{R}_q} |\Delta_{h,q,\alpha}^m f(t)|^{p-s} \int_{-\infty}^{\infty} |\Delta_{h,q,\alpha}^m f(t)|^s p \, d\mu_{q,\alpha}(t) \leqslant C_1 h \omega_m^{p-s}(f,h)_{\infty,q,\alpha},$$

whence

$$\omega_m(f,t)_{p,q,\alpha} \leqslant C_1 t^{1/p} \omega_m^{1-s/p}(f,t)_{\infty,q,\alpha}, \quad t \in \mathbb{R}_q^+. \tag{17}$$

Substituting (17) into integral from the condition of Theorem 2, we conclude that under conditions of Theorem 3 the function $\lambda(t)|\mathcal{F}_{q,\alpha}^d(f)(t)|^r$ belongs to $L_{q,\alpha}^1(\mathbb{R}_q)$. □

Theorem 4 shows the sharpness of Theorem 2 in the case $p = 2$ under some restrictions.

Theorem 4. *Let $\alpha \geqslant 0$, $m \in \mathbb{N}$, $\omega \in B \cap B_m$, $r \in (0,2)$. If $\lambda(t)$ is even positive on $\mathbb{R}_q$ function, such that the integral*

$$\int_1^{\infty} \lambda(t) t^{-r(\alpha+1)} \omega^r(t^{-1}) \, d\mu_{q,\alpha}(t)$$

diverges, then there exists $f_0 \in L_{q,\alpha}^2(\mathbb{R}_q)$ with property $\omega_m(f_0,t)_{2,q,\alpha} = O(\omega(t))$, $t \in \mathbb{R}_q^+$, but $\lambda(t)|\mathcal{F}_{q,\alpha}^d(f_0)(t)|^r \notin L_{q,\alpha}^1(\mathbb{R}_q)$.

Proof. Let us consider the function

$$f_0(t) = \sum_{j=1}^{\infty} \omega(q^j) q^{j(\alpha+1)} (D_{q^{-j},q,\alpha}(t) - D_{q^{1-j},q,\alpha}(t)),$$

where $D_{y,q,\alpha}$ are defined in Lemma 5. If $g \in L_{q,\alpha}^2(\mathbb{R}_q)$ is even, then the equality $\mathcal{F}_{q,\alpha}^d(\mathcal{F}_{q,\alpha}^d(g)) = g$ holds. Therefore,

$$\mathcal{F}_{q,\alpha}^d(f_0)(y) = \sum_{j=1}^{\infty} \omega(q^j) q^{j(\alpha+1)} \chi_{\{q^{1-j} \leqslant |\cdot| \leqslant q^{-j}\}}(y)$$

and (see the proof of Lemma 5)

$$\int_{|y| \geqslant q^{-m}, y \in \mathbb{R}_q} |\mathcal{F}_{q,\alpha}^d(f_0)(t)|^2 \, d\mu_{q,\alpha}(y) =$$

$$= 2 \sum_{j=m+1}^{\infty} \omega^2(q^j) q^{2(\alpha+1)j} \int_{q^{1-j}}^{q^{-j}} 1 \, d\mu_{q,\alpha}(y) =$$

$$= C_1 \sum_{j=m+1}^{\infty} \omega^2(q^j) q^{2(\alpha+1)j} (q^{-j(2\alpha+2)} - q^{(1-j)(2\alpha+2)}) \leqslant C_1 \sum_{j=m+1}^{\infty} \omega^2(q^j) \leqslant$$

$$\leqslant C_1 \sum_{j=m+1}^{\infty} \frac{1}{q^{j-1} - q^j} \int_{q^j}^{q^{j-1}} \omega^2(u) \, du \leqslant C_2 \sum_{j=m+1}^{\infty} \int_{q^j}^{q^{j-1}} \frac{\omega^2(u)}{u} \, du =$$

$$= C_2 \int_0^{q^m} \frac{\omega^2(u)}{u} \, du \leqslant C_3 \omega(q^m).$$

Here we use the fact that the conditions $\omega \in B$ and $\omega^2 \in B$ are equivalent (see condition (S) in [2, Lemma 2]) and obtain that the condition (**??**) from Lemma 4 is valid for f_0. By Lemma 4, we have $\omega_m(f_0, t)_{2,q,\alpha} = O(\omega(t))$, $t \in \mathbb{R}_q^+$. On the other hand, $\omega \in B_m$ satisfies Δ_2-condition $\omega(2t) \leqslant C\omega(t)$ and

$$\int_1^{\infty} \lambda(t) |\mathcal{F}_{q,\alpha}^d(f_0)(t)|^r \, d\mu_{q,\alpha}(t) = \sum_{j=1}^{\infty} \omega^r(q^j) q^{jr(\alpha+1)} \int_{q^{1-j}}^{q^{-j}} \lambda(t) \, d\mu_{q,\alpha}(t) \geqslant$$

$$\geqslant \sum_{j=1}^{\infty} \int_{q^{1-j}}^{q^{-j}} \lambda(t) \omega^r(t^{-1}) t^{-r(\alpha+1)} \, d\mu_{q,\alpha}(t) = +\infty$$

and, moreover, $\lambda(t) |\mathcal{F}_{q,\alpha}^d(f_0)(t)|^r \notin L_{q,\alpha}^1(\mathbb{R}_q)$. □

Acknowledgement. The author expresses his gratitude to anonymous referee for valuable notes and suggestions.

The work of the author is supported by the Program of development of regional scientific-educational center "Mathematics of future technologies" (project no. 075-02-2025-1635).

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Received April 01, 2025.
In revised form, June 24, 2025.
Accepted June 27, 2025.
Published online July 11, 2025.

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