

UDC 517.53, 517.55

S. MEHTA, K. S. CHARAK

NORMALITY OF A FAMILY OF HOLOMORPHIC CURVES THAT PARTIALLY SHARE WANDERING HYPERPLANES WITH THEIR DERIVATIVES

Abstract. In this paper, we prove that a family of holomorphic curves in $\mathbb{P}^N(\mathbb{C})$ that partially share moving as well as wandering hyperplanes with their derivatives is normal. By associating a moving hyperplane in $\mathbb{P}^1(\mathbb{C})$ to any holomorphic function, we also obtain a normality criterion for a family of meromorphic functions that partially share wandering holomorphic functions with their derivatives. Further, we devise a tractable representation of complex-valued holomorphic functions on a domain D as functions from D to $\mathbb{P}^2(\mathbb{C})$ to obtain a normality criterion that leads to a counterexample to the converse of Bloch's principle.

Key words: *normal family, complex projective space, holomorphic curves, moving hyperplanes, partially shared functions*

2020 Mathematical Subject Classification: *Primary 32A19, 32H30; Secondary 30D45, 32H02*

1. Introduction. A family $\mathcal{F}$ of holomorphic functions on a domain D in the complex plane is said to be normal if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence that converges locally uniformly in D to an analytic function f or a subsequence that converges locally uniformly in D to ∞ . Normality of families of meromorphic functions is defined likewise, where the convergence is required to be with respect to the spherical metric of the Riemann sphere. The theory of normal families was pioneered by P. Montel who in 1912 obtained the following fundamental result:

Theorem 1. [20] *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subseteq \mathbb{C}$ that omit three distinct values a, b, c in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Then $\mathcal{F}$ is normal in D .*

Since then, normal families have found wide-ranging applications in the fields of dynamics of rational and transcendental functions, bicomplex analysis, harmonic mappings, functional analysis, and complex projective geometry (see [1], [2], [5], [12], [28]).

A significant milestone in this theory is the following result by W. Schwick [23], which drew a connection between normality and shared values. Recall that two nonconstant meromorphic functions f and g on a domain D share a value $w \in \hat{\mathbb{C}}$ if $f^{-1}(w) = g^{-1}(w)$.

Theorem 2. [23] *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subseteq \mathbb{C}$, and a, b, c be three distinct finite complex numbers. If for every $f \in \mathcal{F}$, f and f' share the values a, b, c , then $\mathcal{F}$ is normal in D .*

Over the last few decades, there have been various improvements of Theorem 2 (see, for example, [4], [9], [21]). The notion of sharing of values has been subsequently used by Sun [24] to generalize Theorem 1, where omission of three distinct values is replaced by sharing three values by all the functions of the family.

Recall that a function $f: D \rightarrow \mathbb{P}^N(\mathbb{C})$ that is holomorphic (regarded as a map between two complex manifolds) is said to be a holomorphic curve. Theorem 1 has been extended to the family of holomorphic curves in $\mathbb{P}^N(\mathbb{C})$ by J. Dufresnoy [13] where, instead of three distinct points, the omission of $2N + 1$ hyperplanes in general position is required. This requirement of omission has been further reduced to merely sharing of such hyperplanes by Yang et al [26]. An adequate extension of Theorem 2 to the case of holomorphic maps into complex projective space could only be achieved after a proper extension of the notion of derivatives for such maps, which was accomplished by Ye et al [27].

Let f and g be two meromorphic functions on a domain $D \subseteq \mathbb{C}$. The functions f and g are said to share a function α on D if $\{z \in D: f(z) = \alpha(z)\} = \{z \in D: g(z) = \alpha(z)\}$. If $\{z \in D: f(z) = \alpha(z)\} \subseteq \{z \in D: g(z) = \alpha(z)\}$, then we say that f partially shares α with g on D . Recently, the notion of partial sharing has led to some interesting normality criteria (see, for example, [7], [8], [17]).

In 2014, Grahl and Nevo [15], using a simultaneous rescaling version of the Zalcman Lemma, obtained a normality criterion for a family of meromorphic functions, where each member shared three functions with its derivative.

Theorem 3. [15] *Let $\mathcal{F}$ be a family of functions meromorphic in the*

unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\varepsilon > 0$. Assume that for each $f \in \mathcal{F}$ there exist functions $a_1^{(f)}$, $a_2^{(f)}$, and $a_3^{(f)}$ that are either meromorphic or identically infinity in $\mathbb{D}$, and do not have any common poles with f , such that f and f' share the functions $a_1^{(f)}$, $a_2^{(f)}$, and $a_3^{(f)}$, with the property that

$$\sigma(a_j^{(f)(z)}, a_k^{(f)(z)}) \geq \varepsilon \quad \forall z \in \mathbb{D}, j, k \in \{1, 2, 3\} \text{ with } j \neq k.$$

Then $\mathcal{F}$ is normal in $\mathbb{D}$.

The functions $a_1^{(f)}$, $a_2^{(f)}$, $a_3^{(f)}$ are called *wandering shared functions* as they vary for different functions of the family $\mathcal{F}$. After proving our first main result, we shall be in a position to replace sharing in Theorem 3 by partial sharing. But it shall entail some extra restrictions on the *wandering partially shared functions*; that is, they must be bounded holomorphic functions on $\mathbb{D}$.

The extension of results in theory of meromorphic functions to that of holomorphic functions into $\mathbb{P}^N(\mathbb{C})$ has been a regular feature in the field of normal families (see [10], [16], [25]). The results so obtained are more general and can be reduced to the case of meromorphic functions by just taking $N = 1$. In our approach to the last result of this paper, we have followed a different path by starting with a family of complex-valued holomorphic functions and lifting them to curves in $\mathbb{P}^2(\mathbb{C})$. A choice of proper hyperplanes has enabled us to formulate a nice criterion for normality of complex-valued holomorphic functions on D . Now we introduce the necessary preliminaries.

1.1. Projective Space and Hyperplanes. An N -dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$ is the quotient space $(\mathbb{C}^{N+1} \setminus \{0\}) / \sim$, where $(a_0, \dots, a_N) \sim (b_0, \dots, b_N)$ if and only if $(a_0, \dots, a_N) = \lambda(b_0, \dots, b_N)$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. We denote by $[a_0 : \dots : a_N]$ the equivalence class of $(a_0, \dots, a_N)$. The mapping $\pi: \mathbb{C}^{N+1} \setminus \{0\} \rightarrow \mathbb{P}^N(\mathbb{C})$ given by $\pi(a_0, \dots, a_N) = [a_0 : \dots : a_N]$ is called the standard projection mapping. In fact, $\mathbb{P}^N(\mathbb{C})$ is endowed with a canonical metric called the Fubini-Study metric. For details, see [19] and [22, pp. 99–102].

A subset H of $\mathbb{P}^N(\mathbb{C})$ is called a hyperplane if there is an N -dimensional linear subspace $\tilde{H}$ of $\mathbb{C}^{N+1}$, such that $\pi(\tilde{H} \setminus \{0\}) = H$.

Let $z = [z_0 : z_1 : \dots : z_N]$ be a fixed system of homogeneous coordinates and $a_j(\zeta)$, $j = 0, 1, \dots, N$ be holomorphic functions on a domain D not all simultaneously zero at any point in D . Then, corresponding to

each point $\zeta \in D$ there is a hyperplane given by

$$H(\zeta) = \{z \in \mathbb{P}^N(\mathbb{C}) | a_0(\zeta)z_0 + \cdots + a_N(\zeta)z_N = 0\}. \quad (1)$$

We call it a moving hyperplane, and write it as $H(\zeta) = \{\langle z, \alpha(\zeta) \rangle = 0\}$, where $\alpha(\zeta) = (a_0(\zeta), \dots, a_N(\zeta))$ for any $\zeta \in D$, and

$$\langle z, \alpha(\zeta) \rangle := a_0(\zeta)z_0 + \cdots + a_N(\zeta)z_N.$$

In particular, if $\alpha(\zeta) = \alpha = (a_0, a_1, \dots, a_N)$ is a fixed non-zero vector in $\mathbb{C}^{N+1}$ then $H = \{\langle z, \alpha \rangle = 0\}$ is said to be a fixed hyperplane.

In this paper, we shall consider only moving hyperplanes $H(\zeta)$ given by (1) for which we can find holomorphic functions $a_j(\zeta)$ on D with $\|H(\zeta)\| := \max_{0 \leq i \leq N} |a_i(\zeta)| = 1 \ \forall \ \zeta \in D$. The corresponding $\alpha(\zeta)$ is said to give a **normalized representation** of the moving hyperplane $H(\zeta)$.

Remark 1. If $\beta(\zeta)$ also gives a normalized representation of $H(\zeta)$, then there exists a uni-modular constant λ , such that $\alpha(\zeta) = \lambda\beta(\zeta)$ on D .

Remark 2. If $\alpha(\zeta) = (a_0(\zeta), \dots, a_N(\zeta))$ is such that for some k , $0 \leq k \leq N$, $|a_k(\zeta)| \geq \max_{\substack{0 \leq j \leq N \\ j \neq k}} |a_j(\zeta)|$, $\forall \zeta \in D$, then the hyperplane defined by (1) shall always have a normalized representation on D given by

$$\alpha^*(\zeta) = \left(\frac{a_0(\zeta)}{a_k(\zeta)}, \dots, \frac{a_{k-1}(\zeta)}{a_k(\zeta)}, 1, \frac{a_{k+1}(\zeta)}{a_k(\zeta)}, \dots, \frac{a_N(\zeta)}{a_k(\zeta)} \right).$$

Remark 3. Any fixed hyperplane always has a normalized representation in D .

Remark. Henceforth, we consider hyperplanes (moving or fixed) only in their normalized representation on any domain D .

Let $H_1(\zeta), H_2(\zeta), \dots, H_{N+1}(\zeta)$ be moving hyperplanes in $\mathbb{P}^N(\mathbb{C})$ represented as $H_j(\zeta) = \{\langle z, \alpha_j(\zeta) \rangle = 0\}$ for $j = 1, \dots, N+1$. Then, by Remark 1, it is clear that

$$\mathcal{D}(H_1(\zeta), \dots, H_{N+1}(\zeta)) := |\det(\alpha_1^t(\zeta), \dots, \alpha_{N+1}^t(\zeta))|$$

depends only on $H_j(\zeta)$ and not on any choice of $\alpha_j(\zeta)$ for normalized representations of such hyperplanes.

Definition 1. Let $H_1(\zeta), \dots, H_q(\zeta)$ with $q \geq N + 1$, be moving hyperplanes in $\mathbb{P}^N(\mathbb{C})$ given in normalized representation respectively by $\alpha_1(\zeta), \dots, \alpha_q(\zeta)$. Define

$$\mathcal{D}(H_1(\zeta), \dots, H_q(\zeta)) := \prod_{1 \leq j_1 < \dots < j_{N+1} \leq q} |\det(\alpha_{j_1}^t(\zeta), \dots, \alpha_{j_{N+1}}^t(\zeta))|.$$

We say the moving hyperplane family $\{H_1(\zeta), \dots, H_q(\zeta)\}$, $q \geq N + 1$, in $\mathbb{P}^N(\mathbb{C})$ is in general position, at a point ζ_0 of D , if $\mathcal{D}(H_1(\zeta_0), \dots, H_q(\zeta_0)) > 0$.

1.2. Reduced Representation. Let D be a domain in $\mathbb{C}$, $f: D \rightarrow \mathbb{P}^N(\mathbb{C})$ be a holomorphic curve, and U be an open set in D . Any holomorphic map $\tilde{f}: U \rightarrow \mathbb{C}^{N+1}$, such that $\pi(\tilde{f}(z)) \equiv f(z)$ in U , is called a *representation* of f on U , where π is the standard projection map.

Definition 2. For an open set U of D , we call $\tilde{f} = (f_0, \dots, f_N)$ a *reduced representation* of f on U if $\tilde{f}$ is a representation of f and $f_0, \dots, f_N$ are holomorphic functions on U without common zeroes.

For any such representation, we put $\|\tilde{f}(z)\| := \left(\sum_{i=0}^N |f_i(z)|^2 \right)^{\frac{1}{2}}$.

Remark 4. Every holomorphic map of D into $\mathbb{P}^N(\mathbb{C})$ has a reduced representation on D (see [14], Section 5).

Remark 5. Let $\tilde{f} = (f_0, \dots, f_N)$ be a reduced representation of f on a domain D . Then, for any arbitrary nowhere-zero holomorphic function h on D , $(hf_0, \dots, hf_N)$ is also a reduced representation of f . Conversely, for any other reduced representation $(g_0, \dots, g_N)$ of f , each g_i can be written as $g_i = hf_i$ for some nowhere-zero holomorphic function h on D .

Definition 3. A function $f: D \rightarrow \mathbb{P}^N(\mathbb{C})$ with the reduced representation $\tilde{f} = (f_0, \dots, f_N)$ is said to intersect a moving hyperplane $H(\zeta)$ given by (1) at a point $z_0 \in D$ if

$$\langle \tilde{f}(z), H(z) \rangle := a_0(z)f_0(z) + \dots + a_N(z)f_N(z) \quad (2)$$

has a zero at z_0 , and we then write $f(z_0) \in H(z_0)$.

If H is a fixed hyperplane given by $\{a_0z_0 + \dots + a_Nz_N = 0\}$, then we say that $f(z_0) \in H$ if $\langle \tilde{f}(z), H \rangle := a_0f_0(z) + \dots + a_Nf_N(z)$ has a zero at z_0 .

If $\langle \tilde{f}(z), H \rangle \equiv 0$ on D , then we say that $f(D) \subset H$, and if $\langle \tilde{f}(z), H \rangle \neq 0 \forall z \in D$, then f is said to omit H on D .

1.3. Sharing of Moving Hyperplanes. With the same notation as in Definition 3, we now define the sharing of moving hyperplanes, which extends the definition of sharing functions.

Definition 4. Suppose f and g are two holomorphic curves from the domain D into $\mathbb{P}^N(\mathbb{C})$ and $H(\zeta)$ be a moving hyperplane in $\mathbb{P}^N(\mathbb{C})$. If there exists some (and thus all) reduced representations $\tilde{f}$ and $\tilde{g}$ of f and g , respectively, such that the holomorphic function $\langle \tilde{f}(z), H(z) \rangle$ shares (or partially shares) zero with $\langle \tilde{g}(z), H(z) \rangle$ on D , then we say that f shares (respectively, partially shares) the moving hyperplane $H(\zeta)$ with g on D .

By Remark 5, $\langle \tilde{f}(z), H(z) \rangle = 0$ is independent of the choice of reduced representation of f . Therefore, the sharing (or partial sharing) of moving hyperplanes is well-defined.

1.4. Normal Families of Holomorphic Curves in Complex Projective Space.

Definition 5. A family $\mathcal{F}$ of holomorphic curves from a domain D into $\mathbb{P}^N(\mathbb{C})$ is said to be normal in D if any sequence in $\mathcal{F}$ contains a subsequence that converges uniformly, with respect to the Fubini-Study metric, on compact subsets of D to a holomorphic curve from D into $\mathbb{P}^N(\mathbb{C})$; and $\mathcal{F}$ is said to be normal at a point a in D if $\mathcal{F}$ is normal on some neighbourhood of a in D .

Using the Fubini-Study metric on $\mathbb{P}^N(\mathbb{C})$, we see that a sequence $\{f_n\}_{n=1}^\infty$ of holomorphic maps of D into $\mathbb{P}^N(\mathbb{C})$ converges uniformly on compact subsets of D to a holomorphic map f if and only if each f_n for any $a \in D$ has a reduced representation $\tilde{f}_n = (f_{n0}, \dots, f_{nN})$ on some fixed neighbourhood U of a in D , such that $\{f_{ni}\}_{n=1}^\infty$ converges uniformly on U to a holomorphic function f_i (for $i = 0, 1, \dots, N$) on U with the property that $\tilde{f} = (f_0, f_1, \dots, f_N)$ is a reduced representation of f on U .

1.5. Extension of Derivative. In order to extend Theorem 2 to a family of holomorphic curves, a suitable notion of derivative was provided by Ye et al [27] as follows:

Let $f = [f_0 : \dots : f_N]$ be a holomorphic map of D into $\mathbb{P}^N(\mathbb{C})$, $\mu \in \{0, 1, \dots, N\}$ with $f_\mu \not\equiv 0$, and $d(z)$ be a holomorphic function in D , such that f_μ^2/d and $W(f_\mu, f_i)/d$, ($i = 0, 1, \dots, N; i \neq \mu$) are holomorphic functions in D without common zeroes. Here, as usual, $W(f_\mu, f_i)$ denotes the Wronskian of f_μ and f_i .

Definition 6. [27] The holomorphic map induced by the map

$$(W(f_\mu, f_0), \dots, W(f_\mu, f_{\mu-1}), f_\mu^2, W(f_\mu, f_{\mu+1}), \dots, W(f_\mu, f_N)) : D \rightarrow \mathbb{C}^{N+1}$$

is called the μ -th derived holomorphic map of f and we write

$$\nabla_\mu f = [W(f_\mu, f_0)/d, \dots, W(f_\mu, f_{\mu-1})/d, f_\mu^2/d, W(f_\mu, f_{\mu+1})/d, \dots, W(f_\mu, f_N)/d].$$

For simplicity, we will write $\nabla_0 f$ as ∇f .

Remark 6. The definition of $\nabla_\mu f$ does not depend on the choice of a reduced representation of f .

Remark 7. A meromorphic function $f = \frac{f_1(z)}{f_0(z)}$ on D can be regarded as a holomorphic map from D to $\mathbb{P}^1(\mathbb{C})$ given by $f(z) = [f_0(z) : f_1(z)]$ and ∇f is exactly the ordinary derivative of $f(z)$.

In 2015, Ye et al obtained the following result, which led them to an extension of the Schwick's theorem:

Theorem 4. [27] Let $\mathcal{F}$ be a family of holomorphic maps of a domain $D \subseteq \mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$, $H_1, \dots, H_{2N+1}$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position, and δ be a real number with $0 < \delta < 1$. Suppose that for each $f \in \mathcal{F}$ the following conditions are satisfied:

- (i) $\nabla f(z)$ partially shares H_j with $f(z)$ on D , for $j = 1, \dots, 2N + 1$.
- (ii) If $f(z) \in \bigcup_{j=1}^{2N+1} H_j$, then $\frac{|f_0(z)|}{\|f(z)\|} \geq \delta$.
- (iii) If $f(z) \in \bigcup_{j=1}^{2N+1} H_j$, then $\frac{|\langle \nabla f(z), H_k \rangle|}{|f_0(z)|^2} \leq \frac{1}{\delta}$ for $k = 1, \dots, 2N + 1$.

Then $\mathcal{F}$ is normal in D .

2. Main Results. We generalize Theorem 4 to the case where we can take moving hyperplanes that can vary for different functions of the family $\mathcal{F}$.

Theorem 5. Let $\mathcal{F}$ be a family of holomorphic maps of a domain $D \subseteq \mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$ and δ be a real number with $0 < \delta < 1$. Suppose that for each $f \in \mathcal{F}$ and $z \in D$ there exist $2N + 1$ moving hyperplanes $H_1^f(z), \dots, H_{2N+1}^f(z)$ in $\mathbb{P}^N(\mathbb{C})$ in general position with

$$\mathcal{D}(H_1^f(z), \dots, H_{2N+1}^f(z)) \geq \delta, \quad \forall f \in \mathcal{F}, z \in D,$$

such that the following conditions are satisfied:

- (i) $\nabla f(z)$ partially shares $H_j^f(z)$ with $f(z)$ on D for $j = 1, \dots, 2N + 1$.
- (ii) If $f(z) \in \bigcup_{j=1}^{2N+1} H_j^f(z)$, then $\frac{|f_0(z)|}{\|f(z)\|} \geq \delta$.
- (iii) If $f(z) \in \bigcup_{j=1}^{2N+1} H_j^f(z)$, then $\frac{|\langle \nabla f(z), H_k^f(z) \rangle|}{|f_0(z)|^2} \leq \frac{1}{\delta}$, $k = 1, \dots, 2N + 1$.

Then $\mathcal{F}$ is normal in D .

In particular, when f and ∇f share these $2N + 1$ moving hyperplanes, then one can see (as in [27, Theorem 2.3]) that the condition (iii) of Theorem 5 is satisfied and we obtain the following result:

Corollary 1. *Let $\mathcal{F}$ be a family of holomorphic maps of a domain $D \subseteq \mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$ and δ be a real number with $0 < \delta < 1$. Suppose that for each $f \in \mathcal{F}$ and $z \in D$ there exist $2N + 1$ moving hyperplanes $H_1^f(z), \dots, H_{2N+1}^f(z)$ in $\mathbb{P}^N(\mathbb{C})$ in general position with*

$$D(H_1^f(z), \dots, H_{2N+1}^f(z)) \geq \delta, \quad \forall f \in \mathcal{F}, z \in D,$$

such that the following conditions are satisfied:

- (i) $f, \nabla f$ share the moving hyperplanes $H_j^f(z)$ on D for $j = 1, \dots, 2N+1$;
- (ii) if $f(z) \in \bigcup_{j=1}^{2N+1} H_j^f(z)$, then $\frac{|f_0(z)|}{\|f(z)\|} \geq \delta$.

Then $\mathcal{F}$ is normal in D .

Recently, Datt et al [11, Theorem 1.1] have obtained a result similar to Corollary 1 but with an additional assumption on the normality of the family of holomorphic functions $\widetilde{H}_j^f$ induced by the moving hyperplanes $H_j^f(z)$. Inequality (3) of the assumption of [11, Theorem 1.1] clearly requires a sort of normalized representation for such moving hyperplanes, and as can be seen in our proof of Theorem 5, the additional assumption on normality of $\{\widetilde{H}_j^f\}$ is not necessary.

Now, considering the case $N = 1$, we can see that each of the moving hyperplane $H_j^f(z)$, $j = 1, 2$ and 3 is a function from the domain D into $\mathbb{P}^1(\mathbb{C})$ and we obtain the following normality criterion regarding partial sharing of functions:

Theorem 6. Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subseteq \mathbb{C}$ and ε, M be two positive real numbers. Suppose for each $f \in \mathcal{F}$ there exist three holomorphic functions a_1^f, a_2^f , and a_3^f defined on D , such that the following conditions are satisfied:

- (i) $|a_j^f(z)| \leq M$ and $|a_j^f(z) - a_k^f(z)| \geq \varepsilon \forall z \in D$ and $j, k \in \{1, 2, 3\}, j \neq k$;
- (ii) $f'(z)$ partially shares $a_j^f(z)$ with $f(z)$ on D , for $j \in \{1, 2, 3\}$;
- (iii) for any $z \in D$ if $f(z) = a_j^f(z)$ then $|f'(z)| \leq M$ for $j \in \{1, 2, 3\}$.

Then $\mathcal{F}$ is normal in D .

The following examples show that each of the condition in Theorem 6 is essential. We denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$.

Example 1. [Condition (i) is essential] Consider the family $\{f_n(z) = e^{nz} : n \in \mathbb{N}\}$, which is not normal in $\mathbb{D}$. Take $a_j^{f_n}(z) = (j+1)ne^{nz}$ for $j \in \{1, 2, 3\}$. Note that $f'_n(z) = ne^{nz} \neq a_j^{f_n}(z)$ for any $z \in \mathbb{D}$ and $j \in \{1, 2, 3\}$. Also, $f_n(z) \neq a_j^{f_n}(z)$ for any $j \in \{1, 2, 3\}$.

Example 2. [Condition (ii) is essential] Consider the family

$$\left\{ f_n(z) = \frac{1}{\sin nz} \left(\frac{1}{2} + \frac{z}{8n} \right) : n \in \mathbb{N} \right\}$$

of meromorphic functions in $\mathbb{D}$, which is not normal. Take

$$a_1^{f_n}(z) = \frac{1}{2} + \frac{z}{8n}, \quad a_2^{f_n}(z) = -\frac{1}{2} - \frac{z}{8n} \quad \text{and} \quad a_3^{f_n}(z) \equiv 0,$$

which are holomorphic functions on $\mathbb{D}$ satisfying condition (i) of Theorem 6 with $M = 1$ and $\varepsilon = \frac{3}{8}$. We see that

$$f'_n(z) = \frac{-n \cos nz}{\sin^2 nz} \left(\frac{1}{2} + \frac{z}{8n} \right) + \frac{1}{\sin nz} \left(\frac{1}{8n} \right).$$

Note that $f_n(z) = a_1^{f_n}(z)$ implies $\sin nz = 1$ and, consequently, $\cos nz = 0$. Then $|f'_n(z)|$ is clearly bounded by 1 for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Similarly, we can see that $f_n(z) = a_2^{f_n}(z)$ implies $\sin nz = -1$, which leads to $\cos nz = 0$ and, again, $|f'_n(z)|$ is bounded by 1. Also it is clear that $f_n(z)$ omits $a_3^{f_n}(z)$ on $\mathbb{D}$. Thus condition (iii) of Theorem 6 is also satisfied.

Example 3. [Condition (iii) is essential] We consider the family $\{f_n(z) = nz : n \in \mathbb{N}\}$, which is not normal in $\mathbb{D}$. Taking $a_j^{f_n}(z) \equiv \frac{1}{j+1}$,

$j \in \{1, 2, 3\}$, on $\mathbb{D}$, we can see that condition (i) of Theorem 6 is satisfied with $\varepsilon = \frac{1}{12}$ and $M = 1$ whereas condition (ii) is vacuously true.

In case $f(z)$ and $f'(z)$ share the functions $a_j^f(z)$, it is easy to see that the condition (ii) of Theorem 6 is fulfilled and so we have the following Corollary, which is a weaker form of Theorem 3:

Corollary 2. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subseteq \mathbb{C}$. Suppose that for each $f \in \mathcal{F}$ there exist three holomorphic functions a_1^f, a_2^f and a_3^f defined on D and some positive real numbers ε and M , with*

$$|a_j^f(z)| \leq M \text{ and } |a_j^f(z) - a_k^f(z)| \geq \varepsilon \quad \forall z \in D \text{ and } j, k \in \{1, 2, 3\} \text{ with } j \neq k,$$

such that f and f' share the functions a_1^f, a_2^f and a_3^f on D . Then $\mathcal{F}$ is normal in D .

2.1. A Representation of Holomorphic Functions as Curves in $\mathbb{P}^2(\mathbb{C})$. Let $f(z)$ be a holomorphic function on a domain $D \subset \mathbb{C}$. Corresponding to this function, we define a function $F: D \rightarrow \mathbb{P}^2(\mathbb{C})$, whose representation $\tilde{F}: D \rightarrow \mathbb{C}^3 \setminus \{0\}$ is given by

$$\tilde{F}(z) = (1, z, f(z)). \quad (3)$$

We regard it as the *lifting* of the complex-valued holomorphic function f to a holomorphic curve F into $\mathbb{P}^2(\mathbb{C})$. Note that the representation (3) is always a reduced representation with $\tilde{F}_0 \neq 0$ everywhere on D . The derivative of F (or, more precisely, the representation of the derivative of F) is given by

$$\nabla F = (1, 1, f'(z)), \quad (4)$$

which again happens to be in reduced representation for the whole domain D .

It is worthwhile to mention that taking f to be a meromorphic function in representation (3) would have forced us to use two different holomorphic functions and we would not have been able to obtain such a convenient form of derivative as in equation (4).

We show that the normality of a family of complex-valued holomorphic functions is equivalent to the normality of family of corresponding holomorphic curves lifted to $\mathbb{P}^2(\mathbb{C})$.

Theorem 7. Let D be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of complex-valued holomorphic functions on D . Then the family

$$\mathcal{F}^* = \{F: D \rightarrow \mathbb{P}^2(\mathbb{C}) \text{ with } \tilde{F}(z) = (1, z, f(z)): f \in \mathcal{F}\}$$

is normal in D if and only if $\mathcal{F}$ is normal in D .

As a consequence, we obtain the following normality criterion for a family of holomorphic curves that requires, among other conditions, a boundedness condition on derivatives at fixed points of the respective functions.

Theorem 8. Let $\mathcal{F}$ be a family of complex-valued holomorphic functions on a domain $D \subseteq \mathbb{C}$ and M, ε be two positive numbers. Suppose for each $f \in \mathcal{F}$ there exist two holomorphic functions $a_1^f(z)$ and $a_2^f(z)$ defined on D with $|a_j^f(z)| \leq 1, \forall z \in D, j = 1, 2$ and $|a_1^f(z) - a_2^f(z)| \geq \varepsilon \forall z \in D$, and the following conditions are satisfied:

- (i) $f'(z) = a_j^f(z)$ implies $f(z) = a_j^f(z)$, $f(z) = a_j^f(z)$ implies $|f'(z)| \leq M$ for $j \in \{1, 2\}$;
- (ii) z is a fixed point of f wherever f' assumes 1 in D ;
- (iii) $|f'(z)| \leq M$, whenever z is a fixed point of f .

Then $\mathcal{F}$ is a normal family.

The following example shows that condition (i) is essential:

Example 4. Let $\mathcal{F} = \{f_n(z) = e^{nz} + z: n \in \mathbb{N}\}$ be defined on $\mathbb{C}$. Here each $f'_n(z) = ne^{nz} + 1 \neq 1$ for any $z \in \mathbb{C}$. Also, it is clear that no member of this family can have any fixed point. But $\mathcal{F}$ is not a normal family.

Theorem 8 enables us to formulate a counterexample to **the converse of Bloch's principle**. Bloch's principle states that a family $\mathcal{F}$ of functions holomorphic in a domain D and possessing a certain property P is likely to be normal if there is no non-constant entire function possessing the property P (see [3], [6], [18]).

Counterexample. Let $\mathcal{F}$ be a family of holomorphic functions on a domain D and let P be the property that for each $f \in \mathcal{F}$ there exist two holomorphic functions $a_1^f(z)$ and $a_2^f(z)$ defined on D with $|a_j^f(z)| \leq 1, \forall z \in D, j = 1, 2$ and $|a_1^f(z) - a_2^f(z)| \geq \varepsilon \forall z \in D$, and the following conditions are satisfied:

- 1) $f'(z) = a_j^f(z)$ implies $f(z) = a_j^f(z)$, $f(z) = a_j^f(z)$ implies $|f'(z)| \leq M$ for $j \in \{1, 2\}$.

- 2) z is a fixed point of f wherever f' assumes 1 in D .
- 3) $|f'(z)| \leq M$, whenever z is a fixed point of f .

Then, by Theorem 8, $\mathcal{F}$ is normal on D . But the function $f(z) = 2z$ $\forall z \in \mathbb{C}$ is a non-constant entire function and possesses the property P when $a_1^f(z) \equiv -\frac{1}{2}$, $a_2^f = -\frac{1}{2}$, $\varepsilon = 1$, and $M = 2$. This shows that the converse of Bloch's principle fails to hold.

Remark 8. We have not been able to show that the conditions (ii) and (iii) are essential in Theorem 8 and leave this question open for further investigation.

3. Proofs of Main Results. To prove our main results, we require the following lemmas:

Lemma 1. [1] Let $\mathcal{F}$ be a family of holomorphic maps of a domain D in $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$. The family $\mathcal{F}$ is not normal on D if and only if there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{z_n\} \subset D$ with $z_n \rightarrow z_0 \in D$, $\{\rho_n\}$ with $\rho_n > 0$, and $\rho_n \rightarrow 0$, such that $g_n(\xi) := f_n(z_n + \rho_n \xi)$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map g of $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$.

Lemma 2. [22, p.141] Let $f: \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ be a holomorphic map, and $H_1, \dots, H_q$ ($q \geq 2N + 1$) be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. If for each $j = 1, \dots, q$, either $f(\mathbb{C})$ is contained in H_j , or $f(\mathbb{C})$ omits H_j , then f is constant.

Proof of Theorem 5. Suppose $\mathcal{F}$ is not normal on D . Then, by Lemma 1, there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{z_n\} \subset D$ with $z_n \rightarrow z_0 \in D$, $\{\rho_n\}$ with $\rho_n > 0$, and $\rho_n \rightarrow 0$, such that $g_n(\xi) := f_n(z_n + \rho_n \xi)$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map $g: \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$. Corresponding to each such f_n , let $H_j^{f_n}(z)$, $j = 1, 2, \dots, 2N + 1$, be the moving hyperplanes given by the assumption of the theorem. Let each $H_j^{f_n}(z)$ be written in normalized representation as

$$H_j^{f_n}(z) = \{\langle \zeta, \alpha_j^{f_n}(z) \rangle = 0\},$$

where $\alpha_j^{f_n}(z) = (\alpha_{j,0}^{f_n}(z), \alpha_{j,1}^{f_n}(z), \dots, \alpha_{j,N}^{f_n}(z))$ is a non-zero vector in $\mathbb{C}^{N+1}$ for any $z \in D$ with $\max_{k=0}^N |\alpha_{j,k}^{f_n}(z)| = 1$.

From compactness of $A = \{(\alpha_0, \dots, \alpha_n) \in \mathbb{C}^{N+1} : \max_{k=0}^N |\alpha_k| = 1\}$ it follows that for any $j \in \{1, 2, \dots, 2N + 1\}$ there exist a subsequence of $\alpha_j^{f_n}(z)$, which we again call $\alpha_j^{f_n}(z)$, that converges to $\alpha_j(z)$ locally uniformly on

D with $\alpha_j(z) \in A$ for any $z \in D$. Let $H_j(z)$ be the moving hyperplane defined by $\alpha_j(z)$, $j = 1, 2, \dots, 2N + 1$. By continuity of the determinant function, it is clear that for any $z \in D$

$$\mathcal{D}(H_1(z), \dots, H_{2N+1}(z)) \geq \inf\{\mathcal{D}(H_1^{f_n}(z), \dots, H_{2N+1}^{f_n}(z)) : n \in \mathbb{N}\} \geq \delta > 0$$

and, so, hyperplanes $H_1(z), \dots, H_{2N+1}(z)$ are in general position for any $z \in D$ and in particular at z_0 .

Now, we proceed along the line of [27, Proof of Theorem 2.1], but with due consideration for the hyperplanes that are moving on D as well wandering for each $f \in \mathcal{F}$. As $g: \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ is a non-constant function, by Lemma 2 we may assume that $\langle g(\xi), H_1(z_0) \rangle$ does not vanish identically and has at least one zero in $\mathbb{C}$. Let ξ_0 be such a zero. That is,

$$\langle g(\xi_0), H_1(z_0) \rangle = 0 \quad (5)$$

and there exists a small neighbourhood U of ξ_0 , such that $\langle g(\xi), H_1(z_0) \rangle$ has no other zero in U . Moreover, each g_n has a reduced representation

$$\tilde{g}_n(\xi) = (g_{n,0}(\xi), \dots, g_{n,N}(\xi)) = (f_{n,0}(z_n + \rho_n \xi), \dots, f_{n,N}(z_n + \rho_n \xi))$$

on U , such that for each $i = 0, 1, \dots, N$ the sequence $\{g_{n,i} : n \in \mathbb{N}\}$ converges uniformly on U to a holomorphic function g_i with $\tilde{g} = (g_0, \dots, g_N)$ as the reduced representation of g on U . Shrinking the neighbourhood U if necessary, we can see that $\sum_{i=0}^N \alpha_{1,i}^{f_n}(z_n + \rho_n \xi) g_{n,i}(\xi)$ converges uniformly on

U to $\sum_{i=0}^N \alpha_{1,i}(z_0) g_i(\xi)$. Note that from equation (5), $\sum_{i=0}^N \alpha_{1,i}(z_0) g_i(\xi_0) = 0$. So, by Hurwitz's theorem, there exists a sequence ξ_n tending to ξ_0 , such that, for large n , $\sum_{i=0}^N \alpha_{1,i}^{f_n}(z_n + \rho_n \xi_n) g_{n,i}(\xi_n) = 0$, that is,

$$\sum_{i=0}^N \alpha_{1,i}^{f_n}(z_n + \rho_n \xi_n) f_{n,i}(z_n + \rho_n \xi_n) = 0, \quad (6)$$

which implies $f_n(z_n + \rho_n \xi_n) \in H_1^{f_n}(z_n + \rho_n \xi_n)$; by condition (ii) of the assumption, we have

$$|f_{n,0}(z_n + \rho_n \xi_n)| \geq \delta \|\tilde{f}_n(z_n + \rho_n \xi_n)\|.$$

That is, $|g_{n,0}(\xi_n)| \geq \delta \|\tilde{g}_n(\xi_n)\|$. Taking $n \rightarrow \infty$ we get

$$|g_0(\xi_0)| \geq \delta \|\tilde{g}(\xi_0)\|0.$$

Thus, $g_0(\xi_0) \neq 0$ and further shrinking U , if necessary, we may assume that $g_0(\xi) \neq 0 \forall \xi \in U$. This implies $g_{n,0}(\xi) \neq 0$ for $\xi \in U$ when n is sufficiently large. For any such n ,

$$(g_{n,0}^2(\xi), W(g_{n,0}, g_{n,1})(\xi), \dots, W(g_{n,0}, g_{n,N})(\xi))$$

is then a reduced representation of $\nabla g_n(\xi)$ on U . Also note that for any $i = 1, 2, \dots, N$ and $\xi \in U$ the Wronskian

$$W(g_{n,0}, g_{n,i})(\xi) = \rho_n W(f_{n,0}, f_{n,i})(z_n + \rho_n \xi_n). \quad (7)$$

From (6) and condition (iii) of the assumption, we have, for any $k = 2, 3, \dots, 2N + 1$,

$$\frac{|\langle (f_{n,0}^2, W(f_{n,0}, f_{n,1}), \dots, W(f_{n,0}, f_{n,N})), (\alpha_{k,0}^{f_n}, \dots, \alpha_{k,N}^{f_n}) \rangle|}{|f_{n,0}^2|} \leq \frac{1}{\delta},$$

where the left-hand side of the above inequality is evaluated at $(z_n + \rho_n \xi_n)$.

$$\implies \left| \alpha_{k,0}^{f_n}(z_n + \rho_n \xi_n) + \sum_{i=1}^N \frac{\alpha_{k,i}^{f_n}(z_n + \rho_n \xi_n) W(f_{n,0}, f_{n,i})(z_n + \rho_n \xi_n)}{f_{n,0}^2(z_n + \rho_n \xi_n)} \right| \leq \frac{1}{\delta}.$$

Using (7), we get

$$\begin{aligned} & \left| \alpha_{k,0}^{f_n}(z_n + \rho_n \xi_n) + \sum_{i=1}^N \frac{\alpha_{k,i}^{f_n}(z_n + \rho_n \xi_n) W(g_{n,0}, g_{n,i})(\xi_n)}{\rho_n g_{n,0}^2(\xi_n)} \right| \leq \frac{1}{\delta}. \\ \implies & \left| \rho_n \alpha_{k,0}^{f_n}(z_n + \rho_n \xi_n) + \sum_{i=1}^N \frac{\alpha_{k,i}^{f_n}(z_n + \rho_n \xi_n) W(g_{n,0}, g_{n,i})(\xi_n)}{g_{n,0}^2(\xi_n)} \right| \leq \frac{\rho_n}{\delta}. \end{aligned} \quad (8)$$

Define for any $k = 2, 3, \dots, 2N + 1$

$$\phi_k(\xi) = \sum_{i=1}^N \frac{\alpha_{k,i}(z_0) W(g_0, g_i)(\xi)}{g_0^2(\xi)}, \quad \xi \in U.$$

Taking $n \rightarrow \infty$ in equation(8), we obtain

$$\phi_k(\xi_0) = 0 \text{ for } k = 2, 3, \dots, 2N + 1. \quad (9)$$

Now we claim that there are at most N hyperplanes in $\{H_k: k = 2, 3, \dots, 2N + 1\}$, such that $\phi_k(\xi) \equiv 0$ on U . Suppose, on the contrary, that $\phi_2(\xi) \equiv \dots \phi_{N+2}(\xi) \equiv 0$ on U . Since

$$\phi_j(\xi) = \left(\sum_{i=1}^N \alpha_{j,i}(z_0) \frac{g_i}{g_0} \right)' \equiv 0,$$

there exist complex numbers c_j , such that $\sum_{i=1}^N \alpha_{j,i}(z_0) \frac{g_i}{g_0} \equiv c_j$ on U for $j = 2, \dots, N + 2$.

Since $H_1(z_0), \dots, H_{2N+1}(z_0)$ are in general position, the system of equations

$$\sum_{i=1}^N \alpha_{j,i}(z_0) x_i \equiv c_j \text{ for } j = 2, \dots, N + 2$$

has no solution or the solution is unique. Thus $\tilde{g}$ is uniquely determined by $g_0 \neq 0$ in U . Hence, g is constant on U , which is a contradiction. So the claim is true. By the validity of our claim, we can suppose without loss of generality that $\phi_k(\xi) \not\equiv 0$ on U for $k = 2, \dots, N + 1$. Also, for each $k \in \{2, \dots, N + 1\}$,

$$\rho_n \alpha_{k,0}^{f_n}(z_n + \rho_n \xi) + \rho_n \sum_{i=1}^N \frac{\alpha_{k,i}^{f_n}(z_n + \rho_n \xi) W(f_{n,0}, f_{n,i})(z_n + \rho_n \xi)}{f_{n,0}^2(z_n + \rho_n \xi)}$$

converges uniformly to $\phi_k(\xi)$ on U . Again, by Hurwitz's theorem and equation (9), there exists a sequence $\zeta_n \rightarrow \xi_0$, such that

$$\langle \nabla f_n(z_n + \rho_n \zeta_n), H_k^{f_n}(z_n + \rho_n \zeta_n) \rangle = 0.$$

Then, by condition (i) of the assumption,

$$\langle f_n(z_n + \rho_n \zeta_n), H_k^{f_n}(z_n + \rho_n \zeta_n) \rangle = 0.$$

Taking $n \rightarrow \infty$, we get

$$\langle g(\xi_0), H_k(z_0) \rangle = 0 \text{ for } k = 2, \dots, N + 1.$$

Noting equation (5), we have

$$\langle g(\xi_0), H_k(z_0) \rangle = 0 \text{ for } k = 1, 2, \dots, N + 1.$$

This contradicts the fact that the hyperplanes $\{H_k(z_0): k = 1, \dots, N+1\}$ are in general position. Hence $\mathcal{F}$ is normal on D . $\square$

Proof of Theorem 6. We can assume that $M = 1$, for otherwise we may replace the functions $a_j^f(z)$ by $\frac{a_j^f(z)}{M}$, $j \in \{1, 2, 3\}$, and ε by $\frac{\varepsilon}{M}$, and consider the normality of the family $\mathcal{F}' = \left\{ \frac{f}{M} : f \in \mathcal{F} \right\}$ which is equivalent to the normality of $\mathcal{F}$.

Now, for any $f \in \mathcal{F}, z \in D, j \in \{1, 2, 3\}$, let $H_j^f(z)$ be the hyperplane defined by $\langle z, \alpha_j^f(z) \rangle = 0$, where $\alpha_j^f(z) = (a_j^f(z), -1) \in \mathbb{C}^2$ gives a normalized representation for $H_j^f(z)$. Writing the meromorphic function $f(z) = \frac{f_1(z)}{f_0(z)}$ as $f(z) = [f_0(z), f_1(z)]$, we can see that for any $z_0 \in D$, $f(z_0) = a_j^f(z_0)$ if and only if $f(z_0) \in H_j^f(z_0)$, and $f'(z_0) = a_j^f(z_0)$ if and only if $\nabla f(z_0) \in H_j^f(z_0)$. It can easily be computed that

$$\mathcal{D}(H_1^f(z), H_2^f(z), H_3^f(z)) \geq \varepsilon^3 \quad \forall f \in \mathcal{F} \text{ and } z \in D. \quad (10)$$

Further, $f(z) \in \bigcup_{j=1}^3 H_j^f(z)$ implies $f(z) = a_j^f(z)$ for some $j \in \{1, 2, 3\}$ and we have

$$\frac{|f_0(z)|}{\|f(z)\|} = \frac{1}{\sqrt{1 + \left| \frac{f_1(z)}{f_0(z)} \right|^2}} \geq \frac{1}{\sqrt{2}} \quad (11)$$

and

$$\frac{|\langle \nabla f(z), H_k^f(z) \rangle|}{|f_0(z)|^2} = |a_k^f(z) - f'(z)| \leq 2. \quad (12)$$

Taking $\delta = \min\{\varepsilon^3, \frac{1}{2}\}$, we see from equations (10), (11), and (12) that $\mathcal{F}$ satisfies all the conditions of the Theorem 5. Hence $\mathcal{F}$ is normal in D . $\square$

Proof of Theorem 7. To prove this theorem, it is enough to show that a sequence $\{f_n\} \subset \mathcal{F}$ converges locally uniformly on D to a holomorphic function f on D or converges locally uniformly to ∞ if and only if the sequence $\{F_n\}$ with $\tilde{F}_n = (1, z, f_n(z))$ converges locally uniformly on D to a holomorphic function $F: D \rightarrow \mathbb{P}^2(\mathbb{C})$.

First suppose that $\{f_n\} \subset \mathcal{F}$ converges locally uniformly on D to a holomorphic function f ; then, clearly, F_n converges locally uniformly to F , whose reduced representation is $\tilde{F} = (1, z, f(z))$. On the other hand,

if f_n converges locally uniformly to ∞ , then we note that for any $z_0 \in D$, such a neighborhood U exists, that f_n converges uniformly to ∞ on U and for sufficiently large n , $f_n(z) \neq 0 \forall z \in U$. On such a neighborhood U , we can see that

$$\left(\frac{1}{f_n(z)}, \frac{z}{f_n(z)}, 1 \right)$$

is again a reduced representation of F_n that converges uniformly on U to the function $F: U \rightarrow \mathbb{P}^2(\mathbb{C})$ with the reduced representation $\tilde{F}(z) \equiv (0, 0, 1)$.

Conversely, suppose $F_n \rightarrow F$. Let $\zeta_0 \in D$. Then there exists a neighborhood U of ζ_0 and a reduced representation $(\alpha_n(z), \beta_n(z), \gamma_n(z))$ of $F_n(z)$ in U , such that $\alpha_n \rightarrow \alpha$, $\beta_n \rightarrow \beta$ and $\gamma_n \rightarrow \gamma$ uniformly on U , where $(\alpha(z), \beta(z), \gamma(z))$ is a reduced representation of $F(z)$ in U . As $\tilde{F}_n(z) = (1, z, f_n(z))$ is also a reduced representation of $F_n(z)$, there are nowhere-zero holomorphic functions $h_n(z)$ on U , such that for all $n \in \mathbb{N}$:

$$\alpha_n(z) = 1 \cdot h_n(z), \quad (13)$$

$$\beta_n(z) = z \cdot h_n(z), \quad (14)$$

$$\gamma_n(z) = f_n(z) h_n(z). \quad (15)$$

Claim: $\alpha(z)$ is either identically zero or nowhere-zero on U .

Suppose, on the contrary, that $\alpha(z) \not\equiv 0$ and $\alpha(z_0) = 0$ for some $z_0 \in U$. Then, by Hurwitz's theorem, there exists a sequence $\{z_n\} \subset U$ converging to z_0 , such that $\alpha_n(z_n) = 0$ for sufficiently large n . Using this in equation (13), we obtain $h_n(z_n) = 0$, which is a contradiction. This proves our claim.

Now we consider both possible cases:

Case 1: $\alpha(z) \neq 0$ for all $z \in U$.

Shrinking U , if necessary, to a compact neighbourhood of ζ_0 , we can see that there exists $\delta > 0$, such that $|\alpha(z)| \geq \delta, \forall z \in U$. Then, due to uniform convergence of $\alpha_n(z)$ to $\alpha(z)$ in U , there exists $n_0 \in \mathbb{N}$, such that

$$\alpha_n(z) \neq 0 \quad \forall z \in U$$

for $n \geq n_0$.

Using equation (15), we can write for $z \in U$ and $n \geq n_0$:

$$f_n(z) = \frac{\gamma_n(z) \alpha_n(z)}{\alpha_n(z) h_n(z)} \rightarrow \frac{\gamma(z)}{\alpha(z)}.$$

Thus $f_n(z)$ converges uniformly to the holomorphic function $\frac{\gamma(z)}{\alpha(z)}$ on U .

Case 2: $\alpha(z) \equiv 0$ on U .

By equation (13), we have $h_n(z) \rightarrow 0$ uniformly on U . With equation (14), this implies that $\beta_n(z) \rightarrow 0$ uniformly on U . That is, $\beta(z) \equiv 0$ on U . Therefore, $\gamma(z) \neq 0 \forall z \in U$, so that $(\alpha(z), \beta(z), \gamma(z))$ is a reduced representation of $F(z)$ on U . As

$$f_n(z)h_n(z) = \gamma_n(z) \rightarrow \gamma(z) \text{ and } h_n(z) \rightarrow 0$$

uniformly on U , it follows that $f_n \rightarrow \infty$ uniformly on U . $\square$

Proof of Theorem 8. Since normality is a local property, we can take $D = \mathbb{D}$. We will show that the family $\mathcal{F}^*$ as defined in Theorem 7 is normal in $\mathbb{D}$. For any $f \in \mathcal{F}$, we make the following choice of 5 hyperplanes:

$$H_j^f(z) = \{w \in \mathbb{P}^2(\mathbb{C}) \mid \langle w, \alpha_j^f(z) \rangle = 0\}, \quad j = 1, \dots, 5,$$

given in normalized representation by

$$\alpha_1^f(z) = (a_1^f(z), 0, -1),$$

$$\alpha_2^f(z) = (a_2^f(z), 0, -1),$$

$$\alpha_3^f(z) = (0, 1, -1),$$

$$\alpha_4^f(z) = \left(1, \frac{-1}{2}, 0\right),$$

$$\alpha_5^f(z) = \left(1, \frac{-1}{3}, 0\right).$$

We can compute that for any $f \in \mathcal{F}$ and $z \in \mathbb{D}$:

$$\begin{aligned} & \mathcal{D} \left(H_1^f(z), H_2^f(z), H_3^f(z), H_4^f(z), H_5^f(z) \right) \\ &= \frac{1}{6^4} \left| \frac{a_1^f(z)}{2} - 1 \right| \left| \frac{a_2^f(z)}{2} - 1 \right| \left| \frac{a_1^f(z)}{3} - 1 \right| \left| \frac{a_2^f(z)}{3} - 1 \right| \left| a_2^f(z) - a_1^f(z) \right|^3 \\ &\geq \frac{1}{6^4} \left(\frac{1}{2} \right)^2 \left(\frac{2}{3} \right)^2 \varepsilon^3 = \delta^*(\text{say}) > 0. \end{aligned}$$

If we take $\delta = \min \left\{ \delta^*, \frac{1}{\sqrt{3}}, \frac{1}{1+M} \right\}$, then it can be seen that $\mathcal{F}^*$ satisfies all the conditions of Theorem 5. Hence, $\mathcal{F}^*$ is normal, and, so, by Theorem 7, $\mathcal{F}$ is also normal. $\square$

Acknowledgement. We thank the anonymous reviewers for their comments.

References

- [1] Aladro G., Krantz S.G. *A criterion for normality in $\mathbb{C}^n$* . J. Math. Appl., 1991, vol.161, pp. 1–8.
DOI: [https://doi.org/10.1016/0022-247X\(91\)90356-5](https://doi.org/10.1016/0022-247X(91)90356-5)
- [2] Beardon A.F., Minda D. *Normal families: a geometric perspective*. Comput. Methods Funct. Theory, 2014, vol. 14, pp. 331–355.
DOI: <https://doi.org/10.1007/s40315-014-0054-2>
- [3] Bergweiler W. *Bloch’s principle*. Comput. Methods Funct. Theory, 2006, vol. 6, pp. 77–108. DOI: <https://doi.org/10.1007/BF03321119>
- [4] Bharti N. *A normal criterion concerning sequence of functions and their differential polynomials*. Probl. Anal. Issues Anal., 2024, vol. 13(31), no. 2, pp. 3–24. DOI: <https://doi.org/10.15393/j3.art.2024.15810>
- [5] Charak K.S., Ronchon D. Sharma N. *Normal families of Bicomplex holomorphic functions*. Fractals, 2009, vol. 17, no.2, pp. 257–268.
DOI: <https://doi.org/10.1142/S0218348X09004314>
- [6] Charak K.S., Singh V. *Two normality criteria and counterexamples to the converse of Bloch’s principle*. Kodai Mathematical Journal, 2015, vol. 38, no. 3, pp. 672–686. DOI: <https://doi.org/10.2996/kmj/1446210601>
- [7] Charak K.S., Sharma S. *Some normality criteria and a counterexample to the converse of Bloch’s principle*. Bull. Aust. Math. Soc., 2017, vol. 95, pp. 238–249. DOI: <https://doi.org/10.1017/S0004972716001015>
- [8] Charak K.S., Sharma S. *Value distribution and normality of meromorphic functions involving partial sharing of small functions*. Turkish Journal of Mathematics, 2017, vol. 41, no. 2, pp. 404–411.
DOI: <https://doi.org/10.3906/mat-1511-69>
- [9] Chen H.H., Fang M.L. *Shared values and Normal families of meromorphic functions*. J. Math. Anal. Appl., 2001, vol. 260, pp. 124–132.
DOI: <https://doi.org/10.1006/jmaa.2001.7439>
- [10] Datt G. *Improvements to the Montel–Carathéodory theorem for families of $\mathbb{P}^N$ –valued holomorphic curves*. Complex Variables and Elliptic Equations, 2025, vol. 70, no. 2, pp. 335–353.
DOI: <https://doi.org/10.1080/17476933.2024.2310218>
- [11] Datt G. , Gupta N., Khanna N., Pal R. *Normal Families of Holomorphic Curves and Sharing of Moving Hyperplanes Wandering on $\mathbb{P}^n$* . Journal of Function Spaces, 2025, 9969004.
DOI: <https://doi.org/10.1155/jofs/9969004>

- [12] Deng H., Ponnusamy S., Qiao J. *Properties of normal harmonic mappings*. Monatsh. Math., 2020, vol. 193, no. 3, pp. 605–621.
DOI: <https://doi.org/10.1007/s00605-020-01459-8>
- [13] Dufresnoy J. *Théorie nouvelle des familles complexes normales. Applications à l'étude des fonctions algébroides*. Annales scientifiques de l'École Normale Supérieure, 1944, vol. 61, pp. 1–44.
- [14] Fujimoto H. *On meromorphic maps into the complex projective space*. Journal of the Mathematical Society of Japan, 1974, vol. 26, no. 2, pp. 272–288.
DOI: <https://doi.org/10.2969/jmsj/02620272>
- [15] Grahl J., Nevo S. *Exceptional functions wandering on the sphere and normal families*. Israel Journal of Mathematics, 2014, vol. 202, no. 1, pp. 21–34. DOI: <https://doi.org/10.1007/s11856-014-1054-7>
- [16] Hu P. C., Thin N. V. *Generalizations of Montel's normal criterion and Lappan's five-valued theorem to holomorphic curves*. Complex Variables and Elliptic Equations, 2020.
DOI: <https://doi.org/10.1080/17476933.2019.1588260>
- [17] Kumar R., Bharti N. *Normal families concerning partially shared and proximate values*. São Paulo J. Math. Sci., 2023, vol. 17, pp. 1076–1085.
DOI: <https://doi.org/10.1007/s40863-022-00341-9>
- [18] Lahiri I. *A simple normality criterion leading to a counterexample to the converse of the Bloch principle*. New Zealand J. Math., 2005, vol. 34, no. 1, pp. 61–65.
- [19] Lee J. M. *Introduction to complex manifolds*. Graduate studies in Mathematics, American Mathematical Society, 2024.
- [20] Montel P. *Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine*. Annales Scientifiques de l'École Normale Supérieure, 1912, vol. 29, pp. 487–535.
- [21] Pang X. C., Zalcman L. *Normality and shared values*. Ark. Mat., 2000, vol. 38, no. 1, pp. 171–182. DOI: <https://doi.org/10.1007/BF02384496>
- [22] Ru M. *Nevanlinna theory and its relation to Diophantine approximation*. World Scientific, 2001.
- [23] Schwick W. *Sharing values and normality*. Archiv der Mathematik, 1992, vol. 59, pp. 50–54. DOI: <https://doi.org/10.1007/BF01199014>
- [24] Sun D.C. *The shared value criterion for normality*. J. Wuhan Univ. Natur. Sci. Ed., 1994, vol. 3, pp. 9–12.
- [25] Sun D., Huo Y., Chai F. *Normal Criteria for a Family of Holomorphic Curves*. Acta Mathematica Scientia, 2021, vol. 41, no. 6, pp. 1887–1895.
DOI: <https://doi.org/10.1007/s10473-021-0606-2>

- [26] Yang L., Fang C., Pang X.C. *Normal families of holomorphic mappings into complex projective space concerning shared hyperplanes*. Pacific Journal of Mathematics, 2014, vol. 272, no. 1, pp. 245–256.
DOI: <http://dx.doi.org/10.2140/pjm.2014.272.245>
- [27] Ye Y., Pang X.C., Yang L. *An extension of Schwick's theorem for normal families*. Ann. Polon. Math., 2015, vol. 115, pp. 23–31.
DOI: <https://doi.org/10.4064/ap115-1-2>
- [28] Zalcman L. *Normal families: new perspectives*. Bull. Amer. Math. Soc., 1998, vol. 35, pp. 215–230.
DOI: <https://doi.org/10.1090/S0273-0979-98-00755-1>

Received April 19, 2025.

In revised form, October 13, 2025.

Accepted October 26, 2025.

Published online November 15, 2025.

Department of Mathematics
University of Jammu, 180006, India

Sonam Mehta
E-mail: mehtasonam2020@gmail.com

Kuldeep Singh Charak
E-mail: kscharak7@rediffmail.com