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GENERALIZED LOGISTIC NEURAL NETWORK APPROXIMATION OVER FINITE DIMENSION BANACH SPACES

Abstract. The functions under approximation here have as a domain a finite dimensional Banach space with dimension $N \in \mathbb{N}$ and are with values in $\mathbb{R}^N$. Exploiting some topological properties of the above we are able to perform Neural Network multivariate approximation to the above functions. The treatment is quantitative. We produce multivariate Jackson type inequalities involving the modulus of continuity of the function under approximation. The established convergences are pointwise and uniform. Perturbation and symmetrization to our operators lead to enhanced speeds of convergence. The activation function here is the generalized logistic.

Key words: *finite dimensional Banach spaces, neural network operators approximation, perturbation and symmetrization, modulus of continuity, accelerated approximation, generalized logistic activation function*

2020 Mathematical Subject Classification: *41A17, 41A25, 41A99, 46B25*

1. Introduction. The author in [1] and [2], see Chapters 2-5 was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support.

Again the author inspired by [10], continued his studies on neural network approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], by treating both the univariate and multivariate cases.

The author continued this trend of approximation and published the extensive monographs [5] – [9], studying in depth and as wide as possible the neural network approximations. Our activation function here is the generalized logistic function.

The multivariate operators we use are: the normalized, the quasi-interpolation, the Kantorovich and the quadrature types. Our symmetrization technique accelerates immensely the speed of convergences. Our neural networks here are from a Banach space of dimension N to $\mathbb{R}^N$, as they are homeomorphic, and this is the main key to this work.

So all results here are via multivariate Jackson type inequalities, studying quantitatively the rate of convergences to the unit operator.

Here we are dealing with one hidden layer feed-forward neural networks.

A multilayer feed-forward neural network can be defined as follows (with $m \in \mathbb{N}$ hidden layers):

Let $x \in \mathbb{R}^s$; $s \in \mathbb{N}$, where $x = (x_1, \ldots, x_s)$; $\alpha_j, c_j \in \mathbb{R}^s$; $b_j \in \mathbb{R}$, with $0 \leqslant j \leqslant n$, $n \in \mathbb{N}$.

Here $\langle \alpha_j \cdot x \rangle$ is the inner product, thus $\sigma(\langle \alpha_j \cdot x \rangle + b_j) \in \mathbb{R}$; and $N_n(x) \in \mathbb{R}^s$, by $c_j \in \mathbb{R}^s$, as it is coming from $N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot x \rangle + b_j)$.

We define:

$$N_n^{(2)}(x) = \sum_{j=0}^{n} c_j \sigma\Big(\Big\langle \alpha_j \cdot N_n\Big(x\Big)\Big\rangle + b_j\Big) =$$

$$\sum_{j=0}^{n} c_j \sigma\Big(\Big\langle \alpha_j \cdot \Big(\sum_{j=0}^{n} c_j \sigma\Big(\Big\langle \alpha_j \cdot x\Big\rangle + b_j\Big)\Big)\Big\rangle + b_j\Big).$$

Furthermore, we can define

$$N_n^{(3)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot N_n^{(2)}(x)\right\rangle + b_j\right).$$

And, in general we define:

$$N_n^{(m)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot N_n^{(m-1)}(x)\right\rangle + b_j\right), \quad \text{for } m \in \mathbb{N}.$$

For further studies in neural networks read [11] – [14].

2. Basics. Initially we follow [8], pp. 395 – 471.

Our activation function here to be used is the q-deformed and λ-parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R}, \ q,\lambda > 0, \ A > 1. \tag{1}$$

This is the A-generalized logistic function.

For more read Chapter 16 of [8]: "Banach space valued ordinary and fractional neural network approximation based on q-deformed and λ-parametrized A-generalized logistic function".

The Chapters 15, 16 of [8] motivate our current work.

The proposed "symmetrization technique" aims to use half data feed to our neural networks.

We will employ the following density function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}, \ q, \lambda > 0. \tag{2}$$

We have that

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \tag{3}$$

and

$$G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x), \quad \forall \ x \in \mathbb{R}. \tag{4}$$

Adding (3) and (4) we obtain

$$G_{q,\lambda}(-x) + G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x), \quad \forall \ x \in \mathbb{R}, \tag{5}$$

the key to this work.

So that

$$W(x) := \frac{G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x)}{2} \tag{6}$$

is an even function, symmetric with respect to the y-axis.

The global maximum of $G_{q,\lambda}$ is given by (16.18), p. 401 of [8] as

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2\left(A^\lambda + 1\right)}. \tag{7}$$

And, the global max of $G_{\frac{1}{q},\lambda}$ is

$$G_{\frac{1}{q},\lambda}\left(\frac{\log_A \frac{1}{q}}{\lambda}\right) = G_{\frac{1}{q},\lambda}\left(\frac{-\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2\left(A^\lambda + 1\right)}, \tag{8}$$

both sharing the same maximum at symmetric points.

By Theorem 16.1, p. 401 of [8], we have that

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x-i) = 1, \ \ \forall\, x \in \mathbb{R},\ \lambda, q > 0,\ A > 1, \tag{9}$$

and

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x-i) = 1, \ \ \forall\, x \in \mathbb{R},\ \lambda, q > 0,\ A > 1. \tag{10}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} W(x-i) = 1, \ \ \forall\, x \in \mathbb{R}. \tag{11}$$

By Theorem 16.2, p. 402 of [8], we have that

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = 1, \ \ \lambda, q > 0, \ \ A > 1, \tag{12}$$

similarly it holds

$$\int_{-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x)\, dx = 1, \tag{13}$$

so that

$$\int_{-\infty}^{\infty} W(x)\, dx = 1, \tag{14}$$

therefore W is a density function.

By Theorem 16.3, p. 402 of [8], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geqslant n^{1-\alpha} \end{cases}}^{\infty} G_{q,\lambda}(nx-k) <$$

$$2\max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \quad (15)$$

where $\lambda, q > 0,\ A > 1;\ \gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

Similarly, we get that

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geqslant n^{1-\alpha} \end{cases}}^{\infty} G_{\frac{1}{q},\lambda}(nx-k) < \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}. \quad (16)$$

Consequently we obtain that

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geqslant n^{1-\alpha} \end{cases}}^{\infty} W(nx-k) < \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \quad (17)$$

where $\gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

Here $\lceil \cdot \rceil$ denotes the ceiling of the number, and $\lfloor \cdot \rfloor$ its integral part.

We mention

Theorem 16.4 (p. 402, [8]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leqslant \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Then

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: K(q). \quad (18)$$

Similarly, we consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Thus

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx-k)} < \max\left\{\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \frac{1}{G_{q,\lambda}(\lambda_q)}\right\} = K(q). \quad (19)$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) > \frac{1}{K(q)}, \tag{20}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}(nx-k) > \frac{1}{K(q)}. \tag{21}$$

Consequently it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}(nx-k) + G_{\frac{1}{q},\lambda}(nx-k)\right)}{2} > \frac{2}{2K(q)} = \frac{1}{K(q)}, \tag{22}$$

so that

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}(nx-k) + G_{\frac{1}{q},\lambda}(nx-k)\right)}{2}} < K(q), \tag{23}$$

that is

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W(nx-k)} < K(q). \tag{24}$$

We have proved

Theorem 1. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leqslant \lfloor nb\rfloor$. For $q,\lambda > 0$, $A > 1$, we consider $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Also consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W(nx-k)} < K(q). \tag{25}$$

We make

Remark.

I) By Remark 16.5, p. 402 of [8], we have

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx_1-k) \neq 1, \text{ for some } x_1 \in [a,b], \tag{26}$$

and

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}\left(nx_2-k\right)\neq 1,\ \textit{for some } x_2\in\left[a,b\right]. \tag{27}$$

Therefore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(G_{q,\lambda}\left(nx_1-k\right)+G_{\frac{1}{q},\lambda}\left(nx_2-k\right)\right)}{2}\neq 1. \tag{28}$$

Hence

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(G_{q,\lambda}\left(nx_1-k\right)+G_{\frac{1}{q},\lambda}\left(nx_1-k\right)\right)}{2}\neq 1, \tag{29}$$

even if

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}\left(nx_1-k\right)=1, \tag{30}$$

because then

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{G_{q,\lambda}\left(nx_1-k\right)}{2}+\frac{1}{2}\neq 1, \tag{31}$$

equivalently

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{G_{q,\lambda}\left(nx_1-k\right)}{2}\neq\frac{1}{2}, \tag{32}$$

true by

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}\left(nx_1-k\right)\neq 1. \tag{33}$$

II) Let $[a,b]\subset\mathbb{R}$. *For large* n *we always have* $\lceil na\rceil\leqslant\lfloor nb\rfloor$. *Also* $a\leqslant\frac{k}{n}\leqslant b$, *iff* $\lceil na\rceil\leqslant k\leqslant\lfloor nb\rfloor$. *So in general it holds*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W\left(nx-k\right)\leqslant 1. \tag{34}$$

Next, we move on to the multivariate case, see Chapter 15 of [8], pp. 365 – 394, as a model of action.

We make

Remark. *We introduce*

$$Z_q(x_1,\dots,x_N) := Z_q(x) := \prod_{i=1}^{N} W(x_i) = \frac{1}{2^N}\prod_{i=1}^{N}\left(G_{q,\lambda} + G_{\frac{1}{q},\lambda}\right)(x_i), \tag{35}$$

$x = (x_1,\dots,x_N) \in \mathbb{R}^N$, $\lambda, q > 0$, $N \in \mathbb{N}$.

Properties:

(i)

$$Z_q(x) > 0, \ \forall\ x \in \mathbb{R}^N, \tag{36}$$

(ii)

$$\sum_{k=-\infty}^{\infty} Z_q(x-k) := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\dots\sum_{k_N=-\infty}^{\infty} Z_q(x_1-k_1,\dots,x_N-k_N) = 1, \tag{37}$$

$k := (k_1,\dots,k_n) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_q(nx-k) = 1, \tag{38}$$

$\forall\ x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

(iv)

$$\int_{\mathbb{R}^N} Z_q(x)\,dx = 1, \tag{39}$$

that is Z_q is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|,\dots,|x_N|\}$, $x \in \mathbb{R}^N$, $\pm\infty := (\pm\infty,\dots,\pm\infty)$, $\lceil na\rceil := (\lceil na_1\rceil,\dots,\lceil na_N\rceil)$, $\lfloor nb\rfloor := (\lfloor nb_1\rfloor,\dots,\lfloor nb_N\rfloor)$, $a := (a_1,\dots,a_N)$, $b := (b_1,\dots,b_N)$.

We obviously see that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_q(nx-k) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N} W(nx_i-k_i)\right) = \prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} W(nx_i-k_i)\right). \tag{40}$$

(v) We derive that

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} Z_q(nx-k) < \gamma A^{-\lambda\left(n^{1-\beta}-2\right)}, \quad \text{where } 0<\beta<1, \tag{41}$$

$0<\beta<1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N}[a_i,b_i]$.

(vi) It holds

$$0 < \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_q(nx-k)} < (K(q))^N, \tag{42}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

It is clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} Z_q(nx-k) < \gamma A^{-\lambda\left(n^{1-\beta}-2\right)}, \tag{43}$$

$0<\beta<1$, $n \in \mathbb{N}$: $n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N}[a_i,b_i]$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_q(nx-k) \neq 1, \tag{44}$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Here $\left(X, \|\cdot\|_{\gamma}\right)$ *is a Banach space.*

Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $x = (x_1,\ldots,x_N) \in \prod_{i=1}^{N}[a_i,b_i]$, $n \in \mathbb{N}$: $\lceil na_i\rceil \leqslant \lfloor nb_i\rfloor$, $i=1,\ldots,N$.

We introduce and define the following multivariate linear normalized symmetrized neural network operator, let $x := (x_1, \ldots, x_N) \in \prod_{i=1}^{N} [a_i, b_i]$:

$$A_n^s(f, x_1, \ldots, x_N) := A_n^s(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z_q(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right)}. \quad (45)$$

For large enough $n \in \mathbb{N}$ *we always obtain* $\lceil na_i \rceil \leqslant \lfloor nb_i \rfloor$, $i = 1, \ldots, N$. *Also* $a_i \leqslant \frac{k_i}{n} \leqslant b_i$, *iff* $\lceil na_i \rceil \leqslant k_i \leqslant \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ *or* $f \in C_U\left(\mathbb{R}^N, X\right)$, *we define*

$$B_n^s(f, x) := B_n^s(f, x_1, \ldots, x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z_q(nx-k) :=$$

$$\frac{1}{2^N} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right)$$

$$\left(\prod_{i=1}^{N}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right), \quad (46)$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, *the multivariate quasi-interpolation symmetrized neural network.*

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ *or* $f \in C_U\left(\mathbb{R}^N, X\right)$, *we define the multivariate Kantorovich type symmetrized neural network operator*

$$C_n^s(f, x) := C_n^s(f, x_1, \ldots, x_N) := \sum_{k=-\infty}^{\infty}\left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt\right) Z_q(nx-k) =$$

$$\frac{1}{2^N} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty}\left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \cdots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, \ldots, t_N)\, dt_1 \ldots dt_N\right)$$

$$\left(\prod_{i=1}^{N}\left(G_{q,\lambda}\left(nx_i-k_i\right)+G_{\frac{1}{q},\lambda}\left(nx_i-k_i\right)\right)\right), \tag{47}$$

$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$.

Again for $f\in C_B\left(\mathbb{R}^N,X\right)$ *or* $f\in C_U\left(\mathbb{R}^N,X\right)$, *we define the multivariate symmetrized neural network operator of quadrature type* $D_n^s\left(f,x\right)$, $n\in\mathbb{N}$, *as follows:*

Let $\theta:=\left(\theta_1,\ldots,\theta_N\right)\in\mathbb{N}^N$, $r=\left(r_1,\ldots,r_N\right)\in\mathbb{Z}_+^n$, $w_r=w_{r_1,r_2,\ldots,r_N}\geqslant 0$, *such that*

$$\sum_{r=0}^{\theta}w_r=\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}\ldots\sum_{r_N=0}^{\theta_N}w_{r_1,r_2,\ldots,r_N}=1;$$

$k\in\mathbb{Z}^N$, *and*

$$\delta_{nk}\left(f\right):=\delta_{n,k_1,\ldots,k_N}\left(f\right):=\sum_{r=0}^{\theta}w_r f\left(\frac{k}{n}+\frac{r}{n\theta}\right)=$$

$$\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}\ldots\sum_{r_N=0}^{\theta_N}w_{r_1,r_2,\ldots,r_N}f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\ldots,\frac{k_n}{n}+\frac{r_N}{n\theta_N}\right), \tag{48}$$

where $\frac{r}{\theta}:=\left(\frac{r_1}{\theta_1},\ldots,\frac{r_N}{\theta_N}\right)$.

We set

$$D_n^s\left(f,x\right):=D_n^s\left(f,x_1,\ldots,x_N\right):=\sum_{k=-\infty}^{\infty}\delta_{nk}\left(f\right)Z_q\left(nx-k\right)=$$

$$\frac{1}{2^N}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\ldots\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,\ldots,k_N}\left(f\right)$$

$$\left(\prod_{i=1}^{N}\left(G_{q,\lambda}\left(nx_i-k_i\right)+G_{\frac{1}{q},\lambda}\left(nx_i-k_i\right)\right)\right), \tag{49}$$

$\forall\ x\in\mathbb{R}^N$.

Definition 1. *([6], p. 274) Let* M *be a convex and compact subset of* $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$, *and* $\left(X,\|\cdot\|_\gamma\right)$ *be a Banach space. Let* $f\in C\left(M,X\right)$. *We define the first modulus of continuity of* f *as*

$$\omega_1\left(f,\delta\right):=\sup_{\substack{x,y\in M:\\ \|x-y\|_p\leqslant\delta}}\left\|f\left(x\right)-f\left(y\right)\right\|_\gamma,\quad 0<\delta\leqslant diam\left(M\right). \tag{50}$$

If $\delta > diam\,(M)$, then

$$\omega_1\,(f,\delta) = \omega_1\,(f, diam\,(M))\,. \tag{51}$$

Notice $\omega_1\,(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B\,(M,X)$ (continuous and bounded functions) $\omega_1\,(f,\delta)$ is defined similarly.

Lemma 1. *([6], p. 274) We have $\omega_1\,(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C\,(M,X)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

In this study we work only for the case of $p = \infty$.

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1\,(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (50). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$, $\|\cdot\|_\infty$ is the supremum norm.

In this work we treat the case of $\left(X, \|\cdot\|_\gamma\right) = \left(\mathbb{R}^N, \|\cdot\|_2\right)$, $N \in \mathbb{N}$; where $\|\cdot\|_2$ is the Euclidean norm.

Next, we describe the main ideas of this work.

Remark. *Let $(E, \|\cdot\|_1)$ be a Banach space of finite dimension N, i.e.* $\dim E = N$, $N \in \mathbb{N}$.

It is known that any two Banach spaces of the same finite dimension are linearly homeomorphic.

Hence $(E, \|\cdot\|_1)$ is linearly homeomorphic to $\left(\mathbb{R}^N, \|\cdot\|_2\right)$: so let $\phi\colon (E, \|\cdot\|_1) \to \left(\mathbb{R}^N, \|\cdot\|_2\right)$ be the linear homeomorphism.

So let $y \in E$, then $\phi\,(y) = x \in \mathbb{R}^N$, and back to $y = \phi^{-1}\,(x) \in E$, that is $\phi^{-1}\colon \mathbb{R}^N \to E$ is $(1-1)$ and onto map, and both ϕ, ϕ^{-1} are continuous maps.

Let now $f\colon E \to \mathbb{R}^N$ be a continuous map, then $f \circ \phi^{-1}$ is a continuous map from $\mathbb{R}^N$ into $\mathbb{R}^N$. We call $g\,(x) := f\left(\phi^{-1}\,(x)\right)$, $\forall\ x \in \mathbb{R}^N$.

If $\|f\,(y)\|_2 \leqslant M$, $\forall\ y \in E$, so is $\left\|f\left(\phi^{-1}\,(x)\right)\right\|_2 \leqslant M$, $\forall\ x \in \mathbb{R}^N$, where $M > 0$.

That is $\left\|\|f\,(y)\|_2\right\|_\infty < \infty$, implies $\left\|\|f \circ \phi^{-1}\|_2\right\|_\infty < \infty$.

In case f is a uniformly continuous map, since (see below) ϕ, ϕ^{-1} are also uniformly continuous, we obtain that $g = f \circ \phi^{-1}$ is a uniformly continuous map.

If $\prod_{i=1}^{N} [a_i, b_i] \subset \mathbb{R}^N$, we can consider $g|_{\prod_{i=1}^{N} [a_i, b_i]}$, which is continuous on a bounded set, that is a uniformly continuous map.

It is known that every linear function on a finite dimensional normed vector space is uniformly continuous, based on the fact that all norms there are equivalent. Thus, ϕ, ϕ^{-1} are uniformly continuous.

Let $f\colon E \to \mathbb{R}^N$ be continuous. We define the following neural network linear operators ${}_iL_n(f) : E \to \mathbb{R}^N$, $\quad i = 1,2,3,4$; $y \in E$, as follows:

I)

$$ {}_1L_n(f)(y) = {}_1L_n(f)\left(\phi^{-1}(x)\right) := A_n^s\left(f \circ \phi^{-1}, x\right) = $$

$$ \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f \circ \psi^{-1}\right)\left(\frac{k}{n}\right) Z_q(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)} = \tag{52} $$

$$ \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(f\circ\phi^{-1}\right)\left(\frac{k_1}{n},\dots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right)}, $$

$\forall\ x := (x_1,\dots,x_N) \in \prod_{i=1}^{N}[a_i,b_i] \subset \mathbb{R}^N$, $\forall\ y \in \phi^{-1}\left(\prod_{i=1}^{N}[a_i,b_i]\right) \subset E$, $n \in \mathbb{N}$.

II) When $f \in C_B\left(E,\mathbb{R}^N\right)$ or $f \in C_U\left(E,\mathbb{R}^N\right)$ we define:

$$ {}_2L_n(f)(y) = {}_2L_n(f)\left(\phi^{-1}(x)\right) := B_n^s\left(f \circ \phi^{-1}, x\right) = $$

$$ \sum_{k=-\infty}^{\infty} \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) Z_q(nx-k) = \tag{53} $$

$$ \frac{1}{2^N} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \left(f \circ \phi^{-1}\right)\left(\frac{k_1}{n},\dots,\frac{k_N}{n}\right) $$

$$ \left(\prod_{i=1}^{N}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right), $$

$\forall\ x \in \mathbb{R}^N$, $\forall\ y \in E$, $n \in \mathbb{N}$.

III) When $f \in C_B\left(E,\mathbb{R}^N\right)$ or $f \in C_U\left(E,\mathbb{R}^N\right)$ we define:

$$ {}_3L_n(f)(y) = {}_3L_n(f)\left(\phi^{-1}(x)\right) := C_n^s\left(f \circ \phi^{-1}, x\right) = $$

$$ \sum_{k=-\infty}^{\infty}\left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(f \circ \phi^{-1}\right)(t)\,dt\right) Z_q(nx-k) = $$

$$\frac{1}{2^N}\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\left(n^N\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\cdots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}}\left(f\circ\phi^{-1}\right)(t_1,\ldots,t_N)\,dt_1\ldots dt_N\right)$$

$$\left(\prod_{i=1}^{N}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right), \tag{54}$$

$\forall\ x\in\mathbb{R}^N$, $\forall\ y\in E$, $n\in\mathbb{N}$.

IV) Again, when $f\in C_B\left(E,\mathbb{R}^N\right)$ or $f\in C_U\left(E,\mathbb{R}^N\right)$ we define:

$${}_4L_n(f)(y) = {}_4L_n(f)\left(\phi^{-1}(x)\right) := D_n^s\left(f\circ\phi^{-1},x\right) =$$

$$\sum_{k=-\infty}^{\infty}\delta_{nk}\left(f\circ\phi^{-1}\right)Z_q(nx-k) =$$

$$\frac{1}{2^N}\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,\ldots,k_N}\left(f\circ\phi^{-1}\right)$$

$$\left(\prod_{i=1}^{N}\left(G_{q,\lambda}(nx_i-k_i)+G_{\frac{1}{q},\lambda}(nx_i-k_i)\right)\right), \tag{55}$$

$\forall\ x\in\mathbb{R}^N$, $\forall\ y\in E$, $n\in\mathbb{N}$.

We want to study quantitatively the multivariate approximation of ${}_iL_n(f)(y)\to f(y)$, as $n\to\infty$, for $i=1,2,3,4$; $y\in E$.

3. Main Results

We present our first approximation result.

Theorem 2. *Let $(E,\|\cdot\|_1)$ be a Banach space, $\dim E=N\in\mathbb{N}$, and ϕ be the corresponding homeomorphism from E onto $\left(\mathbb{R}^N,\|\cdot\|_2\right)$. Here $f\in C\left(E,\mathbb{R}^N\right)$, ${}_1L_n(f)$ is as in (52), $x\in\prod_{i=1}^{N}[a_i,b_i]\subset\mathbb{R}^N$, and $y\in\phi^{-1}\left(\prod_{i=1}^{N}[a_i,b_i]\right)\subset E$; $0<\beta<1$, $n\in\mathbb{N}$ with $n^{1-\beta}>2$. Then*

1)

$$\|{}_1L_n(f)(y)-f(y)\|_2\leqslant(K(q))^N\times$$

$$\left[\omega_1\left(f\circ\phi^{-1},\frac{1}{n^\beta}\right)+2\left\|\|f\circ\phi^{-1}\|_2\right\|_\infty\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right]=:\lambda_1(n), \tag{56}$$

and

2)
$$\left\| \left\| {}_1L_n(f) - f \right\|_2 \right\|_\infty \leqslant \lambda_1(n). \tag{57}$$

We see that $\lim_{n\to\infty} {}_1L_n(f) \overset{\|\cdot\|_2}{=} f$, *pointwise and uniformly.*
Above ω_1 *is with respect to* $p = \infty$.

Proof. We have that

$$\left\| {}_1L_n(f)(y) - f(y) \right\|_2 =$$
$$\left\| {}_1L_n(f)\left(\phi^{-1}(x)\right) - f\left(\phi^{-1}(x)\right) \right\|_2 =$$
$$\left\| A_n^s\left(f \circ \phi^{-1}, x\right) - f\left(\phi^{-1}(x)\right) \right\|_2 = \tag{58}$$
$$\left\| \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f \circ \psi^{-1}\right)\left(\frac{k}{n}\right) Z_q(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)} - \frac{f\left(\psi^{-1}(x)\right) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)} \right\|_2$$
$$\overset{\text{(by (42))}}{\leqslant} (K(q))^N$$
$$\left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) Z_q(nx-k) - f\left(\phi^{-1}(x)\right) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k) \right\|_2 =$$
$$(K(q))^N \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[\left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - \left(f \circ \phi^{-1}\right)(x) \right] Z_q(nx-k) \right\|_2 \leqslant$$
$$(K(q))^N \left[\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - \left(f \circ \phi^{-1}\right)(x) \right\|_2 Z_q(nx-k) \right] =$$
$$(K(q))^N \Bigg[\sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty \leqslant \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \left\| \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - (f \circ \phi^{-1})(x) \right\|_2 Z_q(nx-k)$$
$$+ \sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \left\| \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - (f \circ \phi^{-1})(x) \right\|_2 Z_q(nx-k) \Bigg] \overset{\text{by (38)}}{\leqslant}$$

$$(K(q))^N \left[\omega_1\left(f\circ\phi^{-1},\frac{1}{n^\beta}\right)+2\left\|\left\|f\circ\phi^{-1}\right\|_2\right\|_\infty\right.$$

$$\left.\left(\sum_{\substack{k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z_q(nx-k)\right)\right] \overset{\text{by }(41)}{\leqslant} \tag{59}$$

$$(K(q))^N\left[\omega_1\left(f\circ\phi^{-1},\frac{1}{n^\beta}\right)+2\left\|\left\|f\circ\phi^{-1}\right\|_2\right\|\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right].$$

□

It follows the next result all over E.

Theorem 3. *Let $(E,\|\cdot\|_1)$ be a Banach space with $\dim E=N\in\mathbb{N}$, and ϕ be the corresponding homeomorphism from E onto $\left(\mathbb{R}^N,\|\cdot\|_2\right)$. Here $f\in C_B\left(E,\mathbb{R}^N\right)$, ${}_2L_n(f)$ is as in (53), $x\in\mathbb{R}^N$, and $y\in E$; $0<\beta<1$, $n\in\mathbb{N}$ with $n^{1-\beta}>2$, ω_1 is for $p=\infty$. Then*

1)

$$\left\|{}_2L_n(f)(y)-f(y)\right\|_2\leqslant$$

$$\omega_1\left(f\circ\phi^{-1},\frac{1}{n^\beta}\right)+2\left\|\left\|f\circ\phi^{-1}\right\|_2\right\|_\infty\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}=:\lambda_2(n), \tag{60}$$

and

2)

$$\left\|\left\|{}_2L_n(f)-f\right\|_2\right\|_\infty\leqslant\lambda_2(n). \tag{61}$$

Given that $f\in\left(C_U\left(E,\mathbb{R}^N\right)\cap C_B\left(E,\mathbb{R}^N\right)\right)$, we obtain $\lim_{n\to\infty}{}_2L_n(f)=f$, uniformly.

Proof. We have that

$$\left\|{}_2L_n(f)(y)-f(y)\right\|_2=$$

$$\left\|{}_2L_n(f)\left(\phi^{-1}(x)\right)-f\left(\phi^{-1}(x)\right)\right\|_2=$$

$$\left\|B_n^s\left(f\circ\phi^{-1},x\right)-f\left(\phi^{-1}(x)\right)\right\|_2\overset{(\text{by }(38),\,(53))}{=}$$

$$\left\|\sum_{k=-\infty}^{\infty}\left(f\circ\phi^{-1}\right)\left(\frac{k}{n}\right)Z_q(nx-k)-f\left(\phi^{-1}(x)\right)\sum_{k=-\infty}^{\infty}Z_q(nx-k)\right\|_2=$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - \left(f \circ \phi^{-1}\right)(x) \right) Z_q(nx-k) \right\|_2 \leqslant$$

$$\sum_{k=-\infty}^{\infty} \left\| \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - \left(f \circ \phi^{-1}\right)(x) \right\|_2 Z_q(nx-k) = \tag{62}$$

$$\sum_{\substack{k=-\infty \\ : \left\|\frac{k}{n}-x\right\|_\infty \leqslant \frac{1}{n^\beta}}}^{\infty} \left\| \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - \left(f \circ \phi^{-1}\right)(x) \right\|_2 Z_q(nx-k)$$

$$+ \sum_{\substack{k=-\infty \\ : \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left\| \left(f \circ \phi^{-1}\right)\left(\frac{k}{n}\right) - \left(f \circ \phi^{-1}\right)(x) \right\|_2 Z_q(nx-k) \Bigg] \overset{(38)}{\leqslant}$$

$$\omega_1\left(f \circ \phi^{-1}, \frac{1}{n^\beta}\right) + 2 \left\| \left\| f \circ \phi^{-1} \right\|_2 \right\|_\infty \left(\sum_{\substack{k=-\infty \\ : \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z_q(nx-k) \right) \overset{(43)}{\leqslant} \tag{63}$$

$$\omega_1\left(f \circ \phi^{-1}, \frac{1}{n^\beta}\right) + 2 \left\| \left\| f \circ \phi^{-1} \right\|_2 \right\|_\infty \gamma A^{-\lambda\left(n^{1-\beta}-2\right)},$$

proving the claim. □

It follows the result for the Kantorovich type operator on E.

Theorem 4. *Let $(E, \|\cdot\|_1)$ be a Banach space with $\dim E = N \in \mathbb{N}$, and ϕ be the corresponding homeomorphism from E onto $\left(\mathbb{R}^N, \|\cdot\|_2\right)$. Here $f \in C_B\left(E, \mathbb{R}^N\right)$, ${}_3L_n(f)$ as in (54), $x \in \mathbb{R}^N$ and $y \in E$; $0 < \beta < 1$, $n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\left\| {}_3L_n(f)(y) - f(y) \right\|_2 \leqslant$$

$$\omega_1\left(f \circ \phi^{-1}, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2 \left\| \left\| f \circ \phi^{-1} \right\|_2 \right\|_\infty \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} =: \lambda_3(n), \tag{64}$$

and

2)

$$\left\|\left\|{}_3L_n(f) - f\right\|_2\right\|_\infty \leqslant \lambda_3(n). \tag{65}$$

Given that $f \in \left(C_U\left(E, \mathbb{R}^N\right) \cap C_B\left(E, \mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\to\infty} {}_3L_n(f) = f$, *uniformly.*

Proof. We observe that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(f \circ \phi^{-1}\right)(t)\, dt =$$

$$\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \ldots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} \left(f \circ \phi^{-1}\right)(t_1, \ldots, t_N)\, dt_1 \ldots dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} \left(f \circ \phi^{-1}\right)\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \ldots, t_N + \frac{k_N}{n}\right) dt_1 \ldots dt_N =$$

$$\int_0^{\frac{1}{n}} \left(f \circ \phi^{-1}\right)\left(t + \frac{k}{n}\right) dt. \tag{66}$$

Therefore we can write

$$\left\|{}_3L_n(f)(y) - f(y)\right\|_2 =$$

$$\left\|{}_3L_n(f)\left(\phi^{-1}(x)\right) - f\left(\phi^{-1}(x)\right)\right\|_2 =$$

$$\left\|C_n^s\left(f \circ \phi^{-1}, x\right) - f\left(\phi^{-1}(x)\right)\right\|_2 \overset{(38)}{=}$$

$$\Bigg\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f \circ \phi^{-1}\right)\left(t + \frac{k}{n}\right) dt\right) Z_q(nx - k) -$$

$$f\left(\phi^{-1}(x)\right)\left(\sum_{k=-\infty}^{\infty} Z_q(nx - k)\right)\Bigg\|_2 =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\left(n^N \int_0^{\frac{1}{n}} \left(f \circ \phi^{-1}\right)\left(t + \frac{k}{n}\right) dt\right) - \left(f \circ \phi^{-1}\right)(x)\right) Z_q(nx - k)\right\|_2 = \tag{67}$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(\left(f \circ \phi^{-1} \right) \left(t + \frac{k}{n} \right) - \left(f \circ \phi^{-1} \right) (x) \right) dt \right) Z_q (nx - k) \right\|_2 \leqslant$$

$$\sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| \left(f \circ \phi^{-1} \right) \left(t + \frac{k}{n} \right) - \left(f \circ \phi^{-1} \right) (x) \right\|_2 dt \right) Z_q (nx - k) =$$

$$\sum_{\begin{cases} k = -\infty \\ : \left\| \frac{k}{n} - x \right\|_\infty \leqslant \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| \left(f \circ \phi^{-1} \right) \left(t + \frac{k}{n} \right) - \left(f \circ \phi^{-1} \right) (x) \right\|_2 dt \right) \cdot Z_q (nx - k) +$$

$$\sum_{\begin{cases} k = -\infty \\ : \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| \left(f \circ \phi^{-1} \right) \left(t + \frac{k}{n} \right) - \left(f \circ \phi^{-1} \right) (x) \right\|_2 dt \right) \cdot Z_q (nx - k) \tag{68}$$

$$\overset{\text{(by (50), (43))}}{\leqslant} \omega_1 \left(f \circ \phi^{-1}, \frac{1}{n} + \frac{1}{n^\beta} \right) + 2 \left\| \left\| f \circ \phi^{-1} \right\|_2 \right\|_\infty \gamma A^{-\lambda \left(n^{1-\beta} - 2 \right)},$$

proving the claim. □

The final result is for the Quadrature type operator on E.

Theorem 5. *Let $(E, \|\cdot\|_1)$ be a Banach space with $\dim E = N \in \mathbb{N}$, and ϕ be the corresponding homeomorphism from E onto $\left(\mathbb{R}^N, \|\cdot\|_2\right)$. Here $f \in C_B\left(E, \mathbb{R}^N\right)$, ${}_4L_n(f)$ is as in (55), $x \in \mathbb{R}^N$ and $y \in E$; $0 < \beta < 1$, $n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\left\| {}_4L_n (f)(y) - f(y) \right\|_2 \leqslant$$

$$\omega_1 \left(f \circ \phi^{-1}, \frac{1}{n} + \frac{1}{n^\beta} \right) + 2 \left\| \left\| f \circ \phi^{-1} \right\|_2 \right\|_\infty \gamma A^{-\lambda \left(n^{1-\beta} - 2 \right)} =: \lambda_4 (n), \tag{69}$$

and

2)

$$\left\| \left\| {}_4L_n (f) - f \right\|_2 \right\|_\infty \leqslant \lambda_4 (n). \tag{70}$$

Given that $f \in \left(C_U\left(E,\mathbb{R}^N\right) \cap C_B\left(E,\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\to\infty} {}_4L_n(f) = f$, *uniformly.*

Proof. We have

$$\left\|{}_4L_n(f)(y) - f(y)\right\|_2 =$$

$$\left\|{}_4L_n(f)\left(\phi^{-1}(x)\right) - f\left(\phi^{-1}(x)\right)\right\|_2 =$$

$$\left\|D_n^s\left(f\circ\phi^{-1},x\right) - f\left(\phi^{-1}(x)\right)\right\|_2 \overset{(55)}{=}$$

$$\left\|\sum_{k=-\infty}^{\infty}\delta_{nk}\left(f\circ\phi^{-1}\right)Z_q(nx-k) - f\left(\phi^{-1}(x)\right)\right\|_2 \overset{(38)}{=}$$

$$\Bigg\|\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left(f\circ\phi^{-1}\right)\left(\frac{k}{n}+\frac{r}{n\theta}\right)\right)Z_q(nx-k) -$$

$$\left(\sum_{k=-\infty}^{\infty}Z_q(nx-k)\right)\left(f\circ\phi^{-1}\right)(x)\Bigg\|_2 = \tag{71}$$

$$\left\|\sum_{k=-\infty}^{\infty}\left(\left(\sum_{r=0}^{\theta}w_r\left(f\circ\phi^{-1}\right)\left(\frac{k}{n}+\frac{r}{n\theta}\right)\right) - \left(f\circ\phi^{-1}\right)(x)\right)Z_q(nx-k)\right\|_2 =$$

$$\left\|\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left(\left(f\circ\phi^{-1}\right)\left(\frac{k}{n}+\frac{r}{n\theta}\right) - \left(f\circ\phi^{-1}\right)(x)\right)\right)Z_q(nx-k)\right\|_2 \leqslant$$

$$\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left\|\left(f\circ\phi^{-1}\right)\left(\frac{k}{n}+\frac{r}{n\theta}\right) - \left(f\circ\phi^{-1}\right)(x)\right\|_2\right)Z_q(nx-k) =$$

$$\sum_{\begin{cases} k=-\infty \\ :\left\|\frac{k}{n}-x\right\|_\infty \leqslant \frac{1}{n^\beta}\end{cases}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left\|\left(f\circ\phi^{-1}\right)\left(\frac{k}{n}+\frac{r}{n\theta}\right) - \left(f\circ\phi^{-1}\right)(x)\right\|_2\right) \tag{72}$$

$$\cdot Z_q(nx-k) +$$

$$\sum_{\begin{cases} k=-\infty \\ :\left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}\end{cases}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left\|\left(f\circ\phi^{-1}\right)\left(\frac{k}{n}+\frac{r}{n\theta}\right) - \left(f\circ\phi^{-1}\right)(x)\right\|_2\right)$$

$$\cdot Z_q(nx-k)$$

$$\overset{\text{(by (50), (43))}}{\leqslant} \omega_1\left(f \circ \phi^{-1}, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2\left\|\left\|f \circ \phi^{-1}\right\|_2\right\| \gamma A^{-\lambda\left(n^{1-\beta}-2\right)}, \quad (73)$$

proving the claim. □

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