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ON THREE SUMMATION EQUATIONS FOR FUNCTIONS THAT ARE HOLOMORPHIC IN THE PLANE WITH A CUT ALONG A POLYGONAL LINE

Abstract. We study three four-element summation equations in the class of functions that are holomorphic outside a polygonal line and vanish at infinity. The polygonal line is part of the boundary of a unit square. We seek a solution in the form of a Cauchy-type integral with unknown density satisfying some additional conditions. The regularization of the equation on the polygonal line is achieved by introducing an involutive piecewise-linear shift that reverses the orientation of the line. We rely on the contraction mapping method in a Banach space to prove that the resulting Fredholm equation of the second kind is solvable. Finally, we give the conditions for the equivalence of the regularization and consider some applications to interpolation problems for entire functions.

Key words: *summation equation, regularization method, Carleman boundary-value problem*

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1. Introduction. Let Γ be a polygonal line with vertices $t_1 = -t_3 = 2^{-1}(1-i)$, $t_2 = -t_4 = 2^{-1}(1+i)$, and segments ℓ_j, $j = \overline{1,3}$, listed in the order they occur on the line ($t \in \ell_1$ then $\operatorname{Re} t = 2^{-1}$). Let D be a square with the indicated vertices. Define the transformation $\sigma_k(z) = i^{k-1} - z$, $k = \overline{1,4}$, and the involutive shift $\alpha(t) = \{\sigma_k(t), t \in \ell_k; k = \overline{1,3}\}$. Obviously, $\alpha(t)\colon \ell_k \to \ell_k$ reverses the orientation, and the midpoints of the segments are the fixed points of the shift, discontinuous at the vertices t_2 and t_3. Denote the rectangle with vertices t_2, t_3, $2^{-1}(1-3i)$, and $-2^{-1}(1+3i)$ by D_1.

Our aim in this paper is to investigate the summation equation

$$(Vf)(z) \equiv \sum_{j=1}^{4} \lambda_j f\left[\sigma_j(z)\right] = g(z), \quad z \in D, \tag{1}$$

under the following assumptions:

1) Either the coefficients $\lambda_j = (-1)^{j+1}$, $j = \overline{1,4}$ (Problem A), or $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = -1$ (Problem B), or $\forall j\ \lambda_j = 1$ (Problem C).

2) The solution can be expressed as the Cauchy-type integral

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} (\tau - z)^{-1} \varphi(\tau) d\tau, \tag{2}$$

with a density satisfying the condition

$$\varphi(\tau) = -\theta_\tau \varphi\left[\alpha(\tau)\right], \tag{3}$$

where $\theta_\tau = \left\{\lambda_j, \tau \in \ell_j, j = \overline{1,3}\right\}$. We assume that the functions ϕ are defined on the entire contour Γ and are Holder continuous on the open arcs $l_j, j = \overline{1,3}$. This means that they have finite one-sided limits at the corner points t_2 and t_3. However, these limits, generally speaking, do not coincide. That is, discontinuities of the first kind are possible at the corner points. We denote this class of solutions by B.

3) The independent term is holomorphic in D_1 and its boundary value $g^+(t) \in H_\mu(\partial D_1)$.

Let us explain the problem statement. The transformation $\sigma_4(t)$ is not involved in the definition of the shift $\alpha(t)$. However, when $\lambda_4 = 0$, the problem is not interesting. Consider the sets

$$H_3 = \mathbb{C} \backslash \bigcup_{j=1}^{3} \sigma_j(\Gamma).$$

The set H_3 is connected. Equation (1) holds in some neighborhood of infinity. The independent term can be analytically continued beyond the square D and $g(\infty) = 0$. For details on the triviality and overdetermination of such problems, see Remark 2 in [1] or the Introduction in [2].

The paper consists of two parts. First, we regularize Equation (1). After that, we discuss the conditions of equivalence of the regularization and consider some applications.

2. Regularization. Let us regularize Equation (1) using Cauchy-type integral (2). Thus, we have

$$(1) \Leftrightarrow (E\varphi)(z) \equiv \frac{1}{2\pi i}\int\limits_{\Gamma} A(z,\tau)\varphi(\tau)d\tau = g(z), \quad z \in D, \tag{4}$$

where

$$A(z,\tau) = \sum_{j=1}^{4} \lambda_j \left[\tau - \sigma_j(z)\right]^{-1}.$$

Let $z \in D$ and $z \to t \in \Gamma$. Given that the transformations $\sigma_k(z)$ take the point z to the exterior D, we have $(E^+\phi)(t) = -2^{-1}\theta_t\phi[\alpha(t)] + (E\phi)(t)$, where $t \neq t_k$, $k = \overline{1,4}$.

In view of condition (3), we obtain a formula similar to that of Sokhotski–Plemelj, namely,

$$\left(E^+\varphi\right)(t) = 2^{-1}\varphi(t) + (E\varphi)(t), \quad t \in \Gamma. \tag{5}$$

The singular integral on the right-hand side of (5) is obtained by formally replacing $z \in D$ in (4) with $t \in \Gamma$ and should be understood in the sense of the Cauchy principal value. Replace the variable t in (5) with $\alpha(t)$ and substitute the variable of integration in the singular integral using condition (3) one more time. Thus, $(E^+\varphi)(t) - \theta_t (E^+\varphi)(\alpha(t)) = (T\varphi)(t)$, with

$$(T\varphi)(t) \equiv \varphi(t) + \frac{1}{2\pi i}\int\limits_{\Gamma} K(t,\tau)\varphi(\tau)d\tau = g^+(t) - \theta_t g^+\left[\alpha(t)\right], \tag{6}$$

where the kernel

$$K(t,\tau) = A(t,\tau) + \theta_t\theta_\tau A\left[\alpha(t), \alpha(\tau)\right] \tag{7}$$

is bounded. The latter assertion can be verified directly by considering all the possible options for the relative positions of points τ and t on the segments of the polygonal line (see the specific estimates of the modulus of kernel (7) in the proof of Theorem 1 below). Thus, T is a canonical Fredholm operator, that is, we have regularized Equation (1).

The properties of the integral equation (6) with a kernel having a structure similar to (7) have long been well-known [3]. Let us list those of them that will be needed later.

1) If equation (6) is solvable, then it has a solution satisfying condition (3).

2) It is possible to choose a fundamental system of solutions (f. s. s.) for the corresponding homogeneous equation

$$T\phi = 0 \tag{8}$$

or for the adjoint equation $T'\psi = 0$ in such a way that some of the functions belonging to it satisfy condition (3), and the others satisfy the opposite condition

$$\phi(t) = \theta_t \phi(\alpha(t)). \tag{9}$$

Furthermore, the following statements are valid.

3) If on some interval l_j we have

$$\phi(\tau) = -\phi(\alpha(\tau)) \tag{10}$$

then

$$\int\limits_{l_j} \phi(\tau) d\tau = 0. \tag{11}$$

To verify this, it suffices to replace the integration variable τ with $\alpha(\tau)$.

4) Any solution $\psi(t)$ of the adjoint equation with property (9) is automatically orthogonal to the right-hand side of equation (6). It is clear that $T' = T$. The solutions $\psi(t)$ are defined on the entire contour Γ, are continuous on the open arcs $l_j, j = \overline{1,3}$, and have finite one-sided limits at the corner points t_2 and t_3.

5) If the points τ and t are located on opposite sides of l_1 and l_3, then $K(t,\tau) = 0$. This follows from the equality $\alpha(\tau) + \alpha(t) = -(\tau + t)$.

Let $||\phi|| = \sup|\phi(t)|, t \in \Gamma$.

Theorem 1. *The homogeneous equation (8) has only the trivial solution.*

Proof. To prove this, it is sufficient to show that the following inequality is true:

$$\Big|\int\limits_{\Gamma} K(t,\tau)\phi(\tau)\, d\tau\Big| < 2\pi||\phi||, \;\; t \in \Gamma. \tag{12}$$

It holds if

$$\sum_{j=1}^{3} \sup_{\tau \in l_j} |K(t,\tau)| < 2\pi, \;\; \forall t \in \Gamma. \tag{13}$$

A) $\lambda_j = (-1)^{j+1}$, $j = \overline{1,4}$. We should consider three cases:
1) Let $t \in l_1$, then $\alpha(t) = 1 - t$.

a) If $\tau \in \ell_1$, then $\alpha(\tau) = 1 - \tau$ and

$$K(t,\tau) = (v+1)^{-1} - (v-i)^{-1} - \\ - (v+i)^{-1} + (v-2+i)^{-1} - (v-3)^{-1} + (v-2-i)^{-1},$$

where $v = \tau + t = 1 + i\gamma$, $|\gamma| \leqslant 1$. Therefore,

$$K(t,\tau) = K(\theta),$$

$$K(\theta) = 4\left[(\theta+4)^{-1} - (\theta+2)\left(\theta^2+4\right)^{-1}\right], \ \theta = \gamma^2 \in [0,1]$$

and $|K(t,\tau)| \leqslant 1{,}6$, since the function $|K(\theta)|$ increases on the interval $[0,1]$.

b) If $\tau \in \ell_3$, then $K(t,\tau) = 0$.

c) If $\tau \in \ell_2$, then $\alpha(\tau) = i - \tau$,

$$K(t,\tau) = (v+1)^{-1} - (v+i)^{-1} + (v-i-2)^{-1} - (v-1-2i)^{-1}.$$

Here $v = \tau + t = \mu + 2^{-1}(1+i)$, where $\mu = \beta + i\gamma$ with $|\beta| \leqslant 0{,}5$ and $|\gamma| \leqslant 0{,}5$. Then

$$K(t,\tau) = 8\mu\Big[\Big(\mu + \frac{3}{2} + \frac{i}{2}\Big)\Big(\mu - \frac{i}{2} - \frac{3}{2}\Big)\Big(\mu + \frac{3}{2}i + \frac{1}{2}\Big)\Big(\mu - \frac{1}{2} - \frac{3}{2}i\Big)\Big]^{-1}$$

and $|K(t,\tau)| \leqslant 4\sqrt{2}/5$. The modulus of the kernel attains its largest value at $v = 0$, i.e., $|K(t,\tau)| < 1{,}14$. Since $1{,}6 + 0 + \frac{4\sqrt{2}}{5} < 2\pi$, inequality (12) holds.

2) Let $t \in \ell_2$; then $\alpha(t) = i - t$. From the estimates obtained above, we see that $|K(t,\tau)| < \mu_j$ if $\tau \in \ell_j$ and $\mu_1 = \mu_3 = 1{,}14$, $\mu_2 = 1{,}6$.

3) Due to the symmetry of Γ, the case $t \in l_3$ is analogous to the considered case $t \in l_1$.

For $t \in l_2$, inequality (12) also holds, which completes the consideration of problem A.

B) Now we have $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = -1$. First, we assume that equality (9) is attained for $t \in \ell_1$; then $\alpha(t) = 1 - t$.

C) $\lambda_j = 1$, $j = \overline{1,3} \Rightarrow \theta_t \equiv 1$. The right-hand side of Equation (6) is a function with property (3). We need to prove that the f.s.s. of Equation (8) does not contain any function having property (3) and, therefore, Equation (6) is solvable. Assume the opposite. For such a function, condition (10) and therefore (11) are fulfilled if $j = 1,2,3$. Consider the integrals $A_j(t) = \int\limits_{l_j} K(t,\tau)\phi(\tau)d\tau$; then $|A_j(t)| \leqslant \mu_j$, $j = \overline{1,3}$. Now, using inequality (12), we will prove that the adjoint equation has no solutions that lead to the solvability conditions for the non-homogeneous equation (6).

Let $t \in l_1$.

a) If $\tau \in \ell_1$, then

$$K(t,\tau) = (v-i)^{-1} + (v+i)^{-1} + (v+1)^{-1} - \\ - (v+i-2)^{-1} - (v-i-2)^{-1} - (v-3)^{-1}.$$

Since $v = 1 + i\gamma$, $|\gamma| \leqslant 1$, we have

$$K(t,\tau) = -4f(\theta), f(\theta) = (\theta+2)\left(\theta^2+4\right)^{-1} + (\theta+4)^{-1}, \theta = \gamma^2 \in [0,1].$$

Therefore, $2^{-1}\left[\max f(\theta) - \min f(\theta)\right] = 0{,}1$.

b) If $\tau \in \ell_2$, then

$$K(t,\tau) = (v+1)^{-1} + (v+i)^{-1} - (v-2-i)^{-1} - (v-2i-1)^{-1}.$$

Condition (11) allows us to 'adjust' the kernel (7) by a term depending only on t, so

$$\left|(\tau+t+1)^{-1} - \left(t+2^{-1}i+1\right)^{-1}\right| < 0{,}5,$$

$$\left|(\tau+t+i)^{-1} - \left(t+\frac{3}{2}i\right)^{-1}\right| < 0{,}5,$$

$$\left|(\tau+t-2i-1)^{-1} - \left(t-\frac{3}{2}i-1\right)^{-1}\right| < 0{,}5,$$

$$\left|(\tau+t-2-i)^{-1} - \left(t-2-\frac{i}{2}\right)^{-1}\right| < 0{,}5.$$

Therefore, $|A_2(t)| \leqslant 2\|\varphi\|$.

c) If $\tau \in \ell_3$, then $K(t,\tau) = 0$. Thus, inequality (12) holds for $t \in l_1$.

Let $t \in \ell_2$.

Here $\mu_1 = \mu_3 = 2$, $\mu_2 = 0{,}1$, and $\varphi \equiv 0$. Inequality (12) holds for $t \in \ell_3$; this implies that $\mu_1 = 0$, $\mu_2 = 2$, and $\mu_3 = 0{,}1$. This completes the examination of case C) and the proof of the theorem. □

Remark 1. *The f. s. s. of Equation (3) is empty in cases A) and B), while it may contain some functions with property (9) in case C).*

Remark 2. *The kernel (7) is not only bounded, it has even «better» properties. If the points τ and t are distinct from the vertices t_2 and t_3, then any of its partial derivatives are also bounded. Moreover, it has points of jump discontinuity at the indicated vertices for each variable. Due to the properties of the free term, we obtain that functions $f(z) \in B$. For more details regarding the history of problems of type (1), see the review article* [4].

3. Equivalence of the regularization. Let us determine when the regularization we have carried out is equivalent. For this solution, we have

$$(6) \Rightarrow \left(E^+\varphi\right)(t) - \theta_t \left(E^+\varphi\right)[\alpha(t)] = g^+(t) - \theta_t g^+\left[\alpha(t)\right], \ t \in \Gamma,$$

that is, if the function $\varphi(z) = (E\varphi)(z) - g(z)$, $z \in D$, we have $\varphi^+(t) = \theta_t \psi^+[\alpha(t)]$, $t \in \Gamma$. This is the condition of the Carleman boundary problem. However, this is an underspecified problem [5], since Γ is only a part of the boundary ∂D. For this reason, its set of solutions has «enough cardinality». We have obtained the following result.

Theorem 2. *Summation equation (1) is solvable if and only if the countable set of conditions*

$$\left(E^{(k)}\varphi\right)(0) = g^{(k)}(0), \quad k = 0, 1, 2, \ldots \tag{14}$$

holds.

Here $\varphi(t)$ is the only solution to the integral equation with property (3). The homogeneous equation ($g(z) \equiv 0$) has only the trivial solution.

Remark 3. *Previously, we encountered a similar solvability picture in [6] during the investigation of a certain difference equation.*

Let $F(z)$ be an entire function of exponential type Borel-associated [7, §1, 1.1] with the lower function $f(z) \in B$. Rewrite Equation (1) in the following equivalent form:

$$(V_1 F)(z) \equiv \sum_{j=1}^{4} \lambda_j \int\limits_{\arg \tau = \theta_j} F(\tau) \exp\left[-\sigma_j(z)\tau\right] \, d\tau = g(z), \quad z \in D, \tag{15}$$

where $\theta_1 = \pi$, $\theta_2 = -\theta_4 = -2^{-1}\pi$, and $\theta_3 = 0$. Assume that the independent term can be expanded as

$$g(z) = \sum_{k=0}^{\infty} \frac{c_k z^k}{k!},$$

and the radius of convergence of the power series satisfies the condition $R > 2^{-1}\sqrt{10}$. By equating the corresponding Maclaurin coefficients of the left and right-hand sides in (15), we obtain the following result.

Theorem 3. *Interpolation problem*

$$\frac{d^k}{dz^k} (V_1 F)(z)\Big|_{z=0} = c_k, \quad k = 0, 1, 2, \ldots$$

is solvable in the class of entire functions of exponential type $F(z)$, Borel-associated with lower functions $f(z) \in B$ if and only if the countable set of solvability conditions (14) holds.

Remark 4. *In Theorem 3, we assume beforehand that the conjugated indicator diagram of the lower function is exactly the square D and not a «smaller» convex set $\Omega \subset D$. The latter assumption immediately leads to the trivial and overdetermined problems mentioned in the Introduction. The set of independent terms for which Equation (1) is uninteresting can be described explicitly.*

Remark 5. *We first obtained these three cases of Equation (1) in the series of papers [8], [9], [10], [11], but there they had at most a finite number of solvability conditions. In those papers, we considered the class of functions that are holomorphic outside a quadrilateral instead of a polygonal line Γ as in the present study.*

Finally, let us rewrite Equation (1) in a different form. Since $D_1 \backslash D = \sigma_4(D)$, we can replace the point z in (1) with $\sigma_1(z)$ and obtain the difference equation

$$\lambda_1 f(z+1+i) + \lambda_2 f(z+2i) + \lambda_3 f(z+1+i) + \lambda_4 f(z) = g\left[\sigma_4(z)\right], \quad z \in D_1.$$

We can apply to it many of the powerful classical methods used to investigate convolution operators [12], since the quadrilateral D_1 separates the points 0 and ∞. The independent term $g[\sigma_4(z)]$ does not have to be analytically continuable across a segment of the boundary ∂D_1.

In conclusion, let us consider the problem of generalizing the obtained results. In this article, D is a unit square. Let now D be an arbitrary quadrilateral (not necessarily convex) with vertices t_k, $k = \overline{1,4}$ enumerated in the positive direction of traversing the boundary ∂D. Then equation (6) will still be a Fredholm equation of the second kind. But the validity of Theorem 1 can no longer be guaranteed. Generally speaking, equation (6) may have a finite number of solvability conditions. See, in this regard, the works [8], [9], [10], [11]. In some cases, Theorem 1 remains valid as well. For example, in the case when D is an isosceles trapezoid with vertices $\pm 2 - i$, $\pm 1 + i$ [8]. In this case, it does not matter which three sides are included in the cut.

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