DOI: 10.15393/j3.art.2026.18810

UDC 517.521

J. Hossain, S. Debnath, S. K. Paul

ON $\mathcal{I}_C^Q$-CONVERGENCE OF SEQUENCES OF BI-COMPLEX NUMBERS

Abstract. In this paper, we study the properties of ideals $\mathcal{I}_c^q$ related to the notion of $\mathcal{I}$-convergence of sequences of bi-complex numbers. We prove some results about modified Olivier's theorem for these ideals. For bounded sequences, we show a connection between $\mathcal{I}_c^q$-convergence and regular matrix method of summability of sequences of bi-complex numbers.

Key words: *bi-complex number, ideal, $\mathcal{I}_c^q$-convergence*

2020 Mathematical Subject Classification: *40A35, 40G15*

1. Introduction. In 2001, Kostyrko et al. [9] introduced the concept of ideal convergence, primarily as an extension of statistical convergence. They demonstrated that ideal convergence serves as a generalized form of some other established convergence concepts. In subsequent years, extensive research in this particular direction was conducted by Debnath and Rakshit [3], Choudhury and Debnath [2], [4], Hossain and Debnath [5], alongside numerous other researchers.

In 1892, Segre [15] introduced the notion of bi-complex number, which constitutes an algebra isomorphic to the literariness. In Price's book [10], the most comprehensive study of analysis in bi-complex numbers is available. For an extensive study on bi-complex number, one may refer to [5], [11], [14], [17]. In recent years, many important results have been obtained in this area. Some of them pertain to this work.

The study of convergence is one of the important tools in analysis. Research on the convergence of sequences of bi-complex numbers is still in its infancy and has not gained much ground. The current body of research suggests a pronounced analogy in the convergence behavior of sequences of bi-complex numbers.

Throughout the paper, $\mathbb{C}_2$ represents the set of all bi-complex numbers.

2. Definitions and Preliminaries. Segre [15] defined a bi-complex number as $\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$, where $z_1 = x_1 + i_1 x_2$,

$z_2 = x_3 + i_1 x_4 \in \mathbb{C}$ (set of complex numbers) and $x_1, x_2, x_3, x_4 \in \mathbb{R}$ (set of real numbers), and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. Denote the set of all bi-complex numbers by $\mathbb{C}_2$; it is defined as: $\mathbb{C}_2 = \{\xi \colon \xi = z_1 + i_2 z_2 \colon z_1, z_2 \in \mathbb{C}\}$.

A bi-complex number $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ is designated as a hyperbolic number if $x_2 = 0$ and $x_3 = 0$. The collection of hyperbolic numbers is symbolically represented as $\mathcal{H}$, and the entirety of these hyperbolic numbers is referred to as the $\mathcal{H}$-plane. Equipped with the coordinate-wise addition, real scalar multiplication, and term-by-term multiplication, $\mathbb{C}_2$ becomes a commutative algebra with the identity $1 = 1 + i_1.0 + i_2.0 + i_1 i_2.0$. In $\mathbb{C}_2$, there exist four idempotent elements, namely 0, 1, $e_1 = \dfrac{1 + i_1 i_2}{2}$, and $e_2 = \dfrac{1 - i_1 i_2}{2}$. It is obvious that $e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$. Every bi-complex number $\xi = z_1 + i_2 z_2$ has a unique idempotent representation as $\xi = T_1 e_1 + T_2 e_2$ where $T_1 = z_1 - i_1 z_2$ and $T_2 = z_1 + i_1 z_2$ are called the idempotent components of ξ.

The Euclidean norm $\|\cdot\|$ on $\mathbb{C}_2$ is defined as,

$$\|\xi\|_{\mathbb{C}_2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|T_1|^2 + |T_2|^2}{2}},$$

where $\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = T_1 e_1 + T_2 e_2$ with this norm $\mathbb{C}_2$ is a Banach space.

Definition 1. [16] *The i_1-conjugate of a bi-complex number $\xi = z_1 + i_2 z_2$ is denoted by $\xi^* = \overline{z_1} + i_2 \overline{z_2}$ $\forall$ $z_1, z_2 \in \mathbb{C}$. In this context, $\overline{z_1}$ and $\overline{z_2}$ represent the complex conjugates of z_1 and z_2, respectively.*

The i_2-conjugate of a bi-complex number $\xi = z_1 + i_2 z_2$ is denoted by $\bar{\xi}$ and is defined as $\bar{\xi} = z_1 - i_2 z_2$ $\forall$ $z_1, z_2 \in \mathbb{C}$.

The $i_1 i_2$-conjugate of a bi-complex number $\xi = z_1 + i_2 z_2$ is denoted by ξ' and defined as $\xi' = \overline{z_1} - i_2 \overline{z_2}$ $\forall$ $z_1, z_2 \in \mathbb{C}$; $\overline{z_1}$ and $\overline{z_2}$ are the complex conjugates of z_1 and z_2, respectively.

Definition 2. *If $\xi = z_1 + i_2 z_2$ is not a zero divisor, then the inverse ξ^{-1} exists and is given by $\xi^{-1} = \dfrac{z_1 - i_2 z_2}{z_1^2 + z_2^2}$, where $z_1^2 + z_2^2 \neq 0$.*

Some of the properties of i_1-conjugation, which are obtained by Rochon and Shapiro [11] are listed as follows:

(i) $(\xi + \eta)^* = \xi^* + \eta^*$,

(ii) $(\alpha\xi)^* = \alpha\xi^*$' where $\alpha \in \mathbb{R}$,
(iii) $(\xi^*)^* = \xi$,
(iv) $(\xi\eta)^* = \xi^*\eta^*$,
(v) $(\xi^{-1})^* = (\xi^*)^{-1}$if ξ^{-1} exists,
(vi) $(\frac{\xi}{\eta})^* = \frac{\xi^*}{\eta^*}$.

Rochon and Shapiro also obtained analogous properties of i_2 conjugation.

Definition 3. *Let $S \subseteq \mathbb{N}$ and $S_n = \{k \in S : k \leqslant n\}$. The natural density of S is represented and defined by $\delta(S) = \lim\limits_{n\to\infty} \frac{|S_n|}{n}$, whenever it exists.*

Definition 4. [5] *Consider a nonempty set X. A family of subsets $\mathcal{I} \subset \mathcal{P}(X)$ is termed an ideal on X if it satisfies the following conditions:*

(1) For every $X_1, X_2 \in \mathcal{I}$, the union $X_1 \cup X_2$ belongs to $\mathcal{I}$.
(2) For every $X_1 \in \mathcal{I}$ and every subset X_2 of X_1, X_2 is also in $\mathcal{I}$.

In addition, $\mathcal{I}$ is said to be admissible if $\forall x \in X$, $\{x\} \in \mathcal{I}$ and it is said to be nontrivial if $\mathcal{I} \neq \{\varnothing\}$ and $X \notin \mathcal{I}$.

Example 1. Below are some standard examples of ideals:

(1) The collection of all finite subsets of $\mathbb{N}$ constitutes a nontrivial admissible ideal on $\mathbb{N}$, denoted as $\mathcal{I}_f$.
(2) The set comprising all subsets of $\mathbb{N}$ with natural density zero forms a nontrivial admissible ideal on $\mathbb{N}$. This particular ideal is denoted as $\mathcal{I}_\delta$.
(3) $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum\limits_{a\in A} a^{-1} < \infty\} \subset \mathbb{N}$. Then $\mathcal{I}_c$ forms an admissible ideal on $\mathbb{N}$.
(4) Consider a partitioning of the natural numbers $\mathbb{N}$ into disjoint infinite sets $D_1, D_2, D_3, \ldots$, such that $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ and $D_p \cap D_q = \varnothing$ for $p \neq q$. The set $\mathcal{I}$, comprising all subsets of $\mathbb{N}$ that have finite intersections with the sets D_p, constitutes an ideal on $\mathbb{N}$.
(5) For any $q \in (0,1]$ the set

$$\mathcal{I}_c^{(q)} = \left\{A \subset \mathbb{N} : \sum_{a\in A} a^{-q} < \infty\right\}$$

is an admissible ideal. The ideal

$$\mathcal{I}_c^{(1)} = \left\{A \subset \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\right\}$$

is usually denoted by $\mathcal{I}_c$.

For any $q \in (0,1]$ the set $\mathcal{I}_c^{(q)} = \{A \subset \mathbb{N} \colon \sum_{a \in A} a^{-q} < \infty\}$ is an admissible ideal. The ideal $\mathcal{I}_c^{(1)} = \{A \subset \mathbb{N} \colon \sum_{a \in A} a^{-1} < \infty\}$ is usually denoted by $\mathcal{I}_c$.

It is easy to see that for any $q_1 < q_2$, where $q_1, q_2 \in (0,1)$,

$$\mathcal{I}_f \subset \mathcal{I}_c^{(q_1)} \subset \mathcal{I}_c^{(q_2)} \subset \mathcal{I}_c \subset \mathcal{I}_d. \tag{1}$$

Definition 5. [3] *A family $\mathcal{F} \subset 2^X$ of subsets of a nonempty set X is deemed a filter in X if it satisfies the following conditions:*

(1) The empty set $\varnothing$ does not belong to $\mathcal{F}$.

(2) For all $X_1, X_2 \in \mathcal{F}$, the intersection $X_1 \cap X_2$ is also in $\mathcal{F}$.

(3) For every $X_1 \in \mathcal{F}$ and every superset X_2 of X_1 containing X_1, X_2 is also in $\mathcal{F}$.

Definition 6. [3] *If $\mathcal{I}$ is a proper non-trivial ideal in Y, then $\mathcal{F}(\mathcal{I}) = \{A \subset Y \colon \exists\, B \in \mathcal{I} \colon A = Y - B\}$ constitutes a filter in Y. This filter is commonly referred to as the filter associated with the ideal $\mathcal{I}$.*

Definition 7. *Let $\mathcal{I} \subset P(\mathbb{N})$ denote a non-trivial ideal over $\mathbb{N}$. We define an $\mathcal{I}$-convergence for a real-valued sequence (ξ_n) towards l as follows: for every $\varepsilon > 0$, the set $H(\varepsilon) = \{n \in \mathbb{N} \colon |\xi_n - l| \geqslant \varepsilon\}$ must be an element of $\mathcal{I}$. Here, l is called the $\mathcal{I}$-limit of the sequence (ξ_n) and is denoted by: $\mathcal{I}\text{-}\lim_k \xi_k = l$.*

Definition 8. [7] *An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is classified as a P-ideal (or AP-ideal) if, for every countable family of mutually disjoint sets $A_1, A_2, A_3, \ldots$ from $\mathcal{I}$, there exists a countable family of sets $B_1, B_2, B_3, \ldots$ such that the symmetric differences $A_j \Delta B_j$ are finite sets for every $j \in \mathbb{N}$, and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.*

Definition 9. *Consider an admissible ideal $\mathcal{I}$ in $\mathbb{N}$. We define $\mathcal{I}^*$-convergence for a real-valued sequence (ξ_k) towards l as follows: there exists a set $T = \{t_1 < t_2 < \ldots < t_k < \ldots\}$ in the associated filter $\mathcal{F}(\mathcal{I})$, such that $\lim_{k \in T} \xi_k = l$. Symbolically, we write $\mathcal{I}^*\text{-}\lim_k \xi_k = l$.*

Definition 10. [5] *With respect to the Euclidean norm on $\mathbb{C}_2$, a sequence of bi-complex numbers (ξ_k) is deemed $\mathcal{I}$-convergent to $t \in \mathbb{C}_2$ if, for each*

$\varepsilon > 0$, the set $F(\varepsilon) = \{k \in \mathbb{N}\colon \|\xi_k - t\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{I}$.
Symbolically, we write, $\xi_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathbb{C}_2}} t$.

Definition 11. [5] *Let $\mathcal{I}$ be an admissible ideal. A sequence (ξ_k) of $\mathbb{C}_2$ is deemed $\mathcal{I}^*$-convergent to $\xi \in \mathbb{C}_2$ with respect to the Euclidean norm on $\mathbb{C}_2$ if $\exists$ a set $T = \{t_1 < t_2 < \ldots < t_k < \ldots\}$ in the associated filter $\mathcal{F}(\mathcal{I})$, such that the subsequence (ξ_{t_k}) converges to ξ.*
Symbolically, we write, $\xi_k \xrightarrow{\mathcal{I}^*-\|\cdot\|_{\mathbb{C}_2}} \xi$.

Definition 12. *With respect to the Euclidean norm on $\mathbb{C}_2$, a sequence of bi-complex numbers (ξ_k) is said to be $\mathcal{I}_c^q$-convergent to $t \in \mathbb{C}_2$ if, for each $\varepsilon > 0$, the set $F(\varepsilon) = \{k \in \mathbb{N}\colon \|\xi_k - t\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{I}_c^q$.*
Symbolically, we write $\xi_k \xrightarrow{\mathcal{I}_c^q-\|\cdot\|_{\mathbb{C}_2}} t$.

3. Main Results.

Theorem 1. *The ideal $\mathcal{I}_c^q$ holds the (AP) property in $\mathbb{C}_2$ for any $0<q\leqslant 1$.*

Proof. It suffices to prove that for any sequences $(u_k) \in \mathbb{C}_2$, such that $\mathcal{I}_c^q\text{-}\lim u_k = \xi$, there exists a set $T = \{t_1 < t_2 < \ldots < t_k < \ldots\} \subseteq \mathbb{N}$, such that $\mathbb{N}\backslash T \in \mathcal{I}_c^q$ and $\lim\limits_k u_{t_k} = \xi$. For any positive integer k, let $\varepsilon_k = \frac{1}{2^k}$ and $A_k = \{n \in \mathbb{N}\colon \|u_k - \xi\|_{\mathbb{C}_2} \geqslant \frac{1}{2^k}\}$. As $\mathcal{I}_c^q\text{-}\lim u_k = \xi$, we have $A_k \in \mathcal{I}_c^q$, i.e. $\sum\limits_{a\in A_k} a^{-q} < \infty$. Therefore, there exists an infinite sequence $n_1 < n_2 < \ldots < n_k < \ldots$ of integers, such that for every $k = 1, 2, \ldots$ $\sum\limits_{\substack{a>n_k,\\ a\in A_k}} a^{-q} < \frac{1}{2^k}$. Let $H = \bigcup\limits_{k=1}^{\infty} [(n_k, n_{k+1}) \cap A_k]$. Then

$$\sum_{a\in H} a^{-q} \leqslant \sum_{\substack{a>n_1,\\ a\in A_1}} a^{-q} + \sum_{\substack{a>n_2,\\ a\in A_2}} a^{-q} + \cdots + \sum_{\substack{a>n_k,\\ a\in A_k}} a^{-q} + \cdots < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots < +\infty.$$

Thus, $H \in \mathcal{I}_c^q$. Put $T = \mathbb{N}\backslash H = \{t_1 < t_2 < \ldots < t_k < \ldots\}$.

Now it suffices to prove that $\lim\limits_{k\to\infty} u_{t_k} = \xi$. Let $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$, such that $\frac{1}{2^{k_0}} < \varepsilon$. Let $t_k > n_{k_0}$. Then t_k belongs to some interval (n_j, n_{j+1}), where $j \geqslant k_0$ and $\notin A_j (j \geqslant k_0)$. Hence, $t_k \in \mathbb{N}\backslash A_j$, and then $\|u_{t_k} - \xi\|_{\mathbb{C}_2} < \varepsilon$ for every $t_k > n_{k_0}$, thus $\lim\limits_{k\to\infty} u_{t_k} = \xi$. $\square$

Olivier's like theorem for the ideals $\mathcal{I}_c^q$ in $\mathbb{C}_2$.

In 1827, L. Olivier established an important result concerning the convergence rate for convergent series with positive, decreasing terms. He proved that for any non-increasing sequence $(u_n)_{n=1}^{\infty}$ where $\sum\limits_{n=1}^{\infty} u_n < +\infty$, it necessarily follows that $\lim\limits_{n\to\infty} n \cdot u_n = 0$. However, the monotonicity condition is essential, as demonstrated by the counterexample where $u_n = \frac{1}{n}$ when n is a perfect square ($n = k^2$ for $k = 1, 2, \ldots$) and $u_n = \frac{1}{2^n}$ otherwise. This example shows that without the monotonicity requirement, the conclusion $n \cdot u_n \to 0$ may fail.

Building on this foundation, Šalát and Toma [12] extended this research by characterizing the class of ideals $\mathcal{S}(\mathcal{T})$ for which the implication holds: whenever a series with positive terms converges, the generalized limit $I - \lim\limits_{n\to\infty} n \cdot a_n = 0$ is satisfied. The class $\mathcal{S}(\mathcal{T})$ consists of all admissible ideals $I \subset \mathcal{P}(\mathbb{N})$, such that $I \supset I_c$. From the inclusions in (1), it is clear that the ideals $I_c^{(q)}$ do not belong to the class $\mathcal{S}(\mathcal{T})$. In what follows, we show that it is possible to modify Olivier's condition $\sum\limits_{n=1}^{\infty} a_n < +\infty$ in such a way that the ideal $\mathcal{I}_c^{(q)}$ will play the role of the ideal $\mathcal{I}_c$ in $\mathbb{C}_2$.

Definition 13. *A sequence (u_n) of $\mathbb{C}_2$ is said to be a non-decreasing sequence with respect to the Euclidean norm $\|.\|_{\mathbb{C}_2}$ if the corresponding sequence of real numbers $(\|u_n\|_{\mathbb{C}_2})$ is non-decreasing, that is, $\forall n \in \mathbb{N}$: $\|u_{n+1}\|_{\mathbb{C}_2} \geqslant \|u_n\|_{\mathbb{C}_2}$.*

Lemma 1. *If (u_n) is a non-increasing sequence of $\mathbb{C}_2$ and $\sum\limits_{n=1}^{\infty} \|u_n\|_{\mathbb{C}_2} < +\infty$, then $\mathcal{I}$-$\lim\limits_{n\to\infty} nu_n = 0$.*

Theorem 2. *Let $0 < q \leqslant 1$. For every sequence $(u_n) \in \mathbb{C}_2$, such that $\sum\limits_{n=1}^{\infty} \|u_n^q\|_{\mathbb{C}_2} < +\infty$, $\mathcal{I}_c^q - \lim\limits_{n\to\infty} nu_n = 0$.*

Proof. Let the conclusion of the above theorem be false. Then there exists $\varepsilon > 0$, such that the set $A(\varepsilon) = \{n \in \mathbb{N} \colon n.\|u_n\|_{\mathbb{C}_2} \geqslant \varepsilon\} \notin \mathcal{I}_c^q$. Therefore,

$$\sum_{k=1}^{\infty} m_k^{-q} = +\infty, \tag{2}$$

where $A(\varepsilon) = \{m_1 < m_2 < m_3 < \ldots m_k < \ldots\}$. By the definition of the set $A(\varepsilon)$, we have $m_k.\|u_{m_k}\|_{\mathbb{C}_2} \geqslant \varepsilon > 0$, for each $k \in \mathbb{N}$. From this,

$m_k^q.\|u_{m_k}^q\|_{\mathbb{C}_2} \geqslant \varepsilon^q > 0$, and, so, for each $k \in \mathbb{N}$:

$$\|u_{m_k}^q\|_{\mathbb{C}_2} \geqslant \varepsilon^q.m_k^{-q}. \tag{3}$$

From (2) and (3) we get $\sum_{k=1}^{\infty} \|u_{m_k}^q\|_{\mathbb{C}_2} = +\infty$ and, hence, $\sum_{k=1}^{\infty} \|u_n^q\|_{\mathbb{C}_2} = +\infty$. But this contradicts the assumption of the theorem. □

Let us denote by $S_q(T)$ the class of all admissible ideals $\mathcal{I}$ for which an analogous Theorem 2 holds.

Theorem 3. *Let $0 < q \leqslant 1$. Then, for every sequence (u_n) of $\mathbb{C}_2$, the class $S_q(T)$ consists of all admissible ideals such that $\mathcal{I} \supseteq \mathcal{I}_c^q$.*

Proof. It is sufficient to prove that for any infinite set $T = \{t_1 < \ldots t_k < \ldots\} \in \mathcal{I}_c^q$ we have $T \in \mathcal{I}$, too. Since $T \in \mathcal{I}_c^q$, we have: $\sum_{k=1}^{\infty} t_k^{-q} < +\infty$. Define a sequence $(u_n) \in \mathbb{C}_2$, such that

$$u_k = \begin{cases} \dfrac{e_1 + e_2}{m_k}, & \text{if } n = t_k \text{ and } k \in \mathbb{N}, \\ \dfrac{e_1 + e_2}{10^n}, & \text{if } n \in \mathbb{N} \backslash M. \end{cases}$$

Obviously, $u_n > 0$ and $\sum_{k=1}^{\infty} u_k^q < +\infty$ by the definition of the number u_n. Since $\mathcal{I} \in S_q(T)$, we have $\mathcal{I}$-lim $n.u_n = 0$. This implies that for each $\varepsilon > 0$ we have the set $A(\varepsilon) = \{n \in \mathbb{N} \colon \|.u_n\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{I}$. □

$\mathcal{I}_c^q-$convergence and regular matrix transformations of the sequence of $\mathbb{C}_2$.

Let $A = (a_{nk}), n, k = 1, 2, \ldots$ be an infinite matrix of $\mathbb{C}_2$. The sequence (t_n) of $\mathbb{C}_2$ is said to be A-limitable to the number of s if $\lim_{n\to\infty} s_n = s$, where $s_n = \sum_{k=1}^{\infty} \|a_{nk}t_k\|_{\mathbb{C}_2}$. If (t_n) is A-limitable to the number s, we write $A\text{-}\lim_{n\to\infty} t_n = s$. We denote by $F(A)$ the set of all A-limitable sequences. The set $F(A)$ is called the convergence field. The method defined by the matrix A is said to be regular provided that $F(A)$ contains all convergent sequences and $\lim_{n\to\infty} t_n = t$ implies $A\text{-}\lim_{n\to\infty} t_n = t$. Then A is called a regular matrix. It is well-known that a matrix A is regular if it satisfies the following three conditions:

(a) There exits $K > 0, \forall n = 1, 2, 3, \ldots,\ \sum\limits_{k=1}^{\infty} \|a_{nk}\|_{\mathbb{C}_2} \leqslant K$;

(b) $\forall k = 1{,}2, \ldots,\ \lim\limits_{n\to\infty} \|a_{nk}\|_{\mathbb{C}_2} = 0$;

(c) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} \|a_{nk}\|_{\mathbb{C}_2} = 1$.

We define a class of lower triangular non-negative matrices $\mathcal{T}$ with the properties: $\sum\limits_{k=1}^{\infty} \|a_{nk}\|_{\mathbb{C}_2} = 1, \forall n \in \mathbb{N}$. If $M \subseteq \mathbb{N}$, such that $d(M) = 0$, then $\lim\limits_{n\to\infty} \sum\limits_{k\in M} \|a_{nk}\|_{\mathbb{C}_2} = 0$.

Let us define the class $\mathcal{T}_q$ of lower triangular non-negative matrices in this way:

Definition 14. *Matrix $A = (a_{nk})$ belongs to the class $\mathcal{T}_q$ if and only if it satisfies the following conditions:*

(p_1) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} \|a_{nk}\|_{\mathbb{C}_2} = 1$,

(p_2) *If $M \subseteq \mathbb{N}$ and $M \in \mathcal{I}_c^q$, then* $\lim\limits_{n\to\infty} \sum\limits_{k\in M} \|a_{nk}\|_{\mathbb{C}_2} = 0,\ 0 < q \leqslant 1$.

It is easy to see that every matrix of class $\mathcal{T}_q$ is regular. As the following example shows, the converse does not hold.

Example 2. Let $M = \{n^2 \colon n \in \mathbb{N}\}$ and $q = 1$. Clearly, $M \in \mathcal{I}_c^1 = \mathcal{I}_c$. Now define the matrix A by: $a_{11} = u_1 + u_2; a_{1k} = u_1.u_2, k > 1$; $a_{nk} = \frac{u_1+u_2}{2k.\ln n},\ k \neq \ln n^2,\ k \leqslant n$; $a_{nk} = \frac{u_1+u_2}{l.\ln n},\ k = \ln n^2,\ k \leqslant n$; $a_{nk} = 0,\ k > n$. It is easy to show that A is lower triangular non-negative regular matrix, but does not satisfy the condition (p_2) from Definition 14.

$$\sum_{\substack{k<n^2,\\ k\in M}} \|a_{nk}\|_{\mathbb{C}_2} = \frac{1}{\ln n^2}(1 + \frac{1}{2} + \cdots + \frac{1}{n}) \geqslant \frac{\ln n}{2\ln n} = \frac{1}{2} \nrightarrow 0$$

for $n \to \infty$. Therefore, $A \notin \mathcal{T}_1$.

Lemma 2. *If a bounded sequence (x_n) is not $\mathcal{I}$-convergent, then there exist real numbers $\lambda < \mu$, such that neither the set $\{n \in \mathbb{N} \colon \|x_n\|_{\mathbb{C}_2} < \lambda\}$ nor the set $\{n \in \mathbb{N} \colon \|x_n\|_{\mathbb{C}_2} > \lambda\}$ belongs to ideal $\mathcal{I}$.*

Theorem 4. *Let $0 < q \leqslant 1$. Then the bounded sequence (α_n) of $\mathbb{C}_2$ is $\mathcal{I}_c^q$-convergence to $L \in \mathbb{C}_2$ iff it is A-summable to $L \in \mathbb{C}_2$ for each matrix $A \in \mathcal{T}_q$.*

Proof. Let $\mathcal{I}_c^q$-$\lim \alpha_n = L$ and $A \in \mathcal{T}_q$. As A is regular, there exists a $k \in \mathbb{R}$, such that $\forall\, n \in \mathbb{N}$, $\sum\limits_{k \in M} \|a_{nk}\|_{\mathbb{C}_2} \leqslant K$. To prove the claim, it suffices to show that $\lim\limits_{n\to\infty} b_n = 0$, where $b_n = \sum\limits_{k=1}^{\infty} \|a_{nk}(\alpha_k - L)\|_{\mathbb{C}_2}$. For $\varepsilon > 0$, put $B(\varepsilon) = \{k \in \mathbb{N} \colon \|(\alpha_k - L)\|_{\mathbb{C}_2} \geqslant \varepsilon\}$. By the assumption, we have $B(\varepsilon) \in \mathcal{I}_c^q$. By the condition (p_2), from Definition 14 we have:

$$\lim_{n\to\infty} \sum_{k \in B(\varepsilon)} \|a_{nk}\|_{\mathbb{C}_2} = 0. \tag{4}$$

As the sequence (α_n) is bounded, $\exists\, M > 0$, such that $\forall k \in \mathbb{N}$

$$\|(\alpha_k - L)\|_{\mathbb{C}_2} \leqslant M. \tag{5}$$

Let $\varepsilon > 0$. Then

$$\begin{aligned} \|b_n\|_{\mathbb{C}_2} &\leqslant \sum_{k \in B(\frac{\varepsilon}{2k})} \|a_{nk}\|_{\mathbb{C}_2} \|(\alpha_k - L)\|_{\mathbb{C}_2} + \sum_{k \notin B(\frac{\varepsilon}{2k})} \|a_{nk}\|_{\mathbb{C}_2} \|(\alpha_k - L)\|_{\mathbb{C}_2} \\ &\leqslant M \sum_{k \in B(\frac{\varepsilon}{2k})} \|a_{nk}\|_{\mathbb{C}_2} + \frac{\varepsilon}{2k} \sum_{k \notin B(\frac{\varepsilon}{2k})} \|a_{nk}\|_{\mathbb{C}_2} \leqslant M \sum_{k \in B(\frac{\varepsilon}{2k})} \|a_{nk}\|_{\mathbb{C}_2} + \frac{\varepsilon}{2}. \end{aligned} \tag{6}$$

By part (p_2) of Definition 14, there exists an $n_0 \in \mathbb{N}$, such that

$$\forall\, n > n_0, \sum_{k \in B(\frac{\varepsilon}{2k})} \|a_{nk}\|_{\mathbb{C}_2} < \frac{\varepsilon}{2M};$$

together with (6), we obtain $\lim\limits_{n\to\infty} b_n = 0$.

Conversely, suppose that $\mathcal{I}_c^q$-$\lim \alpha_n = L$ does not hold. We show that there exists a matrix $A \in \mathcal{T}_q$, such that A-$\lim\limits_{n\to\infty} \alpha_n = L$ does not hold neither. If $\mathcal{I}_c^q$-$\lim \alpha_n = L' \neq L$, then from the first part of proof it follows that A-$\lim\limits_{n\to\infty} \alpha_n = L' \neq L$ for any $A \in \mathcal{T}_q$. Thus, we may assume that (α_n) is not $\mathcal{I}_c^q$-convergent, and by the above lemma 2 there exist λ and μ $(\lambda < \mu)$, such that neither the set $U = \{k \in \mathbb{N} \colon \|\alpha_n\|_{\mathbb{C}_2} < \lambda\}$ nor $V = \{k \in \mathbb{N} \colon \|\alpha_n\|_{\mathbb{C}_2} > \mu\}$ belongs to the ideal $\mathcal{I}_c^q$. Clearly, $U \cap V = \varnothing$. If $U \notin \mathcal{I}_c^q$, then $\sum\limits_{i \in U} i^{-q} = +\infty$, and if $V \notin \mathcal{I}_c^q$, then $\sum\limits_{i \in V} i^{-q} = +\infty$.

Let $U_n = U \cap \{1, 2, 3, \dots\}$ and $V_n = V \cap \{1, 2, 3, \dots\}$. Now we define the matrix $A = (a_{nk})$ by the following way:

$$S_{(1)n} = \sum_{i \in U_n} i^{-q} \text{ for } n \in U, \qquad S_{(2)n} = \sum_{i \in V_n} i^{-q} \text{ for } n \in V,$$

$$S_{(3)n} = \sum_{i=1}^{n} i^{-q} \text{ for } n \notin U \cap V.$$

As U, $V \notin \mathcal{I}_c^q$, we have $\lim_{n\to\infty} S_{(j)n} = +\infty,\ j = 1, 2, 3.$

$$a_{nk} = \begin{cases} \dfrac{e_1 + e_2}{k^q.S_{(1)n}}, & \text{if } n \in U \text{ and } k \in U_n; \\ e_1.e_2, & \text{if } n \in U \text{ and } k \notin U_n; \\ \dfrac{e_1 + e_2}{k^q.S_{(2)n}}, & \text{if } n \in V \text{ and } k \in V_n; \\ e_1.e_2, & \text{if } n \in V \text{ and } k \notin V_n; \\ \dfrac{e_1 + e_2}{k^q.S_{(3)n}}, & \text{if } n \notin U \cap V; \\ e_1.e_2, & \text{if } k > n. \end{cases}$$

Let us check that $A \in \mathcal{T}_q$. Obviously, A is a lower triangular non-negative matrix. Condition (p_1) is clear from the definition of matrix A.

Let $B \in \mathcal{I}_c^q$ and $b = \sum_{k\in B} k^{-q} < +\infty$. Then,

$$\sum_{k\in B} \|a_{nk}\|_{\mathbb{C}_2} \leqslant \frac{1}{S_{(3)n}} \sum_{k\in B\cap\{1,2,3,\dots\}} k^{-q}\chi_B(k) \leqslant \frac{b}{S_{(3)n}} \to 0,$$

for $n \to \infty$. Thus, $A \in \mathcal{T}_q$.

For $n \in U$,

$$\sum_{k=1}^{\infty} \|a_{nk}.\alpha_k\|_{\mathbb{C}_2} = \frac{1}{S_{(1)n}} \sum_{k=1}^{\infty} k^{-q}\chi_U(k)\|\alpha_k\|_{\mathbb{C}_2} < \frac{\lambda}{S_{(1)n}} \sum_{k=1}^{\infty} k^{-q}\chi_U(k) = \lambda.$$

On the other hand, for $n \in V$,

$$\sum_{k=1}^{\infty} \|a_{nk}.\alpha_k\|_{\mathbb{C}_2} = \frac{1}{S_{(2)n}} \sum_{k=1}^{\infty} k^{-q}\chi_V(k)\|\alpha_k\|_{\mathbb{C}_2} > \frac{\mu}{S_{(2)n}} \sum_{k=1}^{\infty} k^{-q}\chi_V(k) = \mu.$$

Therefore, $A\text{-}\lim_{n\to\infty} \alpha_n$ does not exist. □

Corollary 1. *If* $0 < q_1 < q_2 \leqslant 1$, *then* $\mathcal{T}_{q_2} \nsubseteq \mathcal{T}_{q_1}$.

Proof. Let $Q \in \mathcal{I}_c^{q_2} \backslash \mathcal{I}_c^{q_1}$. Suppose $(\alpha_n) = \chi_Q(n)$, $n = 1, 2, 3, \dots$. Clearly, $\mathcal{I}_c^{q_2}\text{-}\lim \alpha_n = 0$ and $\mathcal{I}_c^{q_1}\text{-}\lim \alpha_n$ does not exist. Let A be the matrix constructed from the sequence (α_n) as in the proof of Theorem 4. In particular, $A \in \mathcal{I}_c^{q_1}$ and $A\text{-}\lim_{n\to\infty} x_n$ does not exist. Therefore, $A \notin \mathcal{I}_c^{q_2}$. □

Acknowledgment. The authors are grateful to the reviewers for their valuable suggestions and insightful comments, which helped to improve the quality of this paper.

References

[1] Bera S., Tripathy B. C. *Statistical convergence in a bi-complex valued metric space.* Ural Math. J., 2023, vol. 9, no. 1, pp. 49 – 63. DOI: https://doi.org/10.15826/umj.2023.1.004

[2] Choudhury C., Debnath S. *On $\mathcal{I}$-convergence of sequences in gradual normed linear spaces.* Facta. Univ. Ser. Math. Inform., 2021, vol. 36, no. 3, pp. 595 – 604. DOI: https://doi.org/10.22190/FUMI210108044C.

[3] Debnath S., Rakshit D. *On $\mathcal{I}-$statistical convergence.* Iran. J. Math. Sci. Inform., 2018, vol. 13, no. 2, pp. 101 – 109.

[4] Debnath S., Choudhury C. *On some properties of $\mathcal{I}^K-$ convergence.* Palest. J. Math., 2022, vol. 11, no. 2, pp. 129 – 135.

[5] Debnath S., Hossain J. *On $\mathcal{I}-$ convergence, $\mathcal{I}-$limit point and $\mathcal{I}-$ cluster point of sequences of bi-complex numbers.* Adv. Math. Sci. Appl., 2024, vol. 33, no. 2, pp. 303 – 314.

[6] Fridy J. A., Miler H. I. *A matrix characterization of statistical convergence.* Analysis., 1986, vol. 11, pp. 59 – 66.

[7] Gogola J., Cervenánský J. *$\mathcal{I}_c^q$- convergence of real numbers.* Zborník vedeckých prác MtF STU v Trnave., 2005, vol. 18, pp. 15 – 18.

[8] Hossain J., Debnath S. On $\mathcal{I}^{\mathcal{K}}$-convergence of sequences of bi-complex numbers. Probl. Anal. Issues. Anal., 2024, vol. 13(31), no. 3, pp. 43 – 55. DOI: https://doi.org/10.15393/j3.art.2024.16070

[9] Kostyrko P., Salat T., Wilczynski W. *$\mathcal{I}-$convergence.* Real. Anal. Exch., 2000/2001, vol. 26, no. 2, pp. 669 – 686.

[10] Price G. B. *An Introduction to Multi-complex Spaces and Functions: Monographs and Text books in Pure and Applied Mathematics.* Marcel Dekker. Inc. New York, 1991.

[11] Rochon D., Shapiro M. *On algebraic properties of bi complex and hyperbolic numbers.* Anal. Univ. Oradea, Fasc. Math., 2004, vol. 11, pp. 71 – 110.

[12] Salat T., Toma V. *A classical Olivier's theorem and Statistical Convergence.* Annales Math. Blaise Pascal., 2001, vol. 1, pp. 10 – 18.

[13] Sağır B., Değirmen N., Duyar C. *On Bicomplex Matrix Transformations Between c_0 and c in bicomplex setting.* U.P.B. Sci. Bull., Series A., 2023, vol. 85, no. 4, pp. 115 – 128.

[14] Scorza D. G. *Sulla rappresentazione delle funzioni di variabile bicomplessa totalmente derivabili.* Ann. Mat., 1934, vol. 5, pp. 597 – 665.

[15] Segre C. *Le rappresentation reali delle forme complesse e gli enti iperalgebrici.* Math. Ann., 1892, vol. 40, pp. 413 – 467.
DOI: https://doi.org/10.1007/BF01443559 (in Italiano).

[16] Spampinato N. *Estensione nel campo bicomplesso di due teoremi, del levi-Civita e del severi, per le funxione olomorfe di due variabili bicomplesse I, II.* Reale Accad. Naz. Lincei., 1935, vol. 22, no. 6, pp. 96 – 102.

[17] Srivastava R. K., Srivastava N. K. *On a class of entire bi-complex spaces.* South East Asian. J. Math. Sci., 2007, vol. 5, no. 3, pp. 47 – 68.

Received July 10, 2025.
In revised form, October 27, 2025.
Accepted October 31, 2025.
Published online December 13, 2025.

J. Hossain
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala, India.
Email: juweljoya123@gmail.com

S. Debnath (Corresponding Author)
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala, India.
Email: shyamalnitamath@gmail.com

S. K. Paul
Department of Electrical Engineering
Dhalai District Polytechnic
Ambassa Dhalai Tripura India
Email: sanatkumarpaul@gmail.com