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ON NUMERICAL RADIUS INEQUALITIES FOR OPERATOR PRODUCTS IN HILBERT SPACES

Abstract. We establish several numerical radius inequalities for products of two operators on a Hilbert space. Some of the obtained inequalities improve well-known results. More precisely, we show that if $A, B \in B(H)$ double commute, then

$$w(AB) \leqslant w(A)w(B) + \frac{1}{2}w(AB - BA^*).$$

In particular, if in addition $AB = BA^*$, we prove that

$$w(AB) \leqslant w(A)w(B).$$

Key words: *numerical range, numerical radius inequalities*
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1. Introduction. Throughout this paper, let H be a complex Hilbert space. We denote by $B(H)$ the C^*-algebra of all bounded linear operators on H. For $A \in B(H)$, the *numerical range* of A is defined by

$$W(A) = \{\langle Ax, x\rangle \colon x \in H, \ \|x\| = 1\},$$

and the *numerical radius* of A is given by

$$w(A) = \sup\{|z| \colon z \in W(A)\}.$$

The numerical range $W(A)$, like the spectrum $\sigma(A)$, is a subset of the complex plane. A classical result of Hausdorff and Toeplitz [5], [11] states that $W(A)$ is convex and bounded, and its closure contains the convex hull of $\sigma(A)$. Moreover, $W(A)$ is closed if $\dim(H) < \infty$, but not necessarily when $\dim(H) = \infty$.

It is well known that the numerical radius defines a norm on $B(H)$ equivalent to the operator norm $\|\cdot\|$. Specifically, for all $A \in B(H)$,

$$w(A) \leqslant \|A\| \leqslant 2w(A), \tag{1}$$

and both inequalities are sharp: equality holds on the left if A is normal, and on the right if $A^2 = 0$.

For $A \in B(H)$, define the distance from A to the scalar operators by

$$d(A) = \inf\{\|A - \lambda I\| \colon \lambda \in \mathbb{C}\},$$

where I denotes the identity operator on H. According to [10],

$$d^2(A) = \sup_{x\in\mathbb{S}}\{\|Ax\|^2 - |\langle Ax, x\rangle|^2\}, \tag{2}$$

where $\mathbb{S} = \{x \in H \colon \|x\| = 1\}$ denotes the unit sphere of H. In order to compute the norm of the inner derivation δ_A associated with $A \in B(H)$, Stampfli [10] introduced the so called maximal numerical range $W_{\max}(A)$ defined as follows:

$$W_{\max}(A) = \{\lim_n \langle Ax_n, x_n\rangle \colon x_n \in H,\ \|x_n\| = 1,\ \lim_n \|Ax_n\| = \|A\|\}.$$

In the same paper, it was shown that $W_{\max}(A)$ is nonempty, closed, convex, and contained in the closure of the numerical range. In [6], the author proves that

$$\|A\| \leqslant \sqrt{w^2(A) + d^2(A)}. \tag{3}$$

It is known that the numerical radius is not submultiplicative. If $A, B \in \mathcal{B}(H)$, then

$$w(AB) \leqslant 4w(A)w(B). \tag{4}$$

When $AB = BA$, it always holds that

$$w(AB) \leqslant 2w(A)w(B). \tag{5}$$

If A and B double commute, then

$$w(AB) \leqslant w(A)\|B\|. \tag{6}$$

If A is an isometry and $AB = BA$, or a unitary operator that commutes with another operator B, then

$$w(AB) \leqslant w(B). \tag{7}$$

For more details, see [4]. In [1] the authors prove that for all $A, B \in B(H)$,

$$w(AB) \leqslant \|A\|w(B) + \frac{1}{2}w(AB - BA^*). \tag{8}$$

If $A, B \in B(H)$ double commute, we show that

$$w(AB) \leqslant w(A)w(B) + \frac{1}{2}w(AB - BA^*).$$

In particular, if $A, B \in B(H)$ double commute and $AB = BA^*$, we prove that

$$w(AB) \leqslant w(A)w(B).$$

The authors in [1] present an improvement of inequality (4) by establishing that

$$w(AB) \leqslant (\|A\| + d(A))w(B). \tag{9}$$

If A and B double commute, then inequality (9) can be improved by showing that

$$w(AB) \leqslant \sqrt{c_{\max}^2(A) + d^2(A)}\,w(B),$$

where $c_{\max}(A) = \inf\limits_{\lambda \in W_{\max}(A)} |\lambda|$.

In [3], the authors show that for all $A, B \in \mathcal{B}(H)$,

$$w(A^*B + BA) \leqslant 2\|A\|w(B). \tag{10}$$

Based on the inequality (10), the authors in [1] remark that

$$w(A^*B \pm BA) \leqslant 2\|A\|w(B). \tag{11}$$

For $A, B \in B(H)$, set

$$\mathcal{I}_{A,\,B} = \{\langle Ax, x\rangle\langle Bx, x\rangle \colon x \in \mathbb{S}\}.$$

We prove that

$$W(AB) \subset \mathcal{I}_{A,B} + W_0(A)W_0(B),$$

where $W_0(A) = \{\langle Ax, y\rangle \colon (x, y) \in \mathbb{S}^2, \langle x, y\rangle = 0\}$. In particular, we get

$$w(AB) \leqslant w(A)w(B) + d(A)d(B).$$

For a compact set $K \subset \mathbb{C}$, define

$$|K| = \sup_{z \in K} |z|.$$

2. Main results. We need the following result.

Lemma 1. [2] *Let $A \in \mathcal{B}(H)$. Then*

$$d(A) = \sup\{|\langle Ax, y\rangle| : (x, y) \in \mathbb{S}^2, \langle x, y\rangle = 0\}.$$

Theorem 1. *Let $A, B \in B(H)$. Then*

$$W(AB) \subset \mathcal{I}_{A,B} + W_0(A)W_0(B). \tag{12}$$

In particular,

$$w(AB) \leqslant w(A)w(B) + d(A)d(B). \tag{13}$$

Proof. Let $x \in \mathbb{S}$. Then there exists $y \in \mathbb{S}$, such that $\langle x, y\rangle = 0$ and

$$Bx = \alpha x + \beta y.$$

Hence,

$$\langle ABx, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle + \langle Bx, y\rangle\langle Ay, x\rangle.$$

Thus, $\langle ABx, x\rangle \in \mathcal{I}_{A,B} + W_0(A)W_0(B)$. Then

$$W(AB) \subset \mathcal{I}_{A,B} + W_0(A)W_0(B).$$

By inequality (12) and Lemma 1, we get inequality (13). □

Recall that an operator A is called *hyponormal* if

$$A^*A - AA^* \geqslant 0.$$

Lemma 2. [9] *Let $A \in B(H)$ be a hyponormal operator. Then*

$$\overline{W(A)} = conv(\sigma(A)).$$

Moreover, for all $\lambda \in \mathbb{C}$,

$$w(A - \lambda I) = \|A - \lambda I\|.$$

Let $R(A)$ denote the radius of the smallest disk in the complex plane containing $\sigma(A)$.

Theorem 2. *Let $A, B \in B(H)$ with A hyponormal. Then*

$$w(AB) \leqslant w(A)w(B) + R(A)d(B). \tag{14}$$

Proof. Since A is hyponormal, by Lemma 2

$$w(A - \lambda I) = \|A - \lambda I\|, \ \forall \lambda \in \mathbb{C}.$$

Then

$$r(A - \lambda I) = w(A - \lambda I) = \|A - \lambda I\|, \ \forall \lambda \in \mathbb{C}.$$

It follows that

$$\inf_{\lambda \in \mathbb{C}} r(A - \lambda I) = d(A).$$

Thus,

$$R(A) = d(A).$$

We finish the proof by using Theorem 1. □

Proposition 1. *Let $A, B \in B(H)$ with A self-adjoint. Define*

$$\lambda_1 = \min_{\lambda \in \sigma(A)} \lambda \quad \textit{and} \quad \lambda_2 = \max_{\lambda \in \sigma(A)} \lambda.$$

Then

$$w(AB) \leqslant w(A)w(B) + \frac{\lambda_2 - \lambda_1}{2} d(B). \tag{15}$$

Proof. Since A is self-adjoint, A is hyponormal and

$$R(A) = \frac{\lambda_2 - \lambda_1}{2}.$$

Hence, inequality (15) follows from inequality (14). □

We need the following lemma.

Lemma 3. [12] *Let $A \in B(H)$. Then*

$$w(A) = \sup_{\theta \in \mathbb{R}} \| \operatorname{Re}(e^{i\theta} A)\|.$$

Theorem 3. *Let $A, B \in B(H)$ double commute. Then*

$$w(AB) \leqslant w(A)w(B) + \frac{1}{2} w(AB - BA^*). \tag{16}$$

In particular, if in addition $AB = BA^*$, then

$$w(AB) \leqslant w(A)w(B). \tag{17}$$

Proof. Fix $\theta \in \mathbb{R}$. Since A and $e^{i\theta}B + e^{-i\theta}B^*$ double commute, by (6) we have

$$w\left(A(e^{i\theta}B + e^{-i\theta}B^*)\right) \leqslant w(A)\left\|e^{i\theta}B + e^{-i\theta}B^*\right\|.$$

Hence,

$$\begin{aligned}\left\|\operatorname{Re}\left(e^{i\theta}AB\right)\right\| &= w\left(\operatorname{Re}\left(e^{i\theta}AB\right)\right)\\ &= \frac{1}{2}w\left(e^{i\theta}AB + e^{-i\theta}B^*A^*\right)\\ &= \frac{1}{2}w\left(A(e^{i\theta}B + e^{-i\theta}B^*) + e^{-i\theta}(B^*A^* - AB^*)\right)\\ &\leqslant \frac{1}{2}w\left(A(e^{i\theta}B + e^{-i\theta}B^*)\right) + w\left(e^{-i\theta}(B^*A^* - AB^*)\right)\\ &= \frac{1}{2}w\left(A(e^{i\theta}B + e^{-i\theta}B^*)\right) + w\left(AB - BA^*\right)\\ &\leqslant \frac{1}{2}w(A)\left\|e^{i\theta}B + e^{-i\theta}B^*\right\| + \frac{1}{2}w(AB - BA^*).\end{aligned}$$

Thus

$$\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}AB\right)\right\| \leqslant w(A)\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}B\right)\right\| + \tfrac{1}{2}w(AB - BA^*).$$

By Lemma 3, we get

$$w(AB) \leqslant w(A)w(B) + \frac{1}{2}w(B^*A^* - AB).$$

Inequality (17) follows directly from (16). □

The following lemma is crucial for developing our results.

Lemma 4. *Let K be a compact convex of $\mathbb{C}$. Then there exists $\theta \in [0, \pi]$, such that the projection of $e^{i\theta}K$ on the Ox axis is symmetrical with respect to the origin.*

Proof. Let $P(K)$ the projection of K on the Ox axis. Denote $M(P(K)) = \max P(K)$ and $m(P(K)) = \min P(K)$. We have $M(P(e^{i\pi}K)) = -m(P(K))$ and $m(P(e^{i\pi}K)) = -M(P(K))$. Let $\varphi : [0, \pi] \longrightarrow \mathbb{R}$ be defined by $\varphi(\theta) = M(P(e^{i\theta}K)) + m(P(e^{i\theta}K))$. Since the projection is continuous and K is convex, it follows that φ is continuous. We have $\varphi(0) = -\varphi(\pi)$; there exists $\theta \in [0, \pi]$, such that $\varphi(\theta) = 0$. Thus, the projection of $e^{i\theta}K$ on the Ox axis is symmetrical with respect to the origin. □

Proposition 2. *Let $A \in B(H)$. Then there exists $\theta \in [0, \pi]$, such that*

$$\|\operatorname{Re}(e^{i\theta}A)\| \leqslant \frac{1}{2}\operatorname{diam}(W(T)). \tag{18}$$

Proof. Let $K = \overline{W(A)}$. By Lemma 4, there exists $\theta \in [0, \pi]$, such that the projection of $e^{i\theta}K$ on the Ox axis is symmetrical with respect to the origin. Hence we get

$$\begin{aligned}|\operatorname{Re}(e^{i\theta}K)| &= R(\operatorname{Re}(e^{i\theta}K)) = \frac{1}{2}\operatorname{diam}(\operatorname{Re}(e^{i\theta}K)) \\ &\leqslant \frac{1}{2}\operatorname{diam}(e^{i\theta}K) = \frac{1}{2}\operatorname{diam}(K) \\ &= \frac{1}{2}\operatorname{diam}(W(A)).\end{aligned}$$

Since $\|\operatorname{Re}(e^{i\theta}A)\| = |\operatorname{Re}(e^{i\theta}K)|$, we have

$$\|\operatorname{Re}(e^{i\theta}A)\| \leqslant \frac{1}{2}\operatorname{diam}(W(A)).$$

This completes the proof. □

Theorem 4. *Let $A, B \in B(H)$ double commute. Then there exists $\theta \in [0, \pi]$, such that*

$$\left\|\operatorname{Re}\left(e^{i\theta}AB\right)\right\| \leqslant \frac{1}{2}w(A)\operatorname{diam}(W(B)) + \frac{1}{2}w(AB - BA^*). \tag{19}$$

In particular, if in addition $AB = BA^*$, then

$$\left\|\operatorname{Re}\left(e^{i\theta}AB\right)\right\| \leqslant \frac{1}{2}w(A)\operatorname{diam}(W(B)). \tag{20}$$

Proof. By Theorem 2, there exists $\theta \in [0, \pi]$, such that

$$\|\operatorname{Re}(e^{i\theta}B)\| \leqslant \frac{1}{2}\operatorname{diam}(W(B)).$$

As in the proof of Theorem 34, we get

$$\left\|\operatorname{Re}\left(e^{i\theta}AB\right)\right\| \leqslant \frac{1}{2}w(A)\left\|\operatorname{Re}\left(e^{i\theta}B\right)\right\| + \frac{1}{2}w(AB - BA^*).$$

Hence,

$$\left\|\operatorname{Re}\left(e^{i\theta}AB\right)\right\| \leqslant \frac{1}{2}w(A)\operatorname{diam}(W(B)) + \frac{1}{2}w(AB - BA^*).$$

Inequality (20) follows directly from (19). □

Now, we give a refinement of inequality (3).

Proposition 3. *Let $A \in B(H)$. Then*

$$\|A\| \leqslant \sqrt{c_{\max}^2(A) + d^2(A)}. \tag{21}$$

Proof. Let $\lambda \in W_{\max}(A)$, such that $c_{\max}(A) = |\lambda|$. Hence, there exists a sequence of unit vectors (x_n) that satisfies $\lambda = \lim_{n\to\infty} \langle Ax_n, x_n\rangle$ with $\lim_n \|Ax_n\| = \|A\|$. We have

$$Ax_n = \langle Ax_n, x_n\rangle x_n + \langle Ax_n, y_n\rangle y_n,$$

where $y_n \in \mathbb{S}$ and $\langle x_n, y_n\rangle = 0$. Thus

$$\|Ax_n\|^2 = |\langle Ax_n, x_n\rangle|^2 + |\langle Ax_n, y_n\rangle|^2 \leqslant |\langle Ax_n, x_n\rangle|^2 + d^2(A).$$

Letting n to infinity in the last inequality, we get inequality (21). □

The following Theorem provides a refinement of inequality (9).

Theorem 5. *Let $A, B \in B(H)$ double commute. Then*

$$w(AB) \leqslant \sqrt{c_{\max}^2(A) + d^2(A)}\, w(B). \tag{22}$$

Proof. By inequalities (6) and (21), we infer that

$$\begin{aligned} w(AB) &\leqslant \|A\| w(B) \\ &\leqslant \sqrt{c_{\max}^2(A) + d^2(A)}\, w(B). \end{aligned}$$

The proof is completed. □

Definition 1. *Let $A, B, C \in B(H)$. Set*

$$d_{A,B}(C) = \inf\{\|C - U\| : U \in \Gamma_{A,B}\},$$

where

$$\Gamma_{A,B} = \{\alpha U : U \in B(H) \text{ unitary}: UA = AU, UB = BU \text{ and } \alpha \in \mathbb{C}\}.$$

It is clear that

$$d_{A,B}(C) \leqslant d(C).$$

Now, we give an improvement of inequality (9).

Theorem 6. *Let $A, B \in B(H)$. Then*

$$w(AB) \leqslant (\|A\| + d_{A,B}(A))w(B).$$

Proof. Let $\alpha U \in \Gamma_{A,B}$ with $\alpha \neq 0$. Then, by Theorem 34 and inequality (18), we get

$$\begin{aligned} w(AB) &= w(U^*UAB) \\ &= \frac{1}{|\alpha|^2} w(\alpha U^* A \overline{\alpha} U B) \\ &\leqslant \frac{1}{|\alpha|^2} \left(\|\alpha U^* A\| w(\overline{\alpha} U B) + \frac{1}{2} w(\alpha U^* A \overline{\alpha} U B - \overline{\alpha} U B A^* \overline{\alpha} U) \right) \\ &= \|U^* A\| w(UB) + \frac{1}{2|\alpha|} w(\alpha U^* A U B - U B A^* \overline{\alpha} U) \\ &\leqslant \|A\| w(B) + \frac{1}{2|\alpha|} w((\alpha U^* A - I) U B - U B (\alpha U^* A - I)^*) \\ &\leqslant \|A\| w(B) + \frac{1}{|\alpha|} \|\alpha U^* A - I\| w(UB) \quad \text{(by inequality (11))} \\ &= \|A\| w(B) + \|A - \tfrac{1}{\alpha} U\| w(UB) \\ &= \|A\| w(B) + \|A - \tfrac{1}{\alpha} U\| w(B). \end{aligned}$$

Then

$$w(AB) \leqslant \|A\| w(B) + \|A - \tfrac{1}{\alpha} U\| w(B).$$

Hence,

$$w(AB) \leqslant \|A\| w(B) + \|A - \alpha U\| w(B), \ \forall \alpha \in \mathbb{C}^*. \tag{23}$$

Taking the limit as $\alpha \to 0$ in (23), we obtain

$$w(AB) \leqslant \|A\| w(B) + \|A - 0U\| w(B).$$

It follows that

$$w(AB) \leqslant (\|A\| + d_{A,B}(A))w(B),$$

so the proof is completed. □

Remark 1. *The authors of [7] show that if $A, B \in B(H)$, such that $AB = -BA^*$, then*

$$w(AB) \leqslant d(A)\|B\|.$$

This result is not true. Indeed, let $A = iI$ and $B \neq 0$. Hence, $AB = -BA^*$, but

$$w(AB) = w(B) > 0 = d(A)\|B\|.$$

The same authors in [8] prove that if $A, B \in B(H)$ with $z_0 \in \mathbb{R}$, $d(A) = \|A - z_0 I\|$, then

$$\|AB \pm BA^*\| \leqslant 2d(A)\|B\|. \tag{24}$$

The inequality (24) *is not true. Indeed, let* $A = I$ *and* $B \neq 0$. *Hence* $d(A) = 0 = \|A - z_0 I\|$ *with* $z_0 = 1$ *but*

$$\|AB + BA^*\| = 2\|B\| > 0 = 2d(A)\|B\|.$$

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