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SCHWARZ PROBLEM FOR THE INHOMOGENEOUS POLY-ANALYTIC EQUATION IN A HALF-LENS

Abstract. In this paper, we consider the Schwarz boundary-value problem for the poly-analytic equation in the half-lens domain. First, using the parqueting-reflection principle, we construct the Poly-Schwarz and T-type operators in the half-lens domain, and then we study the properties of these operators. Furthermore, we establish solvability conditions and examine the Schwarz boundary-value problem for the inhomogeneous poly-analytic equation in a half-lens.

Key words: *boundary-value problem, Schwarz problem, poly-analytic equation, half-lens*

2020 Mathematical Subject Classification: *30E25, 31A10, 35C15, 35F15, 35J25, 45E05*

1. Introduction. Boundary-value problems are fundamental in both mathematical analysis and mathematical physics, with significant applications in such areas as complex analysis, the theory of partial differential equation, potential theory, fluid dynamics, wave theory, quantum mechanics, thermodynamics, and electromagnetism [1] – [18]. These problems involve finding solutions to partial differential equations subject to specific boundary conditions on the domain boundary, enabling accurate modeling of physical phenomena. In particular, boundary-value problems for complex partial differential equations focus on the existence, uniqueness, and qualitative behavior of solutions under various boundary conditions applied to different classes of these equations. The Schwarz problem is a classical and important example in this context, arising in complex analysis, potential theory, and mathematical physics, where the goal is to find a solution to a complex partial differential equation that satisfies given boundary conditions on a domain. Polyanalytic equations form an important class of complex partial differential equations that extend the concept

of analytic functions by involving higher-order derivatives with respect to the complex conjugate variable. Analytic functions satisfy the classical Cauchy-Riemann equations, whereas polyanalytic functions generalize this concept by involving higher-order derivatives with respect to the complex conjugate variable, which gives rise to a richer mathematical structure and necessitates more sophisticated analytical techniques. Solving the equations associated with polyanalytic functions often requires advanced techniques, such as the reflection principle and integral operators, making them a rich and active area of research in complex analysis.

In recent years, many researchers have studied boundary-value problems for complex partial differential equations in various domains [1] – [18]. In 2014, Begehr and Vaitekhovich examined the Schwarz problem within a lens domain formed by the intersection of two circles with different radii [3]. In 2024, Darya and Taghizadeh studied the Schwarz boundary-value problem in a lens domain formed by the intersection of two circles with equal radii [4]. In [5], Darya and Taghizadeh solved the Dirichlet problem for the inhomogeneous Poisson equation and the homogeneous Cauchy-Riemann equation in the same domain. Furthermore, in [6], Darya and Taghizadeh investigated the Schwarz problem for the polyanalytic equation in the lens domain. Additionally, in [7], Darya and Taghizadeh studied the Schwarz problem for the poly-analytic equation in the partial eclipse domain.

In this paper, we extend the boundary conditions and investigate the Schwarz boundary-value problem for the inhomogeneous poly-analytic equation in the half-lens domain. First, we introduce the half-lens domain. Then, using the parqueting-reflection principle, we construct a covering for the complex plane. We also derive the Cauchy-Schwarz representation formula using reflection points. This representation formula provides a solution to the Schwarz problem for the Cauchy-Riemann equation in the half-lens domain. In the second section, we define integral operators of the Schwarz-type and the Pompeiu-type in the half-lens domain and study their properties. Finally, we examine the Schwarz boundary-value problem for the Cauchy-Riemann equation. In the third section, we introduce the poly-Schwarz operator and the T-type operator in the half-lens domain and analyze their properties. Finally, we study the Schwarz boundary-value problem for the poly-analytic equation and present an explicit solution.

Let Ω be a half-lens domain in the complex plane, defined by

$$\Omega = \left\{ z \in \mathbb{C} : |z - 1| < \sqrt{2}, |z + 1| < \sqrt{2}, \operatorname{Im} z > 0 \right\},$$

where $C_1 = \{z \in \mathbb{C} : |z - 1| = \sqrt{2}\}$ and $C_2 = \{z \in \mathbb{C} : |z + 1| = \sqrt{2}\}$ are two circles with same radii. The boundary of Ω is denoted by $\partial\Omega$, and the points

$$a_1 = -(\sqrt{2} - 1), \quad a_2 = (\sqrt{2} - 1), \quad a_3 = i$$

are the corner points of Ω . The boundary of this domain consists of three components: two circular arcs, each belonging to one of the intersecting circles and spanning between the intersection points, and one straight line segment that connects these points.

The aim of this paper is to present an explicit solution and solvability conditions for the higher-order Schwarz problem in the half-lens domain. Integral operators are fundamental tools in solving boundary-value problems. The construction of integral operators necessitates efficient techniques, with the parqueting-reflection principle being recognized as one of the most significant and powerful approaches [2]-[12]. The parqueting-reflection principle plays a significant role in constructing integral operators for solving boundary-value problems, especially in domains bounded by circular arcs and straight lines. We reflect a domain across its boundaries to create parqueted structure. The cover can be achieved by a single reflection, several consecutive reflections, or infinitely many repetitive reflections. The points obtained through these reflections are subsequently used to construct integral operators. At this stage, we apply the parqueting-reflection principle to the half-lens domain, resulting in a covering of the complex plane.

Reflecting any z with respect to the real axis gives $z^* = \bar{z}$, and reflecting any z with respect to a circle gives

$$|z - a| = r \Rightarrow (z - a)(\bar{z} - \bar{a}) = r^2 \Rightarrow z^* = \frac{a\bar{z} - a\bar{a} + r^2}{\bar{z} - \bar{a}}.$$

The point $z \in \Omega$ is reflected at C_1 onto $z_1^* = \frac{\bar{z}+1}{\bar{z}-1}$, and both these points are reflected at C_2 onto the points $z_2^* = \frac{-\bar{z}+1}{\bar{z}+1}$, $z_3^* = -\frac{1}{z}$. The reflection of these four points across the real axis is as follows:

$$z_4^* = \bar{z}, \quad z_5^* = \frac{z+1}{z-1}, \quad z_6^* = \frac{-z+1}{z+1}, \quad z_7^* = -\frac{1}{\bar{z}}.$$

These reflections generate a parqueting of the entire complex plane, and the corresponding points will also be essential for constructing integral operators.

Based on Vekua's method [18], if ω is a solution of inhomogeneous Cauchy-Riemann equation, then we have

$$\omega(z) = \phi(z) + T[f](z) = \frac{1}{2\pi i} \int_{\partial\Omega} \omega(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_{\Omega} f(\zeta) \frac{d\bar{\zeta} d\eta}{\zeta - z}$$

where $f \in L_p(\Omega; \mathbb{C})$, $p > 2$, $\zeta = \xi + i\eta$, $\phi(z)$ is analytic, and $T[f](z)$ is the Pompeiu integral operator. When dealing with inhomogeneous Cauchy-Riemann equations or boundary-value problems, such as the Schwarz problem, the Cauchy-Schwarz representation formula provides an explicit integral solution. It is often derived using the Cauchy-Pompeiu formula, which generalizes the Cauchy integral formula to functions that are not necessarily analytic. In the following, we derive the Cauchy-Schwarz representation formula in the half-lens domain.

Theorem 1. *Any function $\omega \in C^1(\Omega; \mathbb{C}) \cap C(\bar{\Omega}; \mathbb{C})$ can be represented as*

$$\begin{aligned} \omega(z) = & \frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} \operatorname{Re} \omega(t) \left[\frac{2}{t-z} + \frac{2z}{tz+1} \right. \\ & \left. + \frac{2(z-1)}{t(z-1)-z-1} + \frac{2(z+1)}{t(z+1)+z-1} \right] dt \\ & + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} \operatorname{Re} \omega(\zeta) \left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{\zeta z+1} - 1 \right. \\ & \left. + \frac{2(z-1)(\zeta-1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta-1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta-1} \\ & + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} \operatorname{Re} \omega(\zeta) \left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{\zeta z+1} - 1 \right. \\ & \left. + \frac{2(z-1)(\zeta+1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta+1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta+1} \\ & + \frac{2}{\pi} \int_{\partial\Omega \cap \partial C_1} \operatorname{Im} \omega(\zeta) \frac{d\zeta}{\zeta-1} + \frac{2}{\pi} \int_{\partial\Omega \cap C_2} \operatorname{Im} \omega(\zeta) \frac{d\zeta}{\zeta+1} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\pi} \int_{\Omega} \left\{ \omega_{\bar{\zeta}}(\zeta) \left[\frac{1}{\zeta - z} + \frac{z}{\zeta z + 1} \right. \right. \\
& \quad \left. \left. + \frac{z + 1}{\zeta(z + 1) - z + 1} + \frac{-z + 1}{\zeta(-z + 1) - z - 1} \right] \right. \\
& \quad \left. + \overline{\omega_{\bar{\zeta}}(\zeta)} \left[-\frac{1}{\bar{\zeta} - z} - \frac{z}{\bar{\zeta} z + 1} \right. \right. \\
& \quad \left. \left. - \frac{z - 1}{\bar{\zeta}(z - 1) + z - 1} - \frac{z + 1}{\bar{\zeta}(z + 1) + z - 1} \right] \right\} d\xi d\eta, \tag{1}
\end{aligned}$$

where $\zeta = \xi + i\eta$.

Proof. The points that are obtained by parqueting-reflection method are $z \in \Omega$ and $z_1^*, z_2^*, z_3^*, z_4^*, z_5^*, z_6^*, z_7^*, \notin \Omega$. So, by the Cauchy–Pompeiu formula [3], [4], [16], we have:

$$\frac{1}{2\pi i} \int_{\partial\Omega} \omega(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_{\Omega} \bar{\omega}_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z} = \begin{cases} \omega(z), & z \in \Omega, \\ 0, & z \notin \Omega. \end{cases} \tag{2}$$

Substituting the points into the Cauchy–Pompeiu formula (2), taking the complex conjugate of the resulting expressions (where $\bar{z}$ appears), and summing eight relations, leads to the Cauchy–Schwarz representation formula. $\square$

Formula (1) presents the solution to the Schwarz boundary-value problem for the inhomogeneous Cauchy–Riemann equation in the half-lens domain.

2. Integral operators for the half-lens. Integral operators play a crucial role in solving boundary-value problems for complex partial differential equations [1], [6], [7], [18]. In this section, we introduce integral operators in the half-lens domain and investigate their properties.

We define **the Schwarz-type operator** on half-lens domain as

$$\begin{aligned}
\delta[\gamma](z) = \frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} \gamma(t) \left[\frac{2}{t - z} + \frac{2z}{tz + 1} \right. \\
\left. + \frac{2(z - 1)}{t(z - 1) - z - 1} + \frac{2(z + 1)}{t(z + 1) + z - 1} \right] dt
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} \gamma(\zeta) \left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{\zeta z+1} - 1 \right. \\
& + \left. \frac{2(z-1)(\zeta-1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta-1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta-1} \\
& + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} \gamma(\zeta) \left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{\zeta z+1} - 1 \right. \\
& + \left. \frac{2(z-1)(\zeta+1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta+1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta+1}, \quad (3)
\end{aligned}$$

where $\gamma \in C(\partial\Omega, \mathbb{R})$, $z \in \Omega$. By the by direct computation,

$$\frac{\partial \delta[\gamma]}{\partial \bar{z}}(z) = 0, \quad z \in \Omega,$$

which imply that the Schwarz-type operator is analytic in the half-lens domain. Next, we discuss boundary behavior of the Schwarz-type operator on half-lens domain.

Theorem 2. *If the given function $\gamma \in C(\partial\Omega; \mathbb{R})$, then*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \delta[\gamma](z) = \gamma(\zeta), \quad \zeta \in \partial\Omega.$$

Proof. To verify the boundary behavior, the real part of $\delta[\gamma](z)$ must be studied. In fact,

$$\begin{aligned}
\operatorname{Re} \delta[\gamma](z) &= \frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} \gamma(t) \left[\frac{1}{t-z} - \frac{1}{t-\bar{z}} - \frac{z}{tz+1} + \frac{\bar{z}}{t\bar{z}+1} \right. \\
& + \frac{z-1}{t(z-1)-z-1} - \frac{\bar{z}-1}{t(\bar{z}-1)-\bar{z}-1} \\
& + \left. \frac{z+1}{t(z+1)+z-1} - \frac{\bar{z}+1}{t(\bar{z}+1)+\bar{z}-1} \right] dt \\
& + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} \gamma(\zeta) \left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta-1)}{\zeta z+1} + \frac{z(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} - 1 \right. \\
& + \left. \frac{(z-1)(\zeta-1)}{\zeta(z-1)-z-1} + \frac{(\bar{z}-1)(\bar{\zeta}-1)}{\bar{\zeta}(\bar{z}-1)-\bar{z}-1} - 1 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(z+1)(\zeta-1)}{\zeta(z+1)+z-1} + \frac{(\bar{z}+1)(\bar{\zeta}-1)}{\bar{\zeta}(\bar{z}+1)+\bar{z}-1} - 1 \Big] \frac{d\zeta}{\zeta-1} \\
& + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} \gamma(\zeta) \left[\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta+1)}{\zeta z+1} + \frac{z(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} - 1 \right. \\
& + \frac{(z-1)(\zeta+1)}{\zeta(z-1)-z-1} + \frac{(\bar{z}-1)(\bar{\zeta}+1)}{\bar{\zeta}(\bar{z}-1)-\bar{z}-1} - 1 \\
& \left. + \frac{(z+1)(\zeta+1)}{\zeta(z+1)+z-1} + \frac{(\bar{z}+1)(\bar{\zeta}+1)}{\bar{\zeta}(\bar{z}+1)+\bar{z}-1} - 1 \right] \frac{d\zeta}{\zeta+1}.
\end{aligned}$$

$\operatorname{Re} \delta[\gamma](z)$ on ∂C_k , $k = 1, 2$, could be written as

$$\begin{aligned}
\operatorname{Re} \delta[\gamma](z) &= \frac{1}{2\pi i} \int_{\partial\Omega \cap C_k} \gamma(\zeta) \left[\frac{\zeta - \varepsilon_k}{\zeta - z} + \frac{\bar{\zeta} - \varepsilon_k}{\bar{\zeta} - \bar{z}} - 1 \right] \frac{d\zeta}{\zeta - \varepsilon_k} \\
&= \frac{1}{2\pi i} \int_{C_k} \Upsilon(\zeta) \left[\frac{\zeta - \varepsilon_k}{\zeta - z} + \frac{\bar{\zeta} - \varepsilon_k}{\bar{\zeta} - \bar{z}} - 1 \right] \frac{d\zeta}{\zeta - \varepsilon_k},
\end{aligned}$$

where

$$\Upsilon(\zeta) = \begin{cases} \gamma(\zeta), & \zeta \in \partial\Omega \cap C_k, \\ 0, & \zeta \in C_k \setminus (\partial\Omega \cap C_k). \end{cases}$$

and $\varepsilon_1 = 1$, $\varepsilon_2 = -1$. By considering the properties of the Poisson kernel for C_k , [10]

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \delta[\gamma](z) = \gamma(\zeta),$$

follows for $\zeta \in \partial\Omega \cap C_k$ up to the corner points a_k and a_3 of the domain Ω , because Υ fails to be continuous there if not accidentally γ vanishes at these points.

Also, $\operatorname{Re} \delta[\gamma](z)$ on $(-\sqrt{2}-1, \sqrt{2}-1)$ could be written as

$$\begin{aligned}
\operatorname{Re} \delta[\gamma](z) &= \frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} \gamma(t) \left[\frac{z - \bar{z}}{|t - z|^2} \right] dt \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Upsilon(t) \left[\frac{z - \bar{z}}{|t - z|^2} \right] dt,
\end{aligned}$$

where

$$\Upsilon(\zeta) = \begin{cases} \gamma(\zeta), & t \in (-(\sqrt{2}-1), \sqrt{2}-1), \\ 0, & t \in \mathbb{R} \setminus (-(\sqrt{2}-1), \sqrt{2}-1). \end{cases}$$

From the properties of the Poisson kernel, we have

$$\lim_{z \rightarrow t} \operatorname{Re} \omega(z) = \gamma(t).$$

We now consider the boundary behavior at the tip a_3 . We represent the constant function 1 as

$$\begin{aligned} 1 = & \frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} \left[\frac{1}{t-z} + \frac{1}{t-\bar{z}} + \frac{z}{tz+1} + \frac{\bar{z}}{t\bar{z}+1} \right. \\ & + \frac{z-1}{t(z-1)-z-1} + \frac{\bar{z}-1}{t(\bar{z}-1)-\bar{z}-1} \\ & + \left. \frac{z+1}{t(z+1)+z-1} + \frac{\bar{z}+1}{t(\bar{z}+1)+\bar{z}-1} \right] dt \\ & + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} \left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta-1)}{\zeta z+1} + \frac{z(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} - 1 \right. \\ & + \frac{(z-1)(\zeta-1)}{\zeta(z-1)-z-1} + \frac{(\bar{z}-1)(\bar{\zeta}-1)}{\bar{\zeta}(\bar{z}-1)-\bar{z}-1} - 1 \\ & + \left. \frac{(z+1)(\zeta-1)}{\zeta(z+1)+z-1} + \frac{(\bar{z}+1)(\bar{\zeta}-1)}{\bar{\zeta}(\bar{z}+1)+\bar{z}-1} - 1 \right] \frac{d\zeta}{\zeta-1} \\ & + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} \left[\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta+1)}{\zeta z+1} + \frac{z(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} - 1 \right. \\ & + \frac{(z-1)(\zeta+1)}{\zeta(z-1)-z-1} + \frac{(\bar{z}-1)(\bar{\zeta}+1)}{\bar{\zeta}(\bar{z}-1)-\bar{z}-1} - 1 \\ & + \left. \frac{(z+1)(\zeta+1)}{\zeta(z+1)+z-1} + \frac{(\bar{z}+1)(\bar{\zeta}+1)}{\bar{\zeta}(\bar{z}+1)+\bar{z}-1} - 1 \right] \frac{d\zeta}{\zeta+1}. \end{aligned}$$

Multiplying this relation by $\gamma(i)$ and subtracting the resulting quantity from $\operatorname{Re} \delta[\gamma](z)$, for $z \in \partial\Omega \cap C_1$, we get

$$\operatorname{Re} \delta[\gamma](z) - \gamma(i) = \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} \tilde{\gamma}(\zeta) \left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1 \right] \frac{d\zeta}{\zeta-1},$$

where $\tilde{\gamma}(\zeta) = \gamma(\zeta) - \gamma(i)$ and $\tilde{\gamma}(i) = 0$. So,

$$\lim_{z \rightarrow i} \operatorname{Re} \delta[\gamma](z) = \gamma(i).$$

Similarly, for $z \in \partial\Omega \cap C_2$,

$$\operatorname{Re} \delta[\gamma](z) - \gamma(i) = \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} \tilde{\gamma}(\zeta) \left[\frac{\zeta + 1}{\zeta - z} + \frac{\bar{\zeta} + 1}{\bar{\zeta} - \bar{z}} - 1 \right] \frac{d\zeta}{\zeta + 1},$$

where $\tilde{\gamma}(\zeta) = \gamma(\zeta) - \gamma(i)$ and $\tilde{\gamma}(i) = 0$. So,

$$\lim_{z \rightarrow i} \operatorname{Re} \delta[\gamma](z) = \gamma(i).$$

By repeating the above trend for a_1 and a_2 repeatedly, the proof is completed. $\square$

The classical Pompeiu operator is a well-known and widely studied object in complex analysis and partial differential equations. Its significance stems from its foundational role in integral representation formulas and its versatility in solving partial differential equations. Next we are going to define Pompeiu operator in the half-lens domain.

We define **the Pompeiu-type operator** for the half-lens domain as

$$\begin{aligned} T_\Omega[f](z) = & -\frac{1}{\pi} \int_{\Omega} \left\{ f(\zeta) \left[\frac{1}{\zeta - z} + \frac{z}{\zeta z + 1} \right. \right. \\ & + \frac{z - 1}{\zeta(z - 1) - z - 1} + \frac{z + 1}{\zeta(z + 1) + z - 1} \Big] \\ & + \overline{f(\zeta)} \left[-\frac{1}{\bar{\zeta} - z} - \frac{z}{\bar{\zeta} z + 1} \right. \\ & \left. \left. - \frac{z - 1}{\bar{\zeta}(z - 1) - z - 1} - \frac{z + 1}{\bar{\zeta}(z + 1) + z - 1} \right] \right\} d\xi d\eta, \end{aligned} \quad (4)$$

where $f \in L_p(\Omega, \mathbb{C})$, $p > 2$, and $\zeta = \xi + i\eta$. By the properties of the classical Pompeiu operator T [15], for $z \in \Omega$, one has

$$\frac{\partial T_\Omega[f](z)}{\partial \bar{z}} = f(z); \quad (5)$$

thus, it remains to verify that the boundary conditions are satisfied.

Theorem 3. If $f \in L_p(\Omega, \mathbb{C})$, $p > 2$, then

$$\{\operatorname{Re} T_\Omega[\gamma]\}(\zeta) = 0, \quad \zeta \in \partial\Omega.$$

Proof. The boundary of the half-lens consists of three distinct components. To study the boundary behavior of the integral, one must perform computations on each segment of the boundary $\partial\Omega$.

For $z \in (-\sqrt{2} - 1, \sqrt{2} - 1)$, i.e., $z = \bar{z}$, we obtain

$$\begin{aligned} & T_\Omega[\gamma](z) \\ &= -\frac{1}{\pi} \int_{\Omega} \left\{ f(\zeta) \left[\frac{1}{\zeta - z} + \frac{z}{\zeta z + 1} + \frac{z - 1}{\zeta(z - 1) - z - 1} + \frac{z + 1}{\zeta(z + 1) + z - 1} \right] \right. \\ & \quad \left. - \overline{f(\zeta)} \left[\frac{1}{\bar{\zeta} - \bar{z}} + \frac{\bar{z}}{\bar{\zeta} \bar{z} + 1} + \frac{\bar{z} - 1}{\bar{\zeta}(\bar{z} - 1) - \bar{z} - 1} + \frac{\bar{z} + 1}{\bar{\zeta}(\bar{z} + 1) + \bar{z} - 1} \right] \right\} d\xi d\eta, \end{aligned}$$

which implies that

$$\{\operatorname{Re} T_\Omega[\gamma]\}(\zeta) = 0, \quad \zeta \in \partial\Omega.$$

Similarly, for $z \in \partial\Omega \cap C_1$, and $z \in \partial\Omega \cap C_2$, we have

$$\{\operatorname{Re} T_\Omega[\gamma]\}(\zeta) = 0, \quad \zeta \in \partial\Omega.$$

This completes the proof. $\square$

In the following, we study the Schwarz boundary-value problem for the inhomogeneous Cauchy–Riemann equation in the half-lens domain.

Theorem 4. *The Schwarz boundary-value problem*

$$\frac{\partial \omega}{\partial \bar{z}} = f \quad \text{in } \Omega, \quad \operatorname{Re} \omega = \gamma \quad \text{on } \partial\Omega, \quad (6)$$

$$\frac{2}{\pi i} \int_{\partial\Omega \cap C_1} \operatorname{Im} \omega(\zeta) \frac{d\zeta}{\zeta - 1} + \frac{2}{\pi i} \int_{\partial\Omega \cap C_2} \operatorname{Im} \omega(\zeta) \frac{d\zeta}{\zeta + 1} = c, \quad (7)$$

with given $f \in L_p(\Omega; \mathbb{C})$, $p > 2$, $\gamma \in C(\partial\Omega, \mathbb{C})$, $c \in \mathbb{R}$, is uniquely solved by

$$\omega(z) = \delta[\gamma](z) + ic + T_\Omega[f](z), \quad (8)$$

where $\delta[\gamma](z)$, $T_\Omega[f](z)$ are the Schwarz-type operator and the Pompeiu-type operator, respectively.

Proof. Using the properties of the Schwartz-type and Pompeiu-type operators, and applying Theorems 2 and 3, we see that the function $\Gamma(z) = \delta[\gamma](z) + ic + T_\Omega[f](z)$ satisfies condition (13). Let $\Phi(z) = \omega(z) - \Gamma(z)$; then $\Phi(z)$ satisfies

$$\frac{\partial \Phi}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \text{Re } \Phi = 0 \quad \text{on } \partial\Omega. \quad (9)$$

Then, from Theorem 1, we know that $\Phi(z) = ic$. Thus, the proof is completed. $\square$

3. Poly-Schwarz operator and T-type operator for the half-lens.

Poly-Schwarz operators play a crucial role in solving boundary-value problems for polyanalytic equations. In the following, we introduce the poly-Schwarz operator associated with the half-lens domain and investigate its properties. For $n \in \mathbb{N}$, we define the **the poly-Schwarz operator** S_n on the half-lens domain as

$$\begin{aligned} S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}](z) &= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \left(\frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} (t - z + \overline{t - z})^k \gamma_k(t) \left[\frac{2}{t - z} + \frac{2z}{tz + 1} \right. \right. \\ &\quad \left. \left. + \frac{2(z-1)}{t(z-1) - z - 1} + \frac{2(z+1)}{t(z+1) + z - 1} \right] dt \right. \\ &\quad + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} (\zeta - z + \overline{\zeta - z})^k \gamma_k(\zeta) \left[\frac{2(\zeta-1)}{\zeta - z} - 1 + \frac{2z(\zeta-1)}{tz + 1} - 1 \right. \\ &\quad \left. + \frac{2(z-1)(\zeta-1)}{\zeta(z-1) - z - 1} - 1 + \frac{2(z+1)(\zeta-1)}{\zeta(z+1) + z - 1} - 1 \right] \frac{d\zeta}{\zeta - 1} \\ &\quad + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} (\zeta - z + \overline{\zeta - z})^k \gamma_k(\zeta) \left[\frac{2(\zeta+1)}{\zeta - z} - 1 + \frac{2z(\zeta+1)}{tz + 1} - 1 \right. \\ &\quad \left. + \frac{2(z+1)(\zeta-1)}{\zeta(z-1) - z - 1} - 1 + \frac{2(z+1)(\zeta+1)}{\zeta(z+1) + z - 1} - 1 \right] \frac{d\zeta}{\zeta + 1} \Bigg), \quad z \in \Omega, \quad (10) \end{aligned}$$

with $\gamma_0, \gamma_1, \dots, \gamma_n \in C(\partial\Omega; \mathbb{R})$. Then $S_1 = \delta$, where δ is the Schwarz-type operator defined by (5). Since

$$S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}](z)$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \left(\frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} \sum_{l=0}^k \binom{k}{l} (2t)^l (-z - \bar{z})^{k-l} \gamma_k(\zeta) \left[\frac{2}{t-z} \right. \right. \\
&\quad \left. \left. + \frac{2z}{tz+1} + \frac{2(z-1)}{t(z-1)-z-1} + \frac{2(z+1)}{t(z+1)+z-1} \right] dt \right. \\
&\quad \left. + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} \sum_{l=0}^k \binom{k}{l} (\zeta + \bar{\zeta})^l (-z - \bar{z})^{k-l} \gamma_k(\zeta) \left[\frac{2(\zeta-1)}{\zeta-z} - 1 \right. \right. \\
&\quad \left. \left. + \frac{2z(\zeta-1)}{tz+1} - 1 + \frac{2(z-1)(\zeta-1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta-1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta-1} \right. \\
&\quad \left. + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} \sum_{l=0}^k \binom{k}{l} (\zeta + \bar{\zeta})^l (-z - \bar{z})^{k-l} \gamma_k(\zeta) \left[\frac{2(\zeta+1)}{\zeta-z} - 1 \right. \right. \\
&\quad \left. \left. + \frac{2z(\zeta+1)}{tz+1} - 1 + \frac{2(z-1)(\zeta+1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta+1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta+1}, \right.
\end{aligned}$$

we have

$$S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}](z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{l=0}^k \binom{k}{l} (-z - \bar{z})^{k-l} S[\hat{\gamma}_{k,l}](z), \quad z \in \Omega, \quad (11)$$

with $\hat{\gamma}_{k,l}(\zeta) = (\zeta + \bar{\zeta})^l \gamma_k(\zeta)$, $\zeta \in \partial\Omega$ for $k = 0, 1, 2, \dots, n-1$ and $l = 0, 1, 2, \dots, k$.

Next we investigate the boundary behavior of the poly-Schwarz operator.

Theorem 5. *If the given functions $\gamma_0, \gamma_1, \dots, \gamma_{n-1} \in C(\partial\Omega, \mathbb{R})$, then*

$$\lim_{z \rightarrow \zeta} \left\{ \operatorname{Re} \frac{\partial^l S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}]}{\partial \bar{z}^l} \right\} (z) = \gamma_l(\zeta), \quad \zeta \in \partial\Omega, \quad l = 0, 1, 2, \dots, n-1. \quad (12)$$

Proof. Let $l = 0$; then, from Theorem 2 and (11), we have

$$\lim_{z \rightarrow \zeta} \{ \operatorname{Re} S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}] \} (z)$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{l=0}^k \binom{k}{l} (-\zeta - \bar{\zeta})^{k-l} \hat{\gamma}_{k,l}(\zeta) = \gamma_0(\zeta), \quad \zeta \in \partial\Omega, \quad (13)$$

and, hence, (12) is true. Let $l > 0$; then

$$\begin{aligned} & \frac{\partial^l S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}]}{\partial \bar{z}^l}(z) \\ &= \sum_{k=l}^{n-1} \frac{(-1)^{k-l}}{(k-l)!} \left(\frac{1}{2\pi i} \int_{-(\sqrt{2}-1)}^{\sqrt{2}-1} (t-z+\overline{t-z})^{k-l} \gamma_k(t) \left[\frac{2}{t-z} + \frac{2z}{tz+1} \right. \right. \\ & \quad \left. \left. + \frac{2(z-1)}{t(z-1)-z-1} + \frac{2(z+1)}{t(z+1)+z-1} \right] dt \right. \\ & \quad \left. + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_1} (\zeta-z+\overline{\zeta-z})^{k-l} \gamma_k(\zeta) \left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{tz+1} - 1 \right. \right. \\ & \quad \left. \left. + \frac{2(z-1)(\zeta-1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta-1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta-1} \right. \\ & \quad \left. + \frac{1}{2\pi i} \int_{\partial\Omega \cap C_2} (\zeta-z+\overline{\zeta-z})^{k-l} \gamma_k(\zeta) \left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{tz+1} - 1 \right. \right. \\ & \quad \left. \left. + \frac{2(z+1)(\zeta-1)}{\zeta(z-1)-z-1} - 1 + \frac{2(z+1)(\zeta+1)}{\zeta(z+1)+z-1} - 1 \right] \frac{d\zeta}{\zeta+1} \right), \quad z \in \Omega, \quad (14) \end{aligned}$$

Hence, we similarly get (12). $\square$

One of the important properties of the Poly-Schwarz operator in solving the Schwarz boundary-value problem is its differentiability property. Next, we examine the differentiability property of the Poly-Schwarz operator.

Theorem 6. *If the given functions $\gamma_0, \gamma_1, \dots, \gamma_{n-1} \in C(\partial\Omega, \mathbb{R})$, then*

$$\frac{\partial^n S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}](z)}{\partial \bar{z}^n} = 0, \quad z \in \Omega. \quad (15)$$

Proof. The Schwarz-type operator is analytic in the half-lens domain. Thus, by (11), we obtain

$$\frac{\partial^n S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}](z)}{\partial \bar{z}^n}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{l=0}^k \binom{k}{l} \left(\frac{\partial^n (-z - \bar{z})^{k-l}}{\partial \bar{z}^n} \right) S[\hat{\gamma}_{k,l}](z) = 0. \quad (16)$$

Therefore, the proof of Theorem 6 is completed. $\square$

The T-type operator is indeed an extension of the Pompeiu operator. In the following, we introduce this operator in the half-lens domain and investigate its properties. We define the **T-type operator** as

$$\begin{aligned} T_l[f](z) = & \frac{(-1)^l}{\pi(l-1)!} \int_{\Omega} (\zeta - z + \overline{\zeta - z})^{l-1} \left\{ f(\zeta) \left[\frac{1}{\zeta - z} + \frac{z}{\zeta z + 1} \right. \right. \\ & + \frac{z+1}{\zeta(z+1) - z + 1} + \frac{-z+1}{\zeta(-z+1) - z - 1} \Big] \\ & + \overline{f(\zeta)} \left[-\frac{1}{\bar{\zeta} - z} - \frac{z}{\bar{\zeta} z + 1} \right. \\ & \left. \left. - \frac{z-1}{\bar{\zeta}(z-1) + z - 1} - \frac{z+1}{\bar{\zeta}(z+1) + z - 1} \right] \right\} d\xi d\eta, \end{aligned} \quad (17)$$

where $z \in \Omega$, $l = 1, 2, 3, \dots$, $f \in L_p(\Omega; \mathbb{C})$, $p > 2$, and $T_1[f](z) = T_{\Omega}[f](z)$, which satisfies

$$\frac{\partial T_1[f](z)}{\partial \bar{z}} = f(z). \quad (18)$$

Let $T_0[f](z) = f(z)$, $z \in \Omega$; then (18) is equivalent to

$$\frac{\partial T_1[f](z)}{\partial \bar{z}} = T_0[f](z), \quad z \in \Omega. \quad (19)$$

Next, we investigate an important property of the T-type operator in the half-lens domain.

Theorem 7. *If $f \in L_p(\Omega; \mathbb{C})$, $p > 2$, then*

$$\frac{\partial T_l[f](z)}{\partial \bar{z}} = T_{l-1}[f](z), \quad z \in \Omega, \quad l = 1, 2, \dots \quad (20)$$

Proof. If we assume that $l = 1$, then (20) is equivalent to (19). For $l > 1$, we have

$$T_l[f](z) = \frac{(-1)^{l-1}}{(l-1)!} \sum_{k=0}^{l-1} \binom{l-1}{k} (-z - \bar{z})^{l-k-1} T_1[f_k](z), \quad z \in \Omega, \quad (21)$$

with

$$f_k(\zeta) = (\zeta + \bar{\zeta})^k f(\zeta), \quad k = 0, 1, 2, \dots, l-1.$$

Hence,

$$\begin{aligned} \frac{\partial T_l[f](z)}{\partial \bar{z}} &= \frac{(-1)^{l-1}}{(l-1)!} \sum_{k=0}^{l-1} \binom{l-1}{k} \left\{ \frac{\partial (-z - \bar{z})^{l-k-1}}{\partial \bar{z}} T_1[f_k](z) \right. \\ &\quad \left. + (-z - \bar{z})^{l-k-1} \frac{\partial T_1[f_k](z)}{\partial \bar{z}} \right\} \\ &= \frac{(-1)^{l-2}}{(l-2)!} \sum_{k=0}^{l-2} \binom{l-2}{k} (-z - \bar{z})^{l-k-2} T_1[f_k](z) \\ &= T_{l-1}[f](z). \end{aligned}$$

Therefore, the proof is complete. $\square$

Next we check the boundary behavior of the T-type operator in Ω .

Theorem 8. *If $f \in L_p(\Omega; \mathbb{C})$, $p > 2$, then*

$$\lim_{z \rightarrow \zeta, z \in \Omega} \operatorname{Re} T_l[f](z) = 0, \quad \zeta \in \partial\Omega, \quad l = 1, 2, \dots \quad (22)$$

Proof. From (21), we have

$$\operatorname{Re} \{T_l[f](z)\} = \frac{(-1)^{l-1}}{(l-1)!} \sum_{k=0}^{l-1} \binom{l-1}{k} (-z - \bar{z})^{l-k-1} \operatorname{Re} \{T_1[f_k](z)\}, \quad z \in \Omega. \quad (23)$$

According to Theorem 3, we obtain

$$\lim_{z \rightarrow \zeta, z \in \Omega} \operatorname{Re} T_1[f_k(z)] = 0, \quad \zeta \in \partial\Omega. \quad (24)$$

Thus, from (23) and (24), we have

$$\lim_{z \rightarrow \zeta, z \in \Omega} \operatorname{Re} T_l[f](z) = 0, \quad \zeta \in \partial\Omega, \quad l = 1, 2, \dots \quad (25)$$

This completes the proof. $\square$

4. Schwarz problem for poly-analytic equation in the half-lens. In this section, we examine the Schwarz boundary-value problem

for the poly-analytic equation in the half-lens domain. Using the properties of the poly-Schwarz operator and the T-type operator, we discuss the Schwarz boundary-value problem for the poly-analytic equation and explicitly obtain its solution.

Theorem 9. *The Schwarz boundary-value problem for the poly-analytic function*

$$\begin{cases} (\partial_{\bar{z}}^n \omega)(z) = 0, & \text{in } \Omega, \\ \{\operatorname{Re}(\partial_{\bar{z}}^k \omega)\}(\zeta) = 0, & \text{on } \partial\Omega, \ k = 0, 1, 2, \dots, n-1, \end{cases} \quad (26)$$

is solvable and its solution can be written as

$$\omega(z) = \sum_{k=0}^{n-1} \frac{(z + \bar{z})^k}{k!} i c_k, \quad (27)$$

with $c_k \in \mathbb{R}$, for $k = 0, 1, 2, \dots, n-1$.

Proof. Suppose that $\partial_{\bar{z}}^n \omega(z) = 0$; then ω is a poly-analytic function. Based on results in [1], the poly-analytic function ω can be written as follows:

$$\omega(z) = \sum_{k=0}^{n-1} \frac{(z + \bar{z})^k}{k!} \psi_k(z), \quad z \in M, \quad (28)$$

where $\psi_k(z)$ is analytic on Ω . The boundary conditions in (26) are equivalent to

$$\lim_{z \rightarrow \zeta, z \in \Omega} \operatorname{Re} \psi_k(z) = 0, \quad \zeta \in \partial\Omega, \quad k = 0, 1, 2, \dots, n-1. \quad (29)$$

Thus, by Theorem 4, we have

$$\psi_k(z) = i c_k, \quad c_k \in \mathbb{R}. \quad (30)$$

This completes the proof of Theorem 9. $\square$

Next, using the Schwarz problem for poly-analytic function, we solve the Schwarz boundary-value problem for the inhomogeneous poly-analytic equation in the half-lens domain.

Theorem 10. *The Schwarz problem for the inhomogeneous poly-analytic equation in the half-lens domain*

$$\begin{cases} (\partial_{\bar{z}}^n \omega)(z) = f(z), & z \in \Omega, \quad f \in L_p(\Omega; \mathbb{C}), \quad p > 2, \\ \operatorname{Re}(\partial_{\bar{z}}^k \omega)(\zeta) = \gamma_k(\zeta), & \zeta \in \partial\Omega, \quad \gamma_k \in C(\partial\Omega; \mathbb{C}), \\ \operatorname{Im} \partial_{\bar{z}}^k \omega(\zeta) = c_k, & \zeta \in \partial\Omega, \quad k = 1, 2, \dots, n-1, \end{cases} \quad (31)$$

is solvable and its solution can be represented as

$$\omega(z) = S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}](z) + T_n[f](z) + \sum_{k=0}^{n-1} \frac{(z + \bar{z})^k}{k!} i c_k, \quad (32)$$

where $c_k \in \mathbb{R}$, for $k = 0, 1, 2, \dots, n-1$ and the operators S_n, T_n are defined by (10) and (17), respectively.

Proof. Let us denote

$$\omega_0(z) = S_n[\gamma_0, \gamma_1, \dots, \gamma_{n-1}](z) + T_n[f](z), \quad z \in \Omega, \quad (33)$$

where the operators S_n, T_n are defined by (10) and (17), respectively. From Theorem 6, and Theorem 7, we obtain

$$\frac{\partial^n \omega_0(z)}{\partial \bar{z}^n} = f(z), \quad z \in \Omega. \quad (34)$$

By Theorem 5 and Theorem 8, we also have

$$\left\{ \operatorname{Re} \frac{\partial^k \omega_0}{\partial \bar{z}^k} \right\}(\zeta) = \gamma_k(\zeta), \quad \zeta \in \partial\Omega, \quad k = 0, 1, \dots, n-1. \quad (35)$$

Thus, ω_0 is a special solution to the Schwarz problem for the poly-analytic equation (31). Now let

$$\omega(z) = \omega_0(z) + \psi(z). \quad (36)$$

Taking (36) into the equation of (31),

$$\begin{cases} (\partial_{\bar{z}}^n \psi)(z) = 0, & \text{in } \Omega, \\ \{\operatorname{Re}(\partial_{\bar{z}}^k \psi)\}(\zeta) = 0, & \text{on } \partial\Omega, k = 0, 1, 2, \dots, n-1. \end{cases} \quad (37)$$

which is just the Schwarz boundary-value problem for the poly-analytic function (26). Based on Theorem 9,

$$\psi(z) = \sum_{k=0}^{n-1} \frac{(z + \bar{z})^k}{k!} i c_k, \quad (38)$$

with $c_k \in \mathbb{R}$, for $k = 0, 1, \dots, n - 1$. Hence, the proof is completed. $\square$

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