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ON SOME CLASS OF FUNCTIONS WITH AN INTEGRAL REPRESENTATION INVOLVING COMPLEX MEASURES

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In this paper we obtain some properties of functions extremal with respect to Fréchet-differentiable functionals defined on $\mathcal{P}'_\alpha$ (see deffinition 1) and, in consequence, estimates of the functional $\mathrm{Re}\{e^{i\lambda}p(z)\}$, $0 \neq z \in K$, $\lambda \in [-\pi, \pi)$, $p \in \mathcal{P}'_\alpha$.

§ 1. Introduction

Let $\mathcal{P}$ denote the well-known class of all functions of the form

$$p(z) = 1 + a_1 z + \ldots + a_k z^k + \ldots \tag{1}$$

holomorphic and satisfying the condition $\mathrm{Re}\, p(z) > 0$ in the disc $K = \{z \in \mathbf{C} : |\mathbf{z}| < \mathbf{1}\}$. As is known (e.g. [5], p. 4), a function $p \in \mathcal{P}$ if and only if

$$p(z) = \int_0^{2\pi} P(e^{-it}, z)\, d\mu(t), \quad z \in K, \tag{2}$$

where

$$P(\varepsilon, z) = \frac{1 + \varepsilon z}{1 - \varepsilon z}, \quad |\varepsilon| = 1, \; z \in K, \tag{3}$$

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$\mu \in M$, $M = \{\mu : \mu$ is a nondecreasing real function defined on the interval $[0, 2\pi]$, and $\int_0^{2\pi} d\mu(t) = 1\}$.

V. Starkov ([9], see also [6]), introduced the class $\mathcal{U}'_\alpha$, $\alpha \geq 1$ of functions of the form

$$f'(z) = \exp\left[-2\int_0^{2\pi} \mathrm{Log}\,(1 - e^{-it}z)d\mu(t)\right], \quad z \in K, \tag{4}$$

where $\mu \in I_\alpha$. I_α denotes the family of all complex functions μ with bounded variation, satisfying the condition

$$\left|\int_0^{2\pi} d\mu(t) - 1\right| + \int_0^{2\pi} |d\mu(t)| \leq \alpha. \tag{5}$$

The classes $\mathcal{U}'_\alpha$ appear in a natural way in the question of approximating the derivatives of functions of a universal linearly invariant family of order α by powers of the derivatives of convex functions (see [4], [9]).

Of course, if in (4) $\mu \in M$, we obtain the class S^c of convex univalent functions. I_1 is a family of nondecreasing real functions such that $\int_0^{2\pi} d\mu(t) \leq 1$.

Definition 1 *Let $\mathcal{P}'_\alpha$, $\alpha \geq 1$, denote the class of functions given by formula (2) where μ are elements of the class I_α.*

Evidently $\mathcal{P} \subset \mathcal{P}'_1$. The class $\mathcal{P}'_\alpha$ was introduced and its basic properties were studied in [3] (see also [1], [2]). In particular, we obtain

THEOREM 1. *The set of functions p of form (2), generated by piecewise constant functions $\mu \in I_\alpha$, is dense in $\mathcal{P}'_\alpha$.*

It has also been shown that the class $\mathcal{P}'_\alpha$ is compact in the topology of almost uniform convergence in K, convex and connected.

THEOREM 2. ([3]) *Let $\{p\}_k$, $k = 0, 1, \ldots$, denote the k-th coefficient of the function p. If $p \in \mathcal{P}'_\alpha$, $\alpha \geq 1$, then the set V_k of values of the functional $H(p) = \{p\}_k$, $k = 1, 2, \ldots$, is the closed disc with centre at the point 0 and with radius 2α. If $\alpha > 1$, then the set V_0 of values of the coefficient $\{p\}_0$ is the ellipse*

$$\frac{\left(\mathrm{Re}\,A - \frac{1}{2}\right)^2}{\frac{\alpha^2}{4}} + \frac{(\mathrm{Im}\,A)^2}{\frac{\alpha^2 - 1}{4}} \leq 1. \tag{6}$$

If $\alpha = 1$, *then* $V_0 = [0,1]$.

THEOREM 3. ([3]) *If* $p \in \mathcal{P}'_\alpha$, $\alpha \geq 1$, *then*

$$\frac{1+r}{1-r} \cdot \frac{1-\alpha}{2} \leq \operatorname{Re} p(z) \leq \frac{1+r}{1-r} \cdot \frac{1+\alpha}{2}, \quad |z| = r, \ z \in K. \tag{7}$$

Estimate (7) is sharp.

In the proof of Theorem 2, use was made of the definition and the elementary properties of the class $\mathcal{P}'_\alpha$ and, in particular, of conditions (2), (3), (5). To prove Theorem 3, we use Theorem 1, condition (5) as well as certain classical inequalities.

There arises a natural question concerning the possibility of obtaining a general characterization of functions extremal with respect to a rather wide class of functionals defined on the class $\mathcal{P}'_\alpha$.

§ 2. General properties of extremal functions

Let $p \in \mathcal{P}'_{\alpha_0}$, $\alpha_0 > 1$. Of course, the function μ corresponding to p belongs to I_{α_0}. From (5) it follows that there can exist an α, $1 \leq \alpha < \alpha_0$, such that $\mu \in I_\alpha$. The best characterization of the function μ and, consequently, of p is given by the number α_* for which

$$\left| \int_0^{2\pi} d\mu(t) - 1 \right| + \int_0^{2\pi} |d\mu(t)| = \alpha_*.$$

Definition 2 *Let* $p \in \mathcal{P}'_{\alpha_0}$. *The number* $\alpha_* \leq \alpha_0$ *such that* $p \in \mathcal{P}'_{\alpha_*}$ *and* $p \notin \mathcal{P}'_{\alpha_* - \varepsilon}$ *for any* $\varepsilon > 0$ *is called a degree of the function* p. *The degree of the function* p *is denoted by* $\deg p$.

Since $\mathcal{P'}_{\alpha_1} \subset \mathcal{P'}_{\alpha_2}$ if $1 \leq \alpha_1 \leq \alpha_2$, we have

Property 1 *If* $\deg p = \alpha_*$, $\alpha_* > 1$, *then* $p \in \mathcal{P}'_\alpha$ *for* $\alpha \geq \alpha_*$ *and* $p \notin \mathcal{P}'_\alpha$ *for* $1 \leq \alpha < \alpha_*$. *If* $\deg p = 1$, *then* $p \in \mathcal{P}'_\alpha$ *for* $\alpha \geq 1$.

We shall prove

THEOREM 4. *Let* F *be a Fréchet-differentiable functional defined on* $\mathcal{P}'_{\alpha_0}$, L_p *its differential at the point* p, *and* p_0 *the extremal function for the problem*

$$\max_{p \in \mathcal{P}'_{\alpha_0}} \operatorname{Re}\{F(p^{(n)})\}, \qquad 1 \leq \alpha_0 < \infty, \quad n = 0, 1, 2, \ldots. \tag{8}$$

If there exists $k > n$ *such that* $L_{p_0^{(n)}}(z^{k-n}) \neq 0$, *then* $\deg p_0 = \alpha_0$.

PROOF. From Definition 2 it is evident that $\deg p_0 \leq \alpha_0$. Assume that $\deg p_0 = \alpha$ and $\alpha < \alpha_0$. Let us consider the function

$$p_h(z) = p_0(z) + khz^k + k|h| = p_0(z) + k|h|(1 + e^{i \arg h} z^k)$$

where $k \in \mathbf{N}$ and $h \neq 0$ is a sufficiently small complex number. Note that $1 + \sigma z^k$, $|\sigma| = 1$, is a function of the class $\mathcal{P}$ for any $k \in \mathbf{N}$. So, we have

$$p_h(z) = \int_0^{2\pi} P(e^{-it}, z) d\mu_0(t) + k|h| \int_0^{2\pi} P(e^{-it}, z) d\nu(t),$$

$\mu_0 \in I_\alpha$, $\nu \in M$, and P is defined by formula (3). Consequently,

$$p_h(z) = \int_0^{2\pi} P(e^{-it}, z) d(\mu_0(t) + k|h|\nu(t)).$$

We shall demonstrate that the function $p_h \in \mathcal{P}'_{\alpha_0}$ for $|h| \leq \frac{\alpha_0 - \alpha}{2k}$. Indeed, condition (5) is satisfied because

$$\begin{aligned}
&\left| \int_0^{2\pi} d(\mu_0(t) + k|h|\nu(t)) - 1 \right| + \int_0^{2\pi} |d(\mu_0(t) + k|h|\nu(t))| \\
&= \left| \int_0^{2\pi} d\mu_0(t) + k|h| \int_0^{2\pi} d\nu(t) - 1 \right| + \int_0^{2\pi} |d(\mu_0(t) + k|h|\nu(t))| \\
&\leq \left| \int_0^{2\pi} d\mu_0(t) - 1 \right| + k|h| + \int_0^{2\pi} |d\mu_0(t)| + k|h| \\
&\leq \alpha + 2k|h| \leq \alpha + 2k \frac{\alpha_0 - \alpha}{2k} = \alpha_0.
\end{aligned}$$

The derivative of order n of the function p_h is expressed by the formula

$$p_h^{(n)}(z) = p_0^{(n)}(z) + hk^2(k-1)\ldots(k-n+1)z^{k-n}, \quad z \in K.$$

Calculating the value of the functional F at the point $p_h^{(n)}$, we get

$$\begin{aligned} F(p_h^{(n)}) &= F(p_0^{(n)} + hk^2(k-1)\dots(k-n+1)z^{k-n}) \\ &= F(p_0^{(n)}) + L_{p_0^{(n)}}(hk^2(k-1)\dots(k-n+1)z^{k-n}) + o(|h|) \\ &= F(p_0^{(n)}) + hk^2(k-1)\dots(k-n+1)L_{p_0^{(n)}}(z^{k-n}) + o(|h|) \end{aligned}$$

where $\lim_{|h|\to 0} \frac{o(|h|)}{|h|} = 0$. By the assumption, the function p_0 is extremal for problem (8), therefore, for all $k > n$,

$$L_{p_0^{(n)}}(z^{k-n}) = 0$$

must take place, which contradicts the assumption. Thereby, the theorem has been proved.

To estimate some functional defined on the family $\mathcal{P}'_\alpha$, we shall make use of the method described by V. Starkov in paper [6] (compare also [7]—[9]).

Let G_α be the class of functions of the form

$$\varphi(z) = \int_0^{2\pi} g(z,t)d\mu(t), \quad z \in K,$$

where $\mu \in I_\alpha$, $g(z,t)$ is a fixed function holomorphic with respect to z in the disc K and 2π-periodical and of the class C' with respect to t. The family G_α is compact in the topology of almost uniform convergence in the disc K.

Let F be a Fréchet-differentiable functional defined on the set B described above. Consider the problem

$$\max_{\varphi\in G_\alpha} \operatorname{Re}\{F(\varphi)\} \tag{9}$$

and denote by $\varphi_0(z) = \int_0^{2\pi} g(z,t)d\mu_0(t)$ an extremal function for (9) (not necessarily the only one).

Denote by $I_\alpha(n)$ a subset of the family I_α of piecewise constant functions which have not more than n points of discontinuity. Let us also define a suitable subset of the family G_α, $G_\alpha(n,\mu_0) = \{\varphi \in G_\alpha : \varphi(z) = \int_0^{2\pi} g(z,t)d\mu(t), \ \ \mu \in I_\alpha(n), \ \ \int_0^{2\pi} d\mu(t) = \int_0^{2\pi} d\mu_0(t)\}$. The class $G_\alpha(n,\mu_0)$ is compact, too.

Let us consider the following problem:

$$\max_{\varphi \in G_\alpha(n,\mu_0)} \operatorname{Re}\{F(\varphi)\} \tag{10}$$

and let $\varphi_n(z) = \int_0^{2\pi} g(z,t)d\mu_n(t)$ denote an extremal function for (10). From the sequence (φ_n) one can choose a subsequence almost uniformly convergent in K to the function $\varphi^{(0)} \in G_\alpha$, with that $\varphi^{(0)}$ is an extremal function for problem (9). In order to get any information about the extremal functions in the full class G_α, we may first consider analogous problems in the classes $G_\alpha(n,\mu_0)$.

Let, for a fixed $n \in \mathbf{N}$, problem (10) be given and let t_j, $j = 1,\ldots,k$, $k \leq n$, be points of discontinuity of the function μ_n. Denote $\arg d\mu_n(t_j) = \Theta_j$. If $k \geq 2$ and if there exist at least two different values of Θ_j, then the points $L_{\varphi_n}[g(z,t_j)]$ lie on the circle with centre c_n and radius s_n. L_{φ_n} denotes here a differential of the functional F at the point φ_n. If, moreover, $s_n > 0$, then

$$\begin{cases} |L_{\varphi_n}[g(z,t_i)] - c_n|^2 = |L_{\varphi_n}[g(z,t_j)] - c_n|^2 & \text{for i,j} = 1,\ldots,\text{k}, \\ (|L_{\varphi_n}[g(z,t)] - c_n|^2)'_t\Big|_{t=t_j} = 0 & \text{for j} = 1,\ldots,\text{k}. \end{cases} \tag{11}$$

In the above case, the equalities

$$(L_{\varphi_n}[g(z,t_j)] - c_n)e^{i\Theta_j} = \pm s_n, \quad j = 1,\ldots,k, \tag{12}$$

are true, too, with that the sign preceding s_n is the same for all j's.

As $s_n = 0$, we get

$$\begin{cases} L_{\varphi_n}[g(z,t_i)] = L_{\varphi_n}[g(z,t_j)] & \text{for i,j} = 1,\ldots,\text{k}, \\ \operatorname{Re}\{e^{i\Theta} L_{\varphi_n}[g'_t(z,t_j)]\} = 0 & \text{for j} = 1,\ldots,\text{k}. \end{cases} \tag{13}$$

Whereas if at all points t_j of discontinuity of the function μ_n we have $\arg d\mu_n(t_j) = \Theta$, then

$$\begin{cases} \operatorname{Re}\{e^{i\Theta} L_{\varphi_n}[g'_t(z,t_j)]\} = 0 & \text{for j} = 1,\ldots,\text{k}, \\ \operatorname{Re}\{e^{i\Theta}(L_{\varphi_n}[g(z,t_i)] - L_{\varphi_n}[g(z,t_j)])\} = 0 & \text{for i,j} = 1,\ldots,\text{k}, \end{cases} \tag{14}$$

with that the first of equalities (14) is true at each point t_j in all the cases under consideration.

We shall give a simple application of Theorem 4, exemplified by the following problem.

Let $p \in \mathcal{P}'_\alpha$, $\alpha \geq 1$, and let $\{p\}_k$, $k = 0, 1, \ldots$, denote, as before, the k-th coefficient of an expansion of the function p in the power series with centre at the point $z = 0$.

Consider the problem

$$\max_{p \in \mathcal{P}'_\alpha} |\{p\}_k| \quad \text{for } k = 1, 2, \ldots. \tag{15}$$

LEMMA 1. *If* $p_0(z) = \int_0^{2\pi} P(e^{-it}, z) d\mu_0(t)$, $z \in K$, *and* P *defined by formula* (3) *is an extremal function for problem* (15), *then* $\int_0^{2\pi} d\mu_0(t) = 1$ *and* $\int_0^{2\pi} |d\mu_0(t)| = \alpha$.

PROOF. Since, for $p \in \mathcal{P}'_\alpha$, $\Theta \in \mathbf{R}$, $p(e^{i\Theta} z) \in \mathcal{P}'_\alpha$, therefore problem (15) is equivalent to

$$\max_{p \in \mathcal{P}'_\alpha} \operatorname{Re} \{p\}_k \quad \text{for } k = 1, 2, \ldots. \tag{16}$$

Consider the function

$$p_\Lambda(z) = \int_0^{2\pi} P(e^{-it}, z) d\mu_\Lambda(t), \quad 0 \leq \Lambda \leq 1, \quad z \in K,$$

where $\mu_\Lambda(t) = \mu_0(t) + \Lambda \mu_1(t)$,

$$\mu_1(t) = \begin{cases} 0 & \text{for t} \in [0, \text{t}_0), \\ -m & \text{for t} \in [\text{t}_0, 2\pi], \end{cases}$$

and

$$m = \int_0^{2\pi} d\mu_0(t) - 1.$$

Of course, $p_\Lambda \in \mathcal{P}'_\alpha$ and

$$p_\Lambda(z) = \int_0^{2\pi} P(e^{-it}, z) d\mu_0(t) - m\Lambda P(e^{-it_0}, z),$$

$0 \leq \Lambda \leq 1$. If p_0 is extremal for problem (16), then

$$\operatorname{Re} \left\{ -m \frac{1 + e^{-it_0} z}{1 - e^{-it_0} z} \right\}_k \leq 0, \quad k = 1, 2, \ldots.$$

Consequently,

$$\mathrm{Re}\,(-2me^{-ikt_0}) \le 0, \quad k = 1, 2, \ldots .$$

The above inequality is true for $m = 0$ only. Then

$$\int_0^{2\pi} d\mu_0(t) = 1. \tag{17}$$

Let $\alpha > 1$. From (5) and (17) it follows that $\int_0^{2\pi} |d\mu_0(t)| \le \alpha$. If $\int_0^{2\pi} |d\mu_0(t)| < \alpha$, then there would exist $\beta < \alpha$ such that $\int_0^{2\pi} |d\mu_0(t)| = \beta$ and $\left| \int_0^{2\pi} d\mu_0(t) - 1 \right| + \int_0^{2\pi} |d\mu_0(t)| = \beta$, which would mean that $\deg p_0 < \alpha$. This contradicts Theorem 4, so, indeed,

$$\int_0^{2\pi} |d\mu_0(t)| = \alpha.$$

If $\alpha = 1$, then from (5) we get at once $\int_0^{2\pi} |d\mu_0(t)| = 1$, which concludes the proof.

§ 3. Estimation of the functional $\mathrm{Re}\,\{e^{i\lambda}p(z)\}, p \in \mathcal{P}'_\alpha$

At present, we shall make use of the method described in the second section of this paper. We shall prove

THEOREM 5. *If $p \in \mathcal{P}'_\alpha$, $\alpha > 1$, $\lambda \in [-\pi, \pi)$, then the estimate*

$$\begin{aligned} \mathrm{Re}\,[e^{i\lambda}p(z)] &\le \frac{1}{2(1-r^2)}\left[(1+r^2)\cos\lambda + 2r\alpha + \sqrt{x}\right], \\ x &= \sqrt{[\alpha(1+r^2)\cos\lambda + 2r]^2 + (1+r^2)^2(\alpha^2-1)\sin^2\lambda}, \end{aligned} \tag{18}$$

$|z| = r$, $z \in K$, takes place. The equality in (18) is obtained for functions of form (2) where μ is a piecewise constant function with one jump point t_0 defined by the conditions

$$\sin t_0 = \frac{(1-r^2)\sin\lambda}{(1+r^2)\alpha + 2r\cos\lambda},$$

$$\begin{aligned}\cos t_0 &= \frac{(\alpha^2-1)(1-r^2)^2}{[(1+r^2)\cos\lambda+2r\alpha+\sqrt{x}\,][(1+r^2)\alpha+2r\cos\lambda]} \\ &+ \frac{(1+r^2)\cos\lambda+2r\alpha}{(1+r^2)\alpha+2r\cos\lambda}\end{aligned} \tag{19}$$

with that

$$\int_0^{2\pi} d\mu(t) = \frac{1}{2}\left[1+\frac{\alpha^2(1+r^2)\cos\lambda+2r\alpha}{\sqrt{x}} - i\,\frac{(1+r^2)(\alpha^2-1)\sin\lambda}{\sqrt{x}}\right]. \tag{20}$$

PROOF. Let us determine

$$\max_{p\in\mathcal{P}'_\alpha} \text{Re}\,[e^{i\lambda}p(z)] \tag{21}$$

where z is any fixed point of the disc K, and $\lambda\in[-\pi,\pi)$. Since if $p\in\mathcal{P}'_\alpha$, then $p(e^{i\theta}z)\in\mathcal{P}'_\alpha$ for $\theta\in\mathbf{R}$, thus one may adopt $z=|z|=r$, $0<r<1$. Consequently, one should examine

$$\max_{\mu\in I_\alpha} \text{Re}\,[\int_0^{2\pi} e^{i\lambda}P(e^{-it},r)d\mu(t)]. \tag{22}$$

As can be seen, this is a problem of type (9) for $F[\varphi(z)]=\varphi(|z|)$ where $\varphi(z)=\int_0^{2\pi} g(z,t)d\mu(t)$, $\mu\in I_\alpha$, while $g(z,t)=e^{i\lambda}P(e^{-it},z)$. The extremal function will be denoted by $\varphi_0(z)=\int_0^{2\pi} g(z,t)d\mu_0(t)$.

First, let us deal with the problem of type (10). For any fixed $n\in\mathbf{N}$, determine

$$\max\left[\int_0^{2\pi} g(r,t)d\mu(t)\right] \tag{23}$$

with respect to $\mu\in I_\alpha(n)$ such that $\int_0^{2\pi} d\mu(t)=\int_0^{2\pi} d\mu_0(t)$. The extremal function in (21) will be denoted by $\varphi_n(z)=\int_0^{2\pi} g(z,t)d\mu_n(t)$.

Let us still notice that $L_{\varphi_n}(\cdot)=F(\cdot)$ because the functional F is linear.

It is known from the definition of the class $I_\alpha(n)$ that the function μ_n has not more than n points of discontinuity. Suppose that it has at least two such points. Then one of conditions (11), (13) or (14) must be satisfied. In the problem under consideration

$$L_{\varphi_n}[g(z,t_j)] = e^{i\lambda}\frac{1+e^{-it_j}r}{1-e^{-it_j}r}, \quad j=1,\dots,k, \quad 2\le k\le n.$$

So, it is evident that conditions (13) can not be satisfied. Assume that conditions (11) take place. Then all points $L_{\varphi_n}[g(z,t_j)]$ lie on the circle with centre at some point c_n and with radius $s_n > 0$. It is easy to see that $c_n = e^{i\lambda}\frac{1+r^2}{1-r^2}$ and $s_n = \frac{2r}{1-r^2}$.

Consider $\mathrm{Re}\left[\int_0^{2\pi} g(r,t)d\mu_n(t)\right]$. The function μ_n belongs to the class $I_\alpha(n)$ of piecewise constant functions, and

$$\begin{aligned}\int_0^{2\pi} g(r,t)d\mu_n(t) &= \sum_{j=1}^{k} g(r,t_j)a_j = \sum_{j=1}^{k} L_{\varphi_n}[g(z,t_j)]a_j, \qquad (24)\\ a_j = |a_j|e^{i\theta_j} &= d\mu_n(t_j), \quad j=1,\dots,k, \quad 1\le k\le n.\end{aligned}$$

Moreover, from condition (12) we have

$$\sum_{j=1}^{k} L_{\varphi_n}[g(z,t_j)]a_j = e^{i\lambda}\frac{1+r^2}{1-r^2}\sum_{j=1}^{n} a_j + \frac{2r}{1-r^2}\sum_{j=1}^{k}|a_j|. \qquad (25)$$

Denote $\sum_{j=1}^{k} a_j = A$. From (5) we get that $\sum_{j=1}^{k}|a_j| \le \alpha - |A-1|$. Thus

$$\mathrm{Re}\left[\int_0^{2\pi} g(r,t)d\mu_n(t)\right] \le W$$

where

$$W = \frac{1+r^2}{1-r^2}\mathrm{Re}\,(e^{i\lambda}A) + \frac{2r}{1-r^2}(\alpha - |A-1|).$$

It remains to determine the greatest value of the expression W.

Note that $A = \{p_n\}_0$ where p_n is the function of the class $\mathcal{P}'_\alpha$ corresponding to μ_n in the integral representation. In virtue of Theorem 2 A is a point of the domain V_0 described by inequality (6). So, we may assume that A lies on some ellipse contained in V_0, with foci at the points 0 and 1. Consequently,

$$A = \frac{1+[\eta(\alpha-1)+1]\cos\psi}{2} + i\,\frac{\sqrt{\eta^2(\alpha-1)^2+2\eta(\alpha-1)}\sin\psi}{2}, \qquad (26)$$

$\psi \in [0, 2\pi], \quad \eta \in [0, 1].$

Hence

$$
\begin{aligned}
W &= \frac{1+r^2}{1-r^2}\left(\frac{1+[\eta(\alpha-1)+1]\cos\psi}{2}\cos\lambda\right. \\
&- \left.\frac{\sqrt{\eta^2(\alpha-1)^2+2\eta(\alpha-1)}\sin\psi}{2}\sin\lambda\right) \\
&+ \frac{2r}{1-r^2}\left(\alpha - \left|\frac{[\eta(\alpha-1)+1]\cos\psi - 1}{2}\right.\right. \\
&+ \left.\left. i\,\frac{\sqrt{\eta^2(\alpha-1)^2+2\eta(\alpha-1)}\sin\psi}{2}\right|\right) \\
&= \frac{1}{2(1-r^2)}\{(1+r^2)\cos\lambda + 2r[2\alpha - \eta(\alpha-1) - 1] \\
&+ [(1+r^2)[\eta(\alpha-1)+1]\cos\lambda + 2r]\cos\psi \\
&- (1+r^2)\sqrt{\eta^2(\alpha-1)^2+2\eta(\alpha-1)}\sin\lambda\sin\psi\}.
\end{aligned}
$$

W is a function of the variables η and ψ defined on the set $[0,1]\times[0,2\pi]$. Using an elementary method, one can demonstrate that its greatest value is equal to

$$
\begin{aligned}
W(1,\psi_0) &= \frac{1}{2(1-r^2)}\left\{(1+r^2)\cos\lambda + 2r\alpha\right. \\
&+ \left.\sqrt{[\alpha(1+r^2)\cos\lambda+2r]^2+(1+r^2)^2(\alpha^2-1)\sin^2\lambda}\right\},
\end{aligned} \tag{27}
$$

where ψ_0 is defined by conditions

$$
\begin{aligned}
\sin\psi_0 &= \frac{-(1+r^2)\sqrt{\alpha^2-1}\sin\lambda}{\sqrt{[\alpha(1+r^2)\cos\lambda+2r]^2+(1+r^2)^2(\alpha^2-1)\sin^2\lambda}}, \\
\cos\psi_0 &= \frac{\alpha(1+r^2)\cos\lambda+2r}{\sqrt{[\alpha(1+r^2)\cos\lambda+2r]^2+(1+r^2)^2(\alpha^2-1)\sin^2\lambda}}.
\end{aligned}
$$

The reasoning carried out implies that the maximum of W is attained at the boundary point of ellipse (6) from Theorem 2. From the proof of this theorem included in paper [3], it follows that to the boundary points of the set V_0 there correspond functions $\mu_n \in I_\alpha$ such that $\arg d\mu_n(t) = \text{const.} = \theta$ for all t for which $d\mu_n(t) \neq 0$. Hence we infer that, for the extremal functions φ_n, the functions μ_n cannot have two distinct values θ_j.

However, if all $\theta_j = \theta = \text{const.}$, then the jump points of the function $\mu_n(t)$ satisfy conditions (14). The first of these equations is an algebraic equation of the second degree with respect to e^{it}. The application of the other equation from (14) and of Rolle's theorem lessens the number of the roots twice. So, in consequence, we have that, in all the cases, the function $\mu_n(t)$ corresponding to the extremal function φ_n has only one jump t_0. Note that $L_{\varphi_n}[g(z, t_0)]$ lies on the circle described earlier with centre c_n and with radius s_n and the estimate obtained is true in this case, too. Moreover $d\mu_n(t_0) = A_0$ where A_0 is defined by formula (26) in which $\eta = 1$, $\psi = \psi_0$; hence condition (20) follows. It can easily be verified that the point t_0 satisfies conditions (19).

As has been observed earlier, from the sequence (φ_n) of extremal functions one may choose a subsequence almost uniformly convergent in K to the extremal function in problem (22). Thereby, estimate (18) is true. Of course, to the functions φ_n there correspond functions $p_n = e^{-i\lambda}\varphi_n$ of the class $\mathcal{P}'_\alpha$.

Remark 1 *We were first seeking for the forms of the extremal functions in the subclasses of the family $\mathcal{P}'_\alpha$, generated by piecewise constant functions μ. Of course, they are extremal in the full class $\mathcal{P}'_\alpha$, as well.*

Remark 2 *It turns out that Theorem 5 can be proved also in the way which was applied in paper [3].*

Since the class $\mathcal{P}'_\alpha$, $\alpha > 1$, is convex and, for each $z_1 \in K$, there exists a function $p_1 \in \mathcal{P}'_\alpha$ such that $p_1(z_1) = 0$, with that $\operatorname{Re} p_1(z_1) > \frac{1+|z_1|}{1-|z_1|} \cdot \frac{1-\alpha}{2}$ (see (7)), therefore Theorem 5 implies

COROLARY 1. L

et $z \neq 0$ be a fixed point of the disc K. Then the set $\mathcal{A}_\alpha$, $\alpha > 1$, of values of the functional $H(p) = p(z)$, $p \in \mathcal{P}'_\alpha$, is a set of the plane (w) whose boundary is given by the equation

$$\begin{aligned} w &= \frac{e^{i\lambda}}{2(1-r^2)} \Big[(1+r^2)\cos\lambda + 2r\alpha \\ &+ \sqrt{[\alpha(1+r^2)\cos\lambda + 2r]^2 + (1+r^2)^2(\alpha^2-1)\sin^2\lambda}\Big], \quad (28) \end{aligned}$$

$|z| = r$, $\lambda \in [-\pi, \pi]$, with that $0 \in \mathcal{A}_\alpha$.

Putting $\lambda = 0$ and $\lambda = -\pi$ in (18), we evidently obtain Theorem 3, whereas substituting $\lambda = \pm\frac{\pi}{2}$, we have

COROLARY 2. *I*

f $p \in \mathcal{P}'_\alpha$, $\alpha > 1$, then the sharp estimate

$$\begin{aligned} - & \frac{2+\alpha+\sqrt{(1+r^2)^2\alpha^2-(1-r^2)^2}}{2(1-r^2)} \le \operatorname{Im} p(z) \\ \le & \frac{2+\alpha+\sqrt{(1+r^2)^2\alpha^2-(1-r^2)^2}}{2(1-r^2)}, \quad |z| = r \end{aligned} \tag{29}$$

holds.

Whereas Corollary 1 and the form of equation (28) imply

COROLARY 3. *I*

n the class $\mathcal{P}'_\alpha$, $\alpha > 1$, the sharp estimate

$$|p(z)| \le \frac{1+r}{1-r} \cdot \frac{1+\alpha}{2}, \quad |z| = r, \quad z \in K, \tag{30}$$

takes place.

Remark 3 *Passing to the limit with $\alpha \to 1^+$ in the estimate from above of (7) as well as in (27) and (28), we obtain estimates of the corresponding functionals in the class $\mathcal{P}'_1$, the bounds of these functionals being, as can be seen, analogous to those in the class $\mathcal{P}$. For obvious reasons, we have lost the estimates of* $\operatorname{Re} p(z)$ *and* $|p(z)|$ *from below. Of course, the formal justifications can be carried out by using, for instance, the fact that for $\alpha = 1$, the set $V_0 = [0, 1]$.*

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