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AN EXHAUSTIVE DESCRIPTION OF A NEW LARGE FAMILY OF THIRD DEGREE SEMICLASSICAL FORMS OF CLASS ONE

Abstract. This paper provides a comprehensive characterization of a large family of third degree semiclassical forms of class one. Specifically, we present an exhaustive description of all strict third degree semiclassical forms of class one that arise through the cubic decomposition $W_{3n+1}(x) = xQ_n(x^3 + px^2 + r)$, $p, r \in \mathbb{C}$, $n \geqslant 0$. By using the Stieltjes function and moments of these forms, we provide necessary and sufficient conditions that characterize and identify all these forms. Also, we establish a connection between these forms and the strict third degree classical forms. Furthermore, we provide explicit expressions for the characteristic elements of the structure relation and the second-order differential equation.

Key words: *orthogonal polynomials, classical and semiclassical forms, Stieltjes function, cubic decomposition, third degree forms, differential equations*

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1. Introduction. The framework of this contribution lies within the theory of semiclassical orthogonal polynomials. Since the seminal paper on semiclassical orthogonal polynomials by J. Shohat [31], many authors have taken an interest in this subject. They arise as a natural extension of the well-known classical orthogonal polynomials (Hermite, $\mathcal{H}$; Laguerre, $\mathcal{L}(\alpha)$; Jacobi, $\mathcal{J}(\alpha, \beta)$, and Bessel, $\mathcal{B}(\alpha)$). P. Maroni developed this theory of orthogonal polynomials from an algebraic structural perspective, and it has been thoroughly investigated over the past forty years [28].

In recent decades, there has been a growing interest in studying the cubic decomposition (CD) of orthogonal polynomial sequences (OPS).

The catalyst for this interest was a question posed by Chihara regarding the existence of an OPS that could be obtained from another OPS through a cubic transformation. Barrucand and Dickinson provided a solution to this question in [2], where they established necessary and sufficient conditions for the orthogonality of a symmetric OPS $\{W_n\}_{n\geqslant 0}$, such that $W_{3n}(x) = P_n(x^3 + bx)$, under the assumption that $\{P_n\}_{n\geqslant 0}$ is a symmetric sequence of monic orthogonal polynomials. The latter mark has been therefore extended by Marcellán and Sansigre in [23] to non-symmetric and non-positive definite orthogonal sequences, considering the cubic transformation x^3. Later on, in [24], [25], the constraints of symmetry and the specific choice of cubic polynomials were removed, and then the authors investigated the orthogonality of sequences $\{W_n\}_{n\geqslant 0}$ satisfying $W_{3n+m}(x) = \theta_m(x)P_n(\pi_3(x))$, where $\pi_3(x)$ is a fixed cubic polynomial, and $\theta_m(x)$ is a fixed polynomial of degree m for $m \in \{0, 1, 2\}$. In [29], the issue of general cubic decompositions of orthogonal polynomials is examined. It is worth noting that the issue of cubic decompositions within the context of semiclassical orthogonal polynomial sequences has not been thoroughly examined in the literature until recently. Only a few recent contributions, specifically [10] and [32], have addressed this problem.

On the other hand, the study of regular forms, such that the corresponding formal Stieltjes function $S(w)(z) := -\sum_{n\geqslant 0} \langle w, x^n \rangle / z^{n+1}$ satisfies an algebraic equation, yields an interesting approach to analyze their properties.

Second degree linear forms are defined via a quadratic equation with polynomial coefficients

$$MS^2(w) + NS(w) + R = 0,$$

and they are introduced in [26]. Examples of second degree linear forms are studied in the classical case (see [3]), in the semiclassical case of class $s = 1$ (see [7], among others), and in the semiclassical case of class $s = 2$ in [5], [13].

Third degree regular linear forms TDRF are characterized by the fact that their formal Stieltjes function satisfies a cubic equation with polynomial coefficients

$$AS^3(w) + BS^2(w) + CS(w) + D = 0.$$

A regular form w is said to be a strict third degree form (STDRF, in short) if it is a TDRF and its Stieltjes function does not satisfy a quadratic equation with polynomial coefficients, i.e., it is not a second degree form. It is important to notice that a third degree form belongs to the Laguerre-Hahn class [9], [20]. Properties and examples of TDRF are given in [4]. On the other hand, based either on spectral perturbations of the linear form (see [8]) or in a cubic decomposition of the corresponding sequences of orthogonal polynomials (see [6], [16]), a constructive approach to some families of TDRF is presented therein.

Thus, a natural question is to search for semiclassical linear forms that are of the third degree according to the hierarchy defined by the class of semiclassical linear forms.

In [4], all classical forms that are TDRFs are determined. It is worth mentioning that the unique classical TDRFs are the Jacobi forms $\mathcal{J}(k+q/3, l-q/3)$, where $q \in \{1,2\}$ and k, l are integer numbers with $k+l \geqslant -1$ (see [4]). Examples of strict third-degree linear forms have been investigated in the semiclassical framework of class $s=1$ (see, for instance, [6], [12], [17]), in the case of class $s=2$ [16], [19], [18], and also in the case of class $s=3$ [14].

Our work is focused on the analysis of semiclassical forms of class $s=1$ that are STDRFs. More precisely, we are interested in their description, using the third degree character, of a large family of forms such that their corresponding sequences of orthogonal polynomials $\{W_n\}_{n\geqslant 0}$ are obtained via the cubic decomposition (CD)

$$W_{3n+1}(x) = xQ_n(x^3+px^2+r), \quad p, r \in \mathbb{C}, \ \ n \geqslant 0, \tag{1}$$

requiring that $\{Q_n\}_{n\geqslant 0}$ is a monic orthogonal polynomial sequence MOPS. This is a particular case of the general cubic decomposition of any monic polynomial sequence presented in [29], where the parameters a, b, c, q are considered zero and the two secondary components $\{b_n^1\}_{n\geqslant 0}$ and $\{b_n^2\}_{n\geqslant 0}$ vanish. The general case with $q \neq 0$ is more complicated to handle and remains an open problem.

In this paper, we present an exhaustive description for all the strict third degree semiclassical forms of class one, arising from the cubic decomposition of the form (1). Our main result is given in Theorem 3, which states, using the Stieltjes function and the moments of those forms, the necessary and sufficient conditions characterizing this family along with presenting their link with the Jacobi forms $\mathcal{V}_q^{k,l} := \mathcal{J}(k+q/3, l-q/3)$, $k+l \geqslant -1, k, l \in \mathbb{Z}, q \in \{1,2\}$, as well as shows that all of them are rational transformations of the product of three shifted forms of Jacobi form $\mathcal{V} := \mathcal{V}_1^{-1,0}$.

The paper is organized as follows. Section 2 contains a review of essential notation, definitions, and results that will be used in the subsequent sections, with special emphasis placed on semiclassical linear forms.

We also collect auxiliary results concerning the strict third-degree classical forms , which are needed for the subsequent analysis.

In Section 3, we begin by revisiting the definitions and main properties of third degree forms. We also present some results on the strict third degree classical forms, $\mathcal{V}_q^{k,l}$, which are needed for our subsequent analysis. Then we provide and characterize all forms that meet our objective. In fact, only one canonical case appears, up to an affine transformation. Specifically, we demonstrate that the forms presented in the four families related to $p \neq 0$ in the classification given in [32] are not STDRFs. Only certain forms from the family corresponding to $p = 0$ qualify as STDRFs.

In Section 4, we exhibit the main results of the paper, which establish the equivalence between the conditions stated in Theorem 3. Based on their strict third degree character, we give an identification of a part of the family of semiclassical regular forms obtained via the cubic decomposition (1). In fact, using the Stieltjes function and the associated moments, we establish necessary and sufficient conditions for a regular form to be simultaneously a strict third-degree form and a semiclassical form of class one, for which the corresponding MOPS $\{W_n\}_{n\geqslant 0}$ arises from the cubic decomposition described above. We further characterize this family of forms by relating them to the classical strict third-degree forms.

Finally, in Section 5, both the characteristic elements of the structure relation and of the second-order differential equation are explicitly given.

2. Notation and basic background. In this section we present the basic definitions, notation, and results—based on Maroni's algebraic approach to orthogonal polynomials [28]—that will be used throughout the paper.

2.1. Basic tools. Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients. Its topological dual space will be denoted by $\mathcal{P}'$. The elements of $\mathcal{P}'$ will be called forms (linear functionals). By $\langle \cdot, \cdot \rangle$, we denote the duality brackets between $\mathcal{P}$ and $\mathcal{P}'$. Given a linear functional $w \in \mathcal{P}'$, the sequence of complex numbers $(w)_n$, $n = 0, 1, 2, \ldots$, denotes the moments of w with respect to the sequence $\{x^n\}_{n\geqslant 0}$, namely, the moment of order n for the form w is denoted by $(w)_n := \langle w, x^n \rangle$. Thus, the form w is completely determined by its moments.

We consider sequences of polynomials $\{B_n\}_{n\geqslant 0}$ satisfying $\deg B_n \leqslant n$

for all $n \geqslant 0$. When $\{B_n\}_{n\geqslant 0}$ spans $\mathcal{P}$ – which occurs precisely when $\deg B_n = n$ for all $n \geqslant 0$ – we refer to it as a *polynomial sequence* (PS). To every sequence $\{B_n\}_{n\geqslant 0}$, one can associate a unique sequence $\{u_n\}_{n\geqslant 0}$, with $u_n \in \mathcal{P}'$, called its *dual sequence*, characterized by $\langle u_n, B_m\rangle = \delta_{n,m}$, $n, m \geqslant 0$, where $\delta_{n,m}$ denotes the Kronecker delta symbol [28].

Let us introduce some useful operations in $\mathcal{P}'$. For any form w, and any $(a, b, c) \in (\mathbb{C} - \{0\}) \times \mathbb{C}^2$, let w', $(x-c)^{-1}w$, $h_a w$, and $\tau_b w$ be the forms defined by duality

$$\langle w', f\rangle := -\langle w, f'\rangle, \qquad \langle (x-c)^{-1}w, f\rangle := \langle w, \theta_c f\rangle = \langle w, \tfrac{f(x)-f(c)}{x-c}\rangle,$$
$$\langle h_a w, f\rangle := \langle w, h_a f\rangle = \langle w, f(ax)\rangle,$$
$$\langle \tau_b w, f\rangle := \langle w, \tau_{-b} f\rangle = \langle w, f(x+b)\rangle, \qquad f \in \mathcal{P}.$$

Let $w \in \mathcal{P}'$, and $g \in \mathcal{P}$. The left multiplication of w by g, denoted by gw, is the functional in $\mathcal{P}'$ defined by

$$\langle gw, f\rangle := \langle w, gf\rangle, \qquad f \in \mathcal{P}.$$

Let $w \in \mathcal{P}'$ and $f \in \mathcal{P}$. The right multiplication of w by f, denoted by wf, is the polynomial defined by

$$(wf)(z) := \Big\langle w, \frac{zf(z) - xf(x)}{z-x}\Big\rangle.$$

This allows us to define the Cauchy product of two forms:

$$\langle vw, f\rangle := \langle v, wf\rangle, \quad v, w \in \mathcal{P}', \quad f \in \mathcal{P}.$$

The above product is commutative, associative, and distributive with respect to the sum of forms.

We define the operator $\sigma_\varpi \colon \mathcal{P}' \to \mathcal{P}'$ by

$$\langle \sigma_\varpi(w), f\rangle := \langle w, \sigma_\varpi(f)\rangle, \quad w \in \mathcal{P}', \quad f \in \mathcal{P}, \tag{2}$$

where the linear operator $\sigma_\varpi \colon \mathcal{P} \to \mathcal{P}$ is defined by $\sigma_\varpi(f)(x) := f(x^3)$ for every $f \in \mathcal{P}$.

We now introduce the operator $\boldsymbol{\varrho} \colon \mathcal{P}' \to \mathcal{P}'$, such that for any linear form ω we get, for $\omega \in \mathcal{P}'$,

$$\big(\boldsymbol{\varrho}(\omega)\big)_{3n} = \big(\boldsymbol{\varrho}(\omega)\big)_{3n+1} = 0, \quad \big(\boldsymbol{\varrho}(\omega)\big)_{3n+2} = (\omega)_n, \quad n \geqslant 0. \tag{3}$$

We will also use the so-called formal Stieltjes function associated with $w \in \mathcal{P}'$ that is defined by [11], [28]

$$S(w)(z) = -\sum_{n \geqslant 0} \frac{(w)_n}{z^{n+1}}.$$

Remark 1. *For any $f \in \mathcal{P}$ and $w \in \mathcal{P}'$, $S(w) = f$, if and only if $w = 0$ and $f = 0$.*

Lemma 1. *For any $f \in \mathcal{P}$ and any $v, w \in \mathcal{P}'$, we have [28]*

$$\begin{aligned} S(vw)(z) &= -zS(v)(z)S(w)(z), \\ S(fw)(z) &= f(z)S(w)(z) + (w\theta_0 f)(z). \end{aligned} \tag{4}$$

Let us recall that a form w is called regular (quasi-definite) if there exists a monic polynomial sequence (MPS) $\{W_n\}_{n\geqslant 0}$ with $\deg W_n = n$, $n \geqslant 0$, such that [11]

$$\langle w, W_n W_m \rangle = r_n \delta_{n,m}, \qquad n, m \geqslant 0,$$

where $\{r_n\}_{n\geqslant 0}$ is a sequence of nonzero complex numbers.

The sequence $\{W_n\}_{n\geqslant 0}$ is then called a monic orthogonal polynomial sequence (shortly indicated as MOPS) with respect to the form w. It is well known that any MOPS is characterized by a linear second-order recurrence relation of the form

$$W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \quad n \geqslant 0, \tag{5}$$

with initial conditions $W_0(x) = 1$ and $W_1(x) = x - \beta_0$, being $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_{n+1}\}_{n\geqslant 0}$ sequences of complex numbers, such that $\gamma_{n+1} \neq 0$ for all $n \geqslant 0$.

Remark 2. *The form w is said to be normalized if $(w)_0 = 1$. From here on, we will consider only normalized forms.*

2.2. Semiclassical linear forms.

Definition 1. *A form $w \in \mathcal{P}'$ is called semiclassical if it is regular and there exist two nonzero polynomials ϕ and ψ, ϕ monic and $\deg \psi \geqslant 1$, such that w satisfies the generalized Pearson distributional differential equation*

$$(\phi w)' + \psi w = 0, \tag{6}$$

or, equivalently, there exist polynomials ϕ, C_0, and D_0, such that $S(w)$ is a (formal) solution of the first order nonhomogeneous linear ordinary differential equation [28]

$$\phi(z)S'(w)(z) = C_0(z)S(w)(z) + D_0(z), \tag{7}$$

where

$$C_0 = -\phi' - \psi, \quad D_0 = -(w\theta_0\phi)' - (w\theta_0\psi). \tag{8}$$

Further, if ϕ, C_0, and D_0 are co-prime, then the class of w is

$$s = \max\{\deg C_0 - 1, \deg D_0\}.$$

In this case, the corresponding MOPS is said to be semiclassical.

Remark 3. *$w \in \mathcal{P}'$ is said to be classical if its class is $s = 0$, i.e., there exist nonzero polynomials ϕ and ψ, with $\deg(\phi) \leqslant 2$ and $\deg(\psi) = 1$, such that (6) holds. In this case, the MOPS associated with w is called a classical MOPS. It is well known that up to an affine transformation on the variable, we obtain Hermite forms $\mathcal{H}$, Laguerre forms $\mathcal{L}(\alpha)$ ($\alpha \neq -n$, $n \geqslant 1$), Jacobi forms $\mathcal{J}(\alpha, \beta)$ $\big(\alpha \neq -n-1, \beta \neq -n-1, \alpha+\beta \neq -n-2$, $n \geqslant 0\big)$, and Bessel forms $\mathcal{B}(\alpha)$ $\big(\alpha \neq -n/2,\ n \geqslant 0\big)$ (see [27]).*

The following lemma plays an important role in proving our result.

Lemma 2. *[5] Let w_1, w_2 be two semiclassical forms satisfying (6) with $\deg\phi = \deg\psi + 1 = t \geqslant 2$. If $(w_1)_i = (w_2)_i$, $0 \leqslant i \leqslant t-2$, then $w_1 = w_2$.*

The semiclassical character of a form is preserved by an affine transformation. Indeed, the shifted form $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$, $a \in \mathbb{C} - \{0\}, b \in \mathbb{C}$, is also semiclassical, having the same class as that of w, and satisfies

$$\big(a^{-\deg\phi}\phi(az+b)\widehat{w}\big)' + a^{1-\deg\phi}\psi(az+b)\widehat{w} = 0.$$

Hence, a displacement does not change neither the semiclassical character nor the class of a semiclassical form [28]. Therefore, we can take canonical functional equations, by re-situating the zeros of $\phi(z)$ and $\psi(z)$ in equation (6). This will be put in evidence in the sequel.

The sequence $\{\widehat{W}_n\}_{n\geqslant 0}$, where $\widehat{W}_n(x) = a^{-n}W_n(ax+b)$, $n \geqslant 0$, is orthogonal with respect to $\widehat{w}$. The recurrence coefficients are given by $\widehat{\beta}_n = a^{-1}(\beta_n - b)$, $\widehat{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$, $n \geqslant 0$ (see [28]).

For any $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$, the moments of the shifted form $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$ are

$$(\widehat{w})_n = n!a^{-n} \sum_{\nu+\mu=n} \frac{(-b)^\nu}{\nu!\mu!}(w)_\mu, \quad n \geqslant 0. \tag{9}$$

The formal Stieltjes function of the shifted form $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$, $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$ satisfies (see [4])

$$S\big(\widehat{w}\big)(z) = aS(w)(az + b). \tag{10}$$

2.3. Cubic decomposition. For any MPS $\{W_n\}_{n\geqslant 0}$, there are three MPSs, $\{P_n\}_{n\geqslant 0}$, $\{Q_n\}_{n\geqslant 0}$, and $\{R_n\}_{n\geqslant 0}$, so that [29]

$$W_{3n}(x) = P_n\big(\varpi(x)\big) + xa^1_{n-1}\big(\varpi(x)\big) + x^2a^2_{n-1}\big(\varpi(x)\big), \quad n \geqslant 0, \tag{11}$$
$$W_{3n+1}(x) = b^1_n\big(\varpi(x)\big) + xQ_n\big(\varpi(x)\big) + x^2b^2_{n-1}\big(\varpi(x)\big), \quad n \geqslant 0, \tag{12}$$
$$W_{3n+2}(x) = c^1_n\big(\varpi(x)\big) + xc^2_n\big(\varpi(x)\big) + x^2R_n(\big(\varpi(x)\big), \quad n \geqslant 0, \tag{13}$$

where

$$\varpi(x) = x^3 + px^2 + r, \;\; p, r \in \mathbb{C},$$

with $\deg a^1_{n-1} \leqslant n-1$, $\deg a^2_{n-1} \leqslant n-1$, $\deg b^1_n \leqslant n$, $\deg b^2_{n-1} \leqslant n-1$, $\deg c^1_n \leqslant n$, $\deg c^2_n \leqslant n$, and $a^1_{-1}(x) = a^2_{-1}(x) = b^2_{-1}(x) = 0$. Such decomposition is called a cubic decomposition CD [29]. Let $\{u_n\}_{n\geqslant 0}$, $\{v_n\}_{n\geqslant 0}$, $\{r_n\}_{n\geqslant 0}$, and $\{w_n\}_{n\geqslant 0}$ be the dual sequences of the MPSs $\{P_n\}_{n\geqslant 0}$, $\{Q_n\}_{n\geqslant 0}$, $\{R_n\}_{n\geqslant 0}$, and $\{W_n\}_{n\geqslant 0}$ respectively.

Let us recall this result that we will need in the sequel.

Proposition 1. *[32] Let $\{W_n\}_{n\geqslant 0}$ be a MOPS with respect to w defined by (11)–(13) and such that $b^1_n = b^2_n = 0$, $n \geqslant 0$. Then the recurrence coefficients of (5) satisfy*

$$\beta_0 = 0, \quad \beta_{3n+1} = \beta_1, \quad \beta_{3n+1} + \beta_{3n+2} + \beta_{3n+3} + p = 0, \quad n \geqslant 0,$$
$$\gamma_{3n+3} = \beta_{3n+3}\beta_{3n+2}, \quad \gamma_{3n+1} + \gamma_{3n+2} = -\beta_1\left(\beta_1 + p\right), \quad n \geqslant 0.$$

Moreover, $\{Q_n\}_{n\geqslant 0}$ is orthogonal with respect to the form v_0. Furthermore, we have

$$\sigma_\varpi(xw) = 0, \tag{14}$$
$$\sigma_\varpi(x^2w) = \gamma_1 v_0, \tag{15}$$
$$\sigma_\varpi(x^3w) = \beta_1\gamma_1 v_0. \tag{16}$$

Recently, all semiclassical forms of class one, such that their corresponding MOPS $\{W_n\}_{n\geqslant 0}$ satisfy $W_{3n+1}(x) = xQ_n(x^3+px^2+r)$, have been determined (see [32]). Only five canonical cases appear, up to an affine transformation.

Theorem 1. *[32] Let $\{W_n\}_{n\geqslant 0}$ be a MOPS fulfilling (11)–(13) with $b_n^1 = b_n^2 = 0$, $n \geqslant 0$, and let w be its corresponding regular form. There exist only five families of semiclassical forms of class one. Precisely:*

1) *The first family: w is a semiclassical form satisfying (6) with*

$$\phi(x) = x^3 + \tau^3, \quad \psi(x) = -(3\alpha+4)x^2 + \tau x - \tau^2,$$
$$(w)_0 = 1, \quad (w)_1 = 0, \qquad \tau \neq 0, \ \text{and} \ \ p = 0,$$

which is regular for

$$\alpha \neq -n \quad \text{and} \quad \alpha \neq -n - 1/3, \quad n \geqslant 1. \tag{17}$$

Moreover, the linear form v_0 is classical, fulfilling

$$(h_{2\tau^{-3}} \circ \tau_{-r+2^{-1}\tau^3})v_0 = \mathcal{J}(\alpha, 1/3). \tag{18}$$

2) *The second family: w is a semiclassical form satisfying (6) with*

$$\begin{aligned} &\phi(x) = x^3 + \tfrac{4}{3}px^2 + \tfrac{1}{9}p^2x - \tfrac{2}{9}p^3, \\ &\psi(x) = -\big(3\alpha + \tfrac{9}{2}\big)x^2 - \big(3\alpha + \tfrac{19}{6}\big)px + \tfrac{2}{9}p^2, \\ &(w)_0 = 1, \quad (w)_1 = 0, \ \ \text{and} \ \ p \neq 0, \end{aligned} \tag{19}$$

which is regular for $\alpha \neq -n$ and $\alpha \neq -n - 1/2$, $n \geqslant 1$.
Moreover, the linear form v_0 is classical, fulfilling

$$(h_{-\frac{27}{2}p^{-3}} \circ \tau_{-(r+\frac{2}{27}p^3)})v_0 = \mathcal{J}(\alpha,1/2).$$

3) *The third family: w is a semiclassical form satisfying (6) with*

$$\phi(x) = x^3 - 4\tau x^2 + \tau^2 x + 6\tau^3,$$
$$\psi(x) = -\big(3\alpha + \tfrac{9}{2}\big)x^2 + 3\tau\big(3\alpha + \tfrac{7}{2}\big)x + 3\tau^2,$$
$$(w)_0 = 1, \quad (w)_1 = 0, \ \ p\tau \neq 0, \ \ \text{and} \ \ p = -3\tau,$$

which is regular for $\alpha \neq -n$ and $\alpha \neq -n - 1/2$, $n \geqslant 1$.
Moreover, the linear form v_0 is classical, fulfilling

$$(h_{2^{-1}\tau^{-3}} \circ \tau_{-r+2\tau^3})v_0 = \mathcal{J}(\alpha,1/2).$$

4) *The fourth family: w is a semiclassical form satisfying* (6) *with*

$$\begin{aligned}
&\phi(x) = x^3 - \tau x^2 + \tfrac{1}{16}\tau^2 x + \tfrac{3}{32}\tau^3,\\
&\psi(x) = -\big(3\alpha + \tfrac{9}{2}\big)x^2 + \tfrac{3}{4}\tau\big(3\alpha + \tfrac{9}{2}\big)x - \tfrac{3}{8}\tau^2,\\
&(w)_0 = 1, \quad (w)_1 = 0, \;\; p\tau \neq 0, \;\; \text{and} \;\; p = -\tfrac{3}{4}\tau,
\end{aligned}$$

which is regular for $\alpha \neq -n$ and $\alpha \neq -n - 1/2$, $n \geqslant 1$. Moreover, the linear form v_0 is classical, fulfilling

$$(h_{32\tau^{-3}} \circ \tau_{-r+32^{-1}\tau^3})v_0 = \mathcal{J}(\alpha,1/2).$$

5) *The fifth family: w is a semiclassical form, satisfying* (6) *with*

$$\begin{aligned}
&\phi(x) = x^3 - 2\tau x^2 - 3\tau^2 x,\\
&\psi(x) = -\big(3\alpha + \tfrac{3}{2}\big)x^2 + \tau\big(3\alpha + \tfrac{1}{2}\big)x6\alpha\tau^2,\\
&(w)_0 = 1, \quad (w)_1 = 0, \;\; p\tau \neq 0, \;\; \text{and} \;\; p = -3\tau,
\end{aligned}$$

which is regular for $\alpha \neq -n$ and $\alpha \neq -n + 1/2$, $n \geqslant 1$. Moreover, the linear form v_0 is classical, fulfilling

$$(h_{2^{-1}\tau^{-3}} \circ \tau_{-r+2\tau^3})v_0 = \mathcal{J}(\alpha, -1/2).$$

Remark 4. *The recurrence coefficients of the MOPS $\{W_n\}_{n\geqslant 0}$ of the first family in Theorem 1 are given by [32]:*

$$\begin{aligned}
&\beta_0 = 0, \qquad \beta_{3n+1} = \tau, \qquad \beta_{3n+2} = -\tau\frac{n + \alpha + 4/3}{2n + \alpha + 7/3},\\
&\beta_{3n+3} = -\tau\frac{n+1}{2n + \alpha + 7/3}, \quad n \geqslant 0,\\
&\gamma_{3n+1} = -\tau^2\frac{n + 1/3}{2n + \alpha + 4/3}, \qquad \gamma_{3n+2} = -\tau^2\frac{n + \alpha + 1}{2n + \alpha + 4/3}, \quad n \geqslant 0,\\
&\gamma_{3n+3} = \tau^2\frac{(n+1)(n + \alpha + 4/3)}{(2n + \alpha + 7/3)^2}, \quad n \geqslant 0.
\end{aligned}$$

3. Third-degree semiclassical forms.

3.1. Third-degree forms. In this section of the paper, we provide a brief overview of the definitions and list some fundamental properties of third-degree regular forms. We also present some results related to strict third-degree classical forms that are necessary for our subsequent analysis.

Definition 2. *[4], [9] The form w is called a third-degree regular form (TDRF) if it is regular and if there exist three polynomials A monic, B, and C, such that*

$$A(z)S^3(w)(z) + B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0,$$

where D depends on A, B, C and w.

Remark 5.

1) *If the corresponding Stieltjes function of a regular form w satisfies a quadratic equation with polynomial coefficients, we say that w is a second-degree form, i.e., (see [26])*

$$B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0,$$

B, C, D satisfy $B \neq 0, C^2 - 4BD \neq 0, D \neq 0$ since regularity of w.

2) *If the form w is a third-degree regular form TDRF but not a second-degree form, it is referred to as a strict third-degree regular form (STDRF) (see [4]).*

3) *One of the most well-known examples of strict third-degree regular forms (STDRFs) is the Jacobi form $\mathcal{V} := \mathcal{J}(-\frac{2}{3}, -\frac{1}{3})$ (see [4]). Indeed, its formal Stieltjes function is $S(\mathcal{V})(z) = -(z+1)^{-2/3}(z-1)^{-1/3}$, and satisfies the cubic equation*

$$(z + 1)^2(z - 1)S^3(\mathcal{V})(z) + 1 = 0. \tag{20}$$

Elementary transformations of linear forms as k-associated and k-anti-associated, perturbations, shift, multiplication and division by a polynomial, inversion, among others, preserve the family of linear forms of third degree [4], [8], [21], [22]. In particular, if one deals with the following rational spectral transformations (see [8], [15]).

Lemma 3. *[4] Let w_1 and w_2 be two regular forms satisfying $M(x)w_1 = N(x)w_2$, where $M(x)$ and $N(x)$ are two polynomials. If one of the two forms is a third-degree form, then so is the other.*

Proposition 2. *[4] Let w be a semiclassical form satisfying (7). Set $A_0(z) = \prod_{i=1}^{m}(z-a_i)^{k_i}$, where $a_1, \ldots, a_m$ are complex numbers and $k_1, \ldots, k_m$ are positive integers, such that $k_1 + \cdots + k_m = \deg A_0$. If w is a STDRF,* *then the rational fraction* $\dfrac{C_0}{A_0}$ *has only simple poles and if* $\alpha_1,\ldots,\alpha_m$ *denote the residues of* $\dfrac{C_0}{A_0}$, *then*

$$\frac{C_0(z)}{A_0(z)}=\sum_{i=1}^{m}\frac{\alpha_i}{z-a_i},\quad 3\alpha_1,\ldots,3\alpha_m\in\mathbb{Z}.$$

3.2. Strict third-degree classical forms. As mentioned in the introduction, the classical forms that are of strict third degree are determined in [4]. More precisely, only some Jacobi forms are STDRFs.

Theorem 2. *[4] Among the classical forms, only the Jacobi forms* $\mathcal{J}(k+q/3,l-q/3)$ *are STDRFs, provided* $k+l\geqslant-1$, $k,l\in\mathbb{Z}$, $q\in\{1,2\}$.

Remark 6. *In the sequel, we denote* $\mathcal{V}_q^{k,l}:=\mathcal{J}(k+q/3,l-q/3)$, *with* $k+l\geqslant-1$, $k,l\in\mathbb{Z}$, $q\in\{1,2\}$.

We state now a lemma that gives a relation between the strict third-degree classical forms $\mathcal{V}_q^{k,l}, q\in\{1,2\}, k,l\in\mathbb{Z}, k+l\geqslant-1$, and the form $\mathcal{V}:=\mathcal{J}(-2/3,-1/3)$. More precisely, it clarifies that all the forms $\mathcal{V}_q^{k,l}$ are rational perturbations of $h_{(-1)^{q-1}}\mathcal{V}$.

Lemma 4. *[16] Let* $q\in\{1,2\}$ *and* $k,l\in\mathbb{Z}$ *with* $k+l\geqslant-1$. *The forms* $\mathcal{V}_q^{k,l}$ *and* $\mathcal{V}$ *are related by*

$$f_q^{k,l}\mathcal{V}_q^{k,l}=g_q^{k,l}h_{(-1)^{q-1}}\mathcal{V},\tag{21}$$

where $f_q^{k,l}$ *and* $g_q^{k,l}$ *are polynomials defined by*

$$f_q^{k,l}(x):=\Big\langle h_{(-1)^{q-1}}\mathcal{V},\ (x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}}\Big\rangle(x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}},\tag{22}$$

$$g_q^{k,l}(x):=\Big\langle\mathcal{V}_q^{k,l},(x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}}\Big\rangle(x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}}.\tag{23}$$

The following remark summarizes some results concerning the forms $\mathcal{V}_q^{k,l}$ introduced earlier, which will be used later in the paper.

Remark 7.

1. *Using the first-order linear differential equation satisfied by the Stieltjes function of the Jacobi form (see [28]), it is a straightforward exercise to prove that* $S\big(\mathcal{V}_q^{k,l}\big)(z)$ *satisfies*

$$\Phi(z)S'\big(\mathcal{V}_q^{k,l}\big)(z)=C_{0,q}^{k,l}(z)S\big(\mathcal{V}_q^{k,l}\big)(z)+D_{0,q}^{k,l}(z),\tag{24}$$

with $\Phi, C_{0,q}^{k,l}$, *and* $D_{0,q}^{k,l}$ *given by*

$$\Phi(z)=z^2-1, \quad C_{0,q}^{k,l}(z)=(k+l)z+l-k-\frac{2q}{3}, \quad D_{0,q}^{k,l}(z)=k+l+1. \tag{25}$$

2. *Recall that the moments of the Jacobi form* $\mathcal{V}_q^{k,l}$ *with* $k+l \geqslant -1$, $k,l \in \mathbb{Z}$, $q \in \{1,2\}$, *are*

$$\left(\mathcal{V}_q^{k,l}\right)_n=\sum_{\nu=0}^{n}\binom{n}{\nu}2^{\nu-1}\frac{\Gamma(k+l+2)}{\Gamma(\nu+k+l+2)}F_{n,\nu}\left(k+\tfrac{q}{3},l-\tfrac{q}{3}\right), n \geqslant 0, \tag{26}$$

where

$$F_{n,\nu}\left(k+\tfrac{q}{3},l-\tfrac{q}{3}\right)=(-1)^{n-\nu}\frac{\Gamma(\nu+k+\frac{q}{3}+1)}{\Gamma(k+\frac{q}{3}+1)}+(-1)^{\nu}\frac{\Gamma(\nu+l-\frac{q}{3}+1)}{\Gamma(l-\frac{q}{3}+1)}, \tag{27}$$

and Γ *is the gamma function (see [27]).*

3.3. Strict third degree semiclassical forms of class one arising via cubic decomposition. This section aims to find all the semiclassical forms of class one that are of strict third degree, such that the corresponding MOPS satisfies (11)–(13) and $b_n^1=b_n^2=0$, $n \geqslant 0$.

During the analysis of admissible forms, we will show that p, the coefficient of x^2 in $\varpi(x)$, must be equal to zero, and all other cases are not admissible. This follows from the proposition below, which rules out forms satisfying the condition $p \neq 0$.

Proposition 3. *The forms arising in the second, third, fourth, and fifth families in Theorem 1 are not STDRFs.*

Proof. We give the proof only for forms in the second family. The proofs for the families three, four, and five can be conducted in the same way.

From (19), the formal Stieltjes function of the form w in the second family in Theorem 1 fulfils (7) with

$$A_0(z)=(z+p)\left(z+\tfrac{2}{3}p\right)\left(z-\tfrac{1}{3}p\right), \quad C_0(z)=\left(3\alpha+\tfrac{3}{2}\right)z^2+\left(3\alpha+\tfrac{1}{2}\right)pz-\tfrac{1}{3}p^2.$$

Therefore,

$$\frac{C_0(z)}{A_0(z)}=\frac{3/2}{x+p}+\frac{2\alpha}{x+\frac{2}{3}p}+\frac{\alpha}{x-\frac{1}{3}p}.$$

By applying Proposition 2, it follows that this form cannot be a STDRF. □

Since the last proposition rules out the possibility of taking $p\neq 0$, we set $p=0$ from now on. Moreover, we can use Proposition 1 to derive the following result concerning the moments of w, which will be useful later.

Lemma 5. *Let $\{W_n\}_{n\geqslant 0}$ be a MOPS with respect to w fulfilling (11)–(13) with $b_n^1=b_n^2=0$, $n\geqslant 0$ and $p=0$. Then*

$$(w)_{3n+1}=0,\quad n\geqslant 0, \tag{28}$$

$$(w)_{3n+2}=\gamma_1\big(\tau_{-r}(v_0)\big)_n,\quad n\geqslant 0, \tag{29}$$

$$(w)_{3n+3}=\beta_1\gamma_1\big(\tau_{-r}(v_0)\big)_n,\quad n\geqslant 0. \tag{30}$$

Proof. For any $i\in\{1,2,3\}$ and for any $n\geqslant 0$, we have

$$\begin{aligned}(w)_{3n+i}&=\langle x^i w, x^{3n}\rangle=\big\langle x^i w,\big(\sigma_\varpi(x)-r\big)^n\big\rangle=\big\langle\sigma_\varpi(x^i w),(x-r)^n\big\rangle\\&=\big\langle\tau_{-r}\big(\sigma_\varpi(x^i w)\big),x^n\big\rangle=\Big(\tau_{-r}\big(\sigma_\varpi(x^i w)\big)\Big)_n.\end{aligned}$$

Hence, the statements follow immediately from (14)–(16). □

We can formulate the following proposition, which establishes a connection between the formal Stieltjes function corresponding to the form w and its second component v_0, under the condition that $\varpi(x)=x^3+r$, $r\in\mathbb{C}$.

Proposition 4. *Let $\{W_n\}_{n\geqslant 0}$ be a MOPS fulfilling (11)–(13) with $b_n^1=b_n^2=0$, $n\geqslant 0$ and $p=0$. The formal Stieltjes functions $S(w)$ and $S(v_0)$ associated with the forms w and v_0 with respect to $\{W_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ (resp.) are related by*

$$zS(w)(z)=\gamma_1(z+\beta_1)S(v_0)(z^3+r)-1. \tag{31}$$

Proof.

$$\begin{aligned}zS\big(w\big)(z)&=-\sum_{n\geqslant 0}\frac{(w)_n}{z^n}\\&\overset{(28)}{=}-1-\sum_{n\geqslant 0}\frac{(w)_{3n+2}}{z^{3n+2}}-\sum_{n\geqslant 0}\frac{(w)_{3n+3}}{z^{3n+3}}\\&\overset{(29)-(30)}{=}-1-\gamma_1\sum_{n\geqslant 0}\frac{\big(\tau_{-r}(v_0)\big)_n}{z^{3n+2}}-\beta_1\gamma_1\sum_{n\geqslant 0}\frac{\big(\tau_{-r}(v_0)\big)_n}{z^{3n+3}}\\&=-1+\gamma_1 zS\big(\tau_{-r}(v_0)\big)(z^3)+\beta_1\gamma_1 S\big(\tau_{-r}(v_0)\big)(z^3)\\&\overset{(10)}{=}-1+\gamma_1 zS\big(v_0\big)(z^3+r)+\beta_1\gamma_1 S\big(v_0\big)(z^3+r)\end{aligned}$$

$$= -1 + \gamma_1(z+\beta_1)S\big(v_0\big)(z^3+r). \qquad \square$$

As a result, we obtain the following characterization, which is crucial for this study.

Proposition 5. *[21] Under the assumption of Proposition 4, let w and v_0 be the regular forms with respect to $\{W_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively. Then the following holds:*

(i) w is a TDRF if and only if v_0 is a TDRF.

(ii) w is a second-degree form if and only if v_0 is a second-degree form.

(iii) w is a STDRF if and only if v_0 is a STDRF.

With the above results, we can now formulate and establish the main result of this section.

Proposition 6. *Among the semiclassical forms of class one, such that their corresponding MOPS satisfies (11)–(13) and $b_n^1 = b_n^2 = 0$, $n \geqslant 0$, only the form denoted by $w^{\{\tau,k\}}$ and satisfying the functional equation*

$$\big((x^3+\tau^3)w^{\{\tau,k\}}\big)' + \big(-3(k+2)x^2 + \tau x - \tau^2\big)w^{\{\tau,k\}} = 0,$$

is a STDRF provided that $\big(w^{\{\tau,k\}}\big)_0 = 1$, $\big(w^{\{\tau,k\}}\big)_1 = 0$, $\tau \in \mathbb{C}-\{0\}$, $p=0$ and $k\in\mathbb{Z}$ with $k \geqslant -1$.

Proof. Using (18) and according to Theorem 2, Proposition 5, and the fact that the shifted form of a STDRF is also a STDRF, we get that w is a STDRF semiclassical form of class one, if and only if $\alpha = k + 2/3$ with $k \in \mathbb{Z}$, $k \geqslant -2$. Moreover, according to the regularity conditions (17) of the form $w^{\{\tau,k\}}$, we further have that $\alpha = k + \frac{2}{3} \neq -n - \frac{1}{3}$ for $n \geqslant 1$, that is, $k \neq -n-1$ for $n \geqslant 1$, and therefore $k \geqslant -1$. Hence the desired result follows. $\square$

4. Several characterizations of third-degree semiclassical linear forms of class one appearing via cubic decomposition. The primary objective of this section is to establish several characterizations for semiclassical linear forms of class one, which are strictly of third degree, and whose corresponding MOPS $\{W_n\}_{n\geqslant 0}$ satisfies the CD (11)–(13) with $b_n^1 = b_n^2 = 0$ for $n \geqslant 0$. This will be achieved by highlighting the connection between these forms and the $\mathcal{V}_2^{k,1}$ forms, their respective Stieltjes function, and moments.

In what follows, we introduce some notation that will be used to formulate and prove the results.

$$a_\tau := 2\tau^{-3}, \quad \gamma_{\tau,k} := -\frac{\tau^2}{3(k+2)}, \quad \tau \in \mathbb{C} - \{0\},\ k \in \mathbb{Z},\ k \geqslant -1, \tag{32}$$

$$X_\tau := a_\tau x^3 + 1, \quad Z_\tau := a_\tau z^3 + 1,$$

$$\widehat{\mathcal{V}_q^{k,l}}^{a_\tau} := (h_{a_\tau^{-1}} \circ \tau_{-1})\mathcal{V}_q^{k,l}, \quad q \in \{1,2\},\ k,l \in \mathbb{Z} \text{ with } k+l \geqslant -1, \tag{33}$$

where $\mathcal{V}_q^{k,l}$ is defined in Remark 6.

Next, we require the following lemma:

Lemma 6. *[17], [21] Let $\widehat{\mathcal{V}}^a := (h_{a^{-1}} \circ \tau_1)\mathcal{V}$, with $a \in \mathbb{C} - \{0\}$. One has*

$$S\big(\widehat{\mathcal{V}}^a\big)(z^3) = z^{-2} S\big(\widehat{\mathcal{V}}_{\lambda,0}\widehat{\mathcal{V}}_{\lambda,1}\widehat{\mathcal{V}}_{\lambda,2}\big)(z), \tag{34}$$

where $\widehat{\mathcal{V}}_{\lambda,\nu} := (h_{A_{\lambda,\nu}^{-1}} \circ \tau_1)\mathcal{V}$, with $A_{\lambda,\nu} = -\frac{2}{j^\nu \lambda}$, $\nu = 0,1,2$, where λ is a cubic root of the complex number $-\frac{2}{a}$ and $j = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Having established the necessary prerequisites, we can now present the principal outcome of this section.

Theorem 3. *Let w be a regular and normalized form. The following statements are equivalent:*

(a) The form w is a strict third-degree semiclassical form of class one, such that its corresponding MOPS $\{W_n\}_{n\geqslant 0}$ satisfies (11) – (13) with $b_n^1 = b_n^2 = 0$, $n \geqslant 0$.

(b) There exist $\tau \in \mathbb{C} - \{0\}$, and $k \in \mathbb{Z}$ with $k \geqslant -1$, such that

$$\sigma_\varpi(x^2 w) = \gamma_{\tau,k}\big(h_{a_\tau^{-1}} \circ \tau_{a_\tau r - 1}\big)\mathcal{V}_2^{k,1},$$

where σ_ϖ is the operator defined in (2) and $r = \varpi(0)$.

(c) There exist $\tau \in \mathbb{C} - \{0\}$, and $k \in \mathbb{Z}$ with $k \geqslant -1$, such that

$$\beta_k x^3 w = \gamma_{\tau,k} a_\tau^{k+2}(x+\tau)x^3(x^3+\tau^3)^{k+1}\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2} \tag{35}$$

and

$$\Big(\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\big((x+\tau)x^2(x^3+\tau^3)^{k+1}\big)\Big)(z)$$
$$= a_\tau^{-k-1}(z+\tau)z^2\big((h_{-1}\mathcal{V})\theta_0 g_2^{k,1}\big)(Z_\tau) + a_\tau^{-k-2}\beta_k, \tag{36}$$

where $\widehat{\mathcal{V}}_{\tau,\nu} := (h_{A_{\tau,\nu}^{-1}} \circ \tau_1)\mathcal{V}$, with $A_{\tau,\nu} = -\frac{2}{j^\nu \tau}$, $\nu = 0,1,2$, $j = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\beta_k := -\dfrac{2^{k+1}\sqrt{3}\Gamma(k+\frac{5}{3})\Gamma(\frac{4}{3})}{\pi(k+2)!}$.

(d) *There exist* $\tau \in \mathbb{C} - \{0\}$, *and* $k \in \mathbb{Z}$ *with* $k \geqslant -1$, *such that*

$$zS(w)(z) = \gamma_{\tau,k}(z+\tau)S\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)(z^3) - 1.$$

(e) *There exist* $\tau \in \mathbb{C} - \{0\}$, *and* $k \in \mathbb{Z}$ *with* $k \geqslant -1$, *such that*

$$xw = \gamma_{\tau,k}(x+\tau)\boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big),$$

where $\boldsymbol{\varrho}$ *is the operator defined in* (3).

(f) *There exist* $\tau \in \mathbb{C} - \{0\}$, *and* $k \in \mathbb{Z}$ *with* $k \geqslant -1$, *such that*

$$\begin{aligned}(w)_n =&\delta_{\lfloor \frac{n+2}{3}\rfloor - \lfloor \frac{n+1}{3}\rfloor, 0}\frac{\lfloor \frac{n-1}{3}\rfloor !\ \gamma_{\tau,k}\tau^{1+\lfloor \frac{n}{3}\rfloor - \lfloor \frac{n+1}{3}\rfloor}}{\left(\frac{2}{\tau^3}\right)^{\lfloor \frac{n-1}{3}\rfloor}} \\ &\times \sum_{\nu+\mu=\lfloor \frac{n-1}{3}\rfloor} \frac{1}{\nu!\mu!}\sum_{i=0}^{\mu}\binom{\mu}{i}2^{i-1}\frac{\Gamma(k+3)}{\Gamma(\nu+k+3)}F_{\mu,i}\big(k+\tfrac{2}{3},\tfrac{1}{3}\big),\end{aligned}$$

for all $n \geqslant 1$, *where* $\lfloor x\rfloor$ *denotes the integer part of* x, $\delta_{n,m}$ *is the Kronecker symbol, and* $F_{\mu,i}\big(k+\frac{2}{3},\frac{1}{3}\big)$ *is defined by* (27).

Proof. ***(a)*** $\Rightarrow$ ***(b)***. Suppose that w is a strict third-degree semiclassical form of class one, such that the corresponding MOPS $\{W_n\}_{n\geqslant 0}$ satisfies (11)–(13) with $b_n^1 = b_n^2 = 0$, $n \geqslant 0$. Proposition 6 guarantees the existence of $\tau \in \mathbb{C} - \{0\}$ and $k \in \mathbb{Z}$ with $k \geqslant -1$, such that $w = w^{\{\tau,k\}}$. Based on the notation provided in Remark 6, (18) becomes

$$v_0 = \left(\tau_{r-\frac{\tau^3}{2}} \circ h_{\frac{\tau^3}{2}}\right)J(k+2/3,1/3) = \left(\tau_{r-\frac{\tau^3}{2}} \circ h_{\frac{\tau^3}{2}}\right)\mathcal{J}(k+2/3,1/3)\mathcal{V}_2^{k,1}.$$

Using the well-known relation, in $\mathcal{P}'$, $\tau_B \circ h_A = h_A \circ \tau_{A^{-1}B}$, $A \in \mathbb{C} - \{0\}$, $B \in \mathbb{C}$, we can rewrite the given expression as

$$v_0 = \left(h_{\frac{\tau^3}{2}} \circ \tau_{\frac{2r}{\tau^3}-1}\right)\mathcal{V}_2^{k,1}. \tag{37}$$

Thus, our statement follows.

(b) $\Rightarrow$ ***(c)***. Combining relations (10), (31) and (37), it is easy to check that

$$zS(w)(z) = a_\tau\gamma_{\tau,k}(z+\tau)S(\mathcal{V}_2^{k,1})(Z_\tau) - 1,$$

with a_τ and $\gamma_{\tau,k}$ defined in (32).

By multiplying both sides of the last equation by $f_2^{k,1}(Z_\tau)$, we can deduce from (4)

$$
\begin{aligned}
&f_2^{k,1}(Z_\tau)zS(w)(z)\\
&=a_\tau\gamma_{\tau,k}(z+\tau)S\big(f_2^{k,1}\mathcal{V}_2^{k,1}\big)(Z_\tau)-a_\tau\gamma_{\tau,k}(z+\tau)\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)(Z_\tau)-f_2^{k,1}(Z_\tau)\\
&\overset{(21)}{=}a_\tau\gamma_{\tau,k}(z+\tau)S\big(g_2^{k,1}(h_{-1}\mathcal{V})\big)(Z_\tau)-a_\tau\gamma_{\tau,k}(z+\tau)\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)(Z_\tau)-f_2^{k,1}(Z_\tau)\\
&\overset{(4)}{=}a_\tau\gamma_{\tau,k}(z+\tau)g_2^{k,1}(Z_\tau)S(h_{-1}\mathcal{V})(Z_\tau)\\
&\quad+a_\tau\gamma_{\tau,k}(z+\tau)\Big(\big((h_{-1}\mathcal{V})\theta_0 g_2^{k,1}\big)-\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)\Big)(Z_\tau)-f_2^{k,1}(Z_\tau)\\
&\overset{(10)}{=}-a_\tau\gamma_{\tau,k}(z+\tau)g_2^{k,1}(Z_\tau)S(\mathcal{V})(-Z_\tau)\\
&\quad+a_\tau\gamma_{\tau,k}(z+\tau)\Big(\big((h_{-1}\mathcal{V})\theta_0 g_2^{k,1}\big)-\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)\Big)(Z_\tau)-f_2^{k,1}(Z_\tau)\\
&=\gamma_{\tau,k}(z+\tau)g_2^{k,1}(Z_\tau)z^{-2}S\big(\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\big)(z)\\
&\quad+a_\tau\gamma_{\tau,k}(z+\tau)\Big(\big((h_{-1}\mathcal{V})\theta_0 g_2^{k,1}\big)-\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)\Big)(Z_\tau)-f_2^{k,1}(Z_\tau).
\end{aligned}
$$

The last line is obtained using Lemma 6 with $\widehat{\mathcal{V}}_{\tau,\nu}:=(h_{A_{\tau,\nu}^{-1}}\circ\tau_1)\mathcal{V}$, with $A_{\tau,\nu}=-\frac{2}{j^\nu\tau}$, $\nu=0,1,2$, $j=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, where $a=-a_\tau$, and $\lambda=\tau$ is a cubic root of the complex number $-\frac{2}{a}=\frac{2}{a_\tau}=\tau^3$.

Now, multiplying by z^2 and using relation (4), the latter becomes

$$
\begin{aligned}
S\Big(x^3f_2^{k,1}(X_\tau)w\Big)(z)&=S\Big(\gamma_{\tau,k}(x+\tau)g_2^{k,1}(X_\tau)\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\Big)(z)\\
&\quad+a_\tau\gamma_{\tau,k}(z+\tau)z^2\Big(\big((h_{-1}\mathcal{V})\theta_0 g_2^{k,1}\big)-\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)\Big)(Z_\tau)\\
&\quad+\big(w\theta_0(x^3f_2^{k,1}(X_\tau))\big)(z)-z^2f_2^{k,1}(Z_\tau)\\
&\quad-\gamma_{\tau,k}\Big(\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\theta_0\big((x+\tau)g_2^{k,1}(X_\tau)\big)\Big)(z),
\end{aligned}
$$

which readily gives

$$
S\Big(x^3f_2^{k,1}(X_\tau)w-\gamma_{\tau,k}(x+\tau)g_2^{k,1}(X_\tau)\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\Big)(z)=N_{\tau,k}(z)\in\mathcal{P},
$$

with

$$
\begin{aligned}
N_{\tau,k}(z):=&+a_\tau\gamma_{\tau,k}(z+\tau)z^2\Big(\big((h_{-1}\mathcal{V})\theta_0 g_2^{k,1}\big)-\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)\Big)(Z_\tau)\\
&+\big(wx^2f_2^{k,1}(X_\tau)\big)(z)-z^2f_2^{k,1}(Z_\tau)\\
&-\gamma_{\tau,k}\Big(\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\theta_0\big((x+\tau)g_2^{k,1}(X_\tau)\big)\Big)(z).
\end{aligned}
$$

It follows from Remark 1 that

$$
x^3f_2^{k,1}(X_\tau)w-\gamma_{\tau,k}(x+\tau)g_2^{k,1}(X_\tau)\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}=0 \quad \text{in } \mathcal{P}', \tag{38}
$$

and

$$N_{\tau,k} = 0, \tag{39}$$

which are exactly the desired relations (35) and (36), respectively. Indeed, let us recall that a Jacobi form $\mathcal{J}(\alpha,\beta)$ has the following integral representation for $\operatorname{Re}(\alpha+1)>0$ and $\operatorname{Re}(\beta+1)>0$ [27]:

$$\langle \mathcal{J}(\alpha,\beta), f\rangle = C_{\alpha,\beta}\int_{-1}^{+1}(1+x)^{\alpha}(1-x)^{\beta}f(x)dx, \quad f\in\mathcal{P}, \tag{40}$$

with

$$C_{\alpha,\beta} = \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}. \tag{41}$$

For any $k\geqslant -1$, it is easy to see that

$$\begin{aligned}
f_2^{k,1}(x) &\overset{(22)}{=} \big\langle h_{-1}\mathcal{V},(x+1)^{k+1}(x-1)\big\rangle\\
&= \big\langle \mathcal{J}\big(-1/3,-2/3\big),(x+1)^{k+1}(x-1)\big\rangle\\
&\overset{(40)}{=} -C_{-\frac{1}{3},-\frac{2}{3}}\int_{-1}^{+1}(1+x)^{k+\frac{2}{3}}(1-x)^{\frac{1}{3}}dx\\
&\overset{(40)}{=} -\frac{C_{-\frac{1}{3},-\frac{2}{3}}}{C_{k+\frac{2}{3},\frac{1}{3}}} \overset{(41)}{=} -\frac{2^{k+1}\sqrt{3}\Gamma(k+\frac{5}{3})\Gamma(\frac{4}{3})}{\pi(k+2)!} =: \beta_k,
\end{aligned}$$

and

$$g_2^{k,1}(x) \overset{(23)}{=} (x+1)^{k+1}(x-1).$$

Hence, the statement follows immediately from (38)−(39), once the following relations

$$\begin{aligned}
\big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)(z) &= 0,\\
\big(wx^2 f_2^{k,1}(X_\tau)\big)(z) &= \beta_k\big(z^2+\gamma_{\tau,k}\big),\\
g_2^{k,1}(X_\tau) &= a_\tau^{k+2}x^3(x^3+\tau^3)^{k+1},
\end{aligned}$$

which are easily checked, are taken into consideration.

(c)* ⇒ *(d). Notice that, according to the proof of the preceding implication, we have already shown that relations (35) and (36) are exactly (38) and (39), respectively. So, applying the linear operator S to (35) and taking into account formula (4), we get

$$z^3 f_2^{k,1}(Z_\tau)S(w)(z) = \gamma_{\tau,k}(z+\tau)g_2^{k,1}(Z_\tau)S\big(\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\big)(z) + M_{\tau,k}(z),$$

with

$$M_{\tau,k}(z) := \Big(w\big(x^2 f_2^{k,1}(X_\tau)\big)\Big)(z) - \gamma_{\tau,k}\Big(\widehat{\mathcal{V}}_{\tau,0}\widehat{\mathcal{V}}_{\tau,1}\widehat{\mathcal{V}}_{\tau,2}\theta_0\big((x+\tau)g_2^{k,1}(X_\tau)\big)\Big)(z).$$

Then we use (34) to obtain

$$\begin{aligned}
z^3 f_2^{k,1}(Z_\tau)S(w)(z) &= a_\tau\gamma_{\tau,k}z^2(z+\tau)g_2^{k,1}(Z_\tau)S(h_{-1}\mathcal{V})(Z_\tau) + M_{\tau,k}(z)\\
&\overset{(4)}{=} a_\tau\gamma_{\tau,k}z^2(z+\tau)S(g_2^{k,1}(h_{-1}\mathcal{V}))(Z_\tau) - a_\tau z^2(z+\tau)((h_{-1}\mathcal{V})\theta_0 g_2^{k,1})(Z_\tau) + M_{\tau,k}(z)\\
&\overset{(21)}{=} a_\tau\gamma_{\tau,k}z^2(z+\tau)S(f_2^{k,1}\mathcal{V}_2^{k,1})(Z_\tau) - a_\tau z^2(z+\tau)((h_{-1}\mathcal{V})\theta_0 g_2^{k,1})(Z_\tau) + M_{\tau,k}(z)\\
&\overset{(4)}{=} a_\tau\gamma_{\tau,k}z^2(z+\tau)f_2^{k,1}(Z_\tau)S\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) + M_{\tau,k}(z)\\
&\quad + a_\tau\gamma_{\tau,k}z^2(z+\tau)\Big(\big((h_{-1}\mathcal{V})\theta_0 g_2^{k,1}\big) - \big(\mathcal{V}_2^{k,1}\theta_0 f_2^{k,1}\big)\Big)(Z_\tau).
\end{aligned}$$

Hence, one has from (39)

$$z^3 f_2^{k,1}(Z_\tau)S(w)(z) = a_\tau\gamma_{\tau,k}z^2(z+\tau)f_2^{k,1}(Z_\tau)S\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) - z^2 f_2^{k,1}(Z_\tau).$$

This clearly implies that $zS(w)(z) = a_\tau\gamma_{\tau,k}(z+\tau)S\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) - 1$, which is what we wanted to prove.

(d)* ⇒ *(e). Based on the definition of the operator $\boldsymbol{\varrho}$ and the enunciated relation, we have

$$zS(w)(z) \overset{(3)}{=} -\gamma_{\tau,k}(z+\tau)\sum_{n\geqslant 0}\frac{\Big(\boldsymbol{\varrho}\Big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\Big)\Big)_n}{z^{n+1}} = \gamma_{\tau,k}(z+\tau)S\Big(\boldsymbol{\varrho}\Big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\Big)\Big)(z).$$

Afterwards, using formula (4) and taking into account the linearity of operator S, the last relation becomes

$$S\Big(xw - \gamma_{\tau,k}(x+\tau)\boldsymbol{\varrho}\Big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\Big)\Big)(z) = \Big(\boldsymbol{\varrho}\Big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\Big)\theta_0\rho_\tau\Big)(z) \in \mathcal{P}.$$

From Remark 1, we infer $xw - \gamma_{\tau,k}(x+\tau)\boldsymbol{\varrho}\Big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\Big) = 0$ in $\mathcal{P}'$, which yields statement (e).

(e)* ⇒ *(f). With the enunciated relation, we have for all $n \geqslant 0$:

$$\begin{aligned}
(w)_{3n+1} &= \gamma_{\tau,k}\Big\langle \boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big), x^{3n}(x+\tau)\Big\rangle\\
&= \gamma_{\tau,k}\big(\boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)\big)_{3n+1} + \gamma_{\tau,k}\tau\big(\boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)\big)_{3n} \overset{(3)}{=} 0,
\end{aligned}$$

$$
\begin{aligned}
(w)_{3n+2} &= \gamma_{\tau,k}\Big\langle \boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big), x^{3n+1}(x+\tau)\Big\rangle \\
&= \gamma_{\tau,k}\Big(\boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)\Big)_{3n+2} + \gamma_{\tau,k}\tau\Big(\boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)\Big)_{3n+1} \overset{(3)}{=} \gamma_{\tau,k}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_n, \\
(w)_{3n+3} &= \gamma_{\tau,k}\Big\langle \boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big), x^{3n+2}(x+\tau)\Big\rangle \\
&= \gamma_{\tau,k}\Big(\boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)\Big)_{3n+3} + \gamma_{\tau,k}\tau\Big(\boldsymbol{\varrho}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)\Big)_{3n+2} \overset{(3)}{=} \gamma_{\tau,k}\tau\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_n.
\end{aligned}
$$

Summing up the last three relations, we gain the following one:

$$
(w)_n = \delta_{\lfloor \frac{n+2}{3}\rfloor - \lfloor \frac{n+1}{3}\rfloor, 0}\,\gamma_{\tau,k}\tau^{1+\lfloor \frac{n}{3}\rfloor - \lfloor \frac{n+1}{3}\rfloor}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_{\lfloor \frac{n-1}{3}\rfloor}, \quad n \geqslant 1, \tag{42}
$$

where $\lfloor x\rfloor$ denotes the integer part of x and $\delta_{n,m}$ is the Kronecker symbol. Hence, the desired result is reached using the formula (26) and taking into account notation (32)–(33).

(f)* ⇒ *(a). The proof will be presented in three steps.

First step. Here, our goal is to show that the form w is a STDRF.

Let us first prove the identity

$$
zS(w)(z) = \gamma_{\tau,k}(z+\tau)S\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)(z^3) - 1. \tag{43}
$$

Based on the equivalence between the enunciated relation and relation (42), it is easy to check that for $n \geqslant 0$

$$
(w)_0 = 1,\ (w)_{3n+1} = 0,\ (w)_{3n+2} = \gamma_{\tau,k}\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_n,\ (w)_{3n+3} = \gamma_{\tau,k}\tau\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_n. \tag{44}
$$

As a consequence,

$$
\begin{aligned}
zS(w)(z) &= -\sum_{n\geqslant 0}\frac{(w)_{3n}}{z^{3n}} - \sum_{n\geqslant 0}\frac{(w)_{3n+1}}{z^{3n+1}} - \sum_{n\geqslant 0}\frac{(w)_{3n+2}}{z^{3n+2}} \\
&\overset{\text{by } (44)}{=} -\gamma_{\tau,k}\tau\sum_{n\geqslant 0}\frac{\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_n}{z^{3n+3}} - \gamma_{\tau,k}z\sum_{n\geqslant 0}\frac{\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_n}{z^{3n+3}} - 1 \\
&= -\gamma_{\tau,k}(z+\tau)\sum_{n\geqslant 0}\frac{\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)_n}{z^{3n+3}} - 1,
\end{aligned}
$$

which matches with (43).

On the other hand, relations (20) and (21) and Lemma 3 provide the fact that the form $\mathcal{V}_2^{k,1}$ is a STDRF and its Stieltjes function $S\big(\mathcal{V}_2^{k,1}\big)(z)$ satisfies the following cubic equation:

$$
A_2^{k,1}(z)S^3\big(\mathcal{V}_2^{k,1}\big)(z) + B_2^{k,1}(z)S^2\big(\mathcal{V}_2^{k,1}\big)(z) + C_2^{k,1}(z)S\big(\mathcal{V}_2^{k,1}\big)(z) + D_2^{k,1}(z) = 0.
$$

Now, the fact that the affine transformation of a STDRF is also a STDRF means that $\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}$ is a STDRF and its Stieltjes function $S\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)(z)$ satisfies the cubic equation

$$\begin{aligned}\widehat{A_2^{k,1}}(z)S^3\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)(z) + \widehat{B_2^{k,1}}(z)S^2\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)(z)\\ + \widehat{C_2^{k,1}}(z)S\big(\widehat{\mathcal{V}_2^{k,1}}^{a_\tau}\big)(z) + \widehat{D_2^{k,1}}(z) = 0, \quad (45)\end{aligned}$$

with

$$\begin{aligned}\widehat{A_2^{k,1}}(z) &= a_\tau^{-3}A_2^{k,1}(a_\tau z+1), \quad \widehat{B_q^{k,l}}(z) = a_\tau^{-2}B_2^{k,1}(a_\tau z+1),\\ \widehat{C_2^{k,1}}(z) &= a_\tau^{-1}C_2^{k,1}(a_\tau z+1), \quad \widehat{D_2^{k,1}}(z) = D_2^{k,1}(a_\tau z+1).\end{aligned}$$

Making $z \leftarrow z^3$ in (45), multiplying this equation by $\gamma_{\tau,k}^3(z+\tau)^3$, and on account of (43), we obtain

$$A_w(z)S^3(w)(z) + B_w(z)S^2(w)(z) + C_w(z)S(w)(z) + D_w(z) = 0,$$

with $A_w(z) = \widehat{A_2^{k,1}}(z^3)$, $B_w(z) = \gamma_{\tau,k}(z+\tau)\widehat{B_2^{k,1}}(z^3)$, $C_w(z) = \gamma_{\tau,k}^2(z+\tau)^2\widehat{C_2^{k,1}}(z^3)$, $D_w(z) = \gamma_{\tau,k}^3(z+\tau)^3\widehat{D_2^{k,1}}(z^3)$. As a consequence, we conclude that w is a STDRF.

Second step. We will prove that the form w is semiclassical of class one. Indeed, according to the assumption, with the notation in (33) and based on (10), relation (43) becomes

$$S\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) = \frac{zS(w)(z)+1}{\gamma_{\tau,k}a_\tau(z+\tau)}. \quad (46)$$

Taking formal derivatives in the last equation, we get

$$zS'(w)(z) + S(w)(z) = 3\gamma_{\tau,k}a_\tau^2z^2(z+\tau)S'\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) + a_\tau\gamma_{\tau,k}S\big(\mathcal{V}_2^{k,1}\big)(Z_\tau). \quad (47)$$

The combination of equations (46) and (47) yields

$$S'\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) = \frac{z(z+\tau)S'(w)(z) + \beta_1 S(w)(z) - 1}{3\gamma_{\tau,k}a_\tau^2z^2(z+\tau)^2}. \quad (48)$$

In the first-order linear differential equation (24), the change of variable $z \leftarrow Z_\tau$ gives

$$\Phi(Z_\tau)S'\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) = C_{0,2}^{k,1}(Z_\tau)S\big(\mathcal{V}_2^{k,1}\big)(Z_\tau) + D_{0,2}^{k,1}(Z_\tau). \quad (49)$$

Injecting (46) and (48) in (49), and multiplying both sides of the resulting equation by $3\gamma_{\tau,k}a_\tau^2 z^2(z+\tau)^2$, one obtains

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{50}$$

where the polynomials ϕ_w, C_w, and D_w are

$$\begin{aligned}
\phi_w(z) &= z(z+\tau)\Phi(Z_\tau),\\
C_w(z) &= -\tau\Phi(Z_\tau) + 3a_\tau z^3(z+\tau)C_{0,2}^{k,1}(Z_\tau),\\
D_w(z) &= 3\gamma_{\tau,k}a_\tau^2 z^2(z+\tau)^2 D_{0,2}^{k,1}(Z_\tau) + 3a_\tau z^2(z+\tau)C_{0,2}^{k,1}(Z_\tau) + \Phi(Z_\tau).
\end{aligned}$$

Now, from (25) and in view of that $a_\tau = \left(\frac{\tau^3}{2}\right)^{-1}$ and $Z_\tau = \left(\frac{\tau^3}{2}\right)^{-1}z^3 + 1$, the Stieltjes function $S(w)(z)$ fulfills (50) with

$$\begin{aligned}
\phi_w(z) =& \left(\frac{\tau^3}{2}\right)^{-2} z^4(z+\tau)\left(z^3+\tau^3\right),\\
C_w(z) =& -\tau\left(\frac{\tau^3}{2}\right)^{-2} z^3\left(z^3+\tau^3\right)\\
&+ 3\left(\frac{\tau^3}{2}\right)^{-1} z^3(z+\tau)\left[(k+1)\left(\left(\frac{\tau^3}{2}\right)^{-1}z^3+1\right) - k - \frac{1}{3}\right],\\
D_w(z) =& 3\gamma_{\tau,k}\left(\frac{\tau^3}{2}\right)^{-2} z^2(z+\tau)^2(k+2)\\
&+ 3\left(\frac{\tau^3}{2}\right)^{-1} z^2(z+\tau)\left[(k+1)\left(\left(\frac{\tau^3}{2}\right)^{-1}z^3+1\right) - k - \frac{1}{3}\right]\\
&+ \left(\frac{\tau^3}{2}\right)^{-2} z^3\left(z^3+\tau^3\right).
\end{aligned}$$

Hence, it is clear that the polynomials ϕ_w, C_w, and D_w have $\left(\frac{\tau^3}{2}\right)^{-2}(z+\tau)z^4$ as a common factor; so, dividing these polynomials by $\left(\frac{\tau^3}{2}\right)^{-2}(z+\tau)z^4$ gives

$$\phi_w(z) = z^3+\tau^3,\ C_w(z) = 3(k+1)z^2-\tau z+\tau^2,\ D_w(z) = (3k+4)z-\tau. \tag{51}$$

Further, we see that the conditions

$$\begin{aligned}
C_w(-\tau) &= (3k+5)\tau^2 \neq 0,\\
C_w(-\tau j) &= (3k+2)\tau^2 j^2 \neq 0,\\
C_w(-\tau j^2) &= (3k+2)\tau^2 j \neq 0,
\end{aligned}$$

hold. Thus, the polynomials ϕ_w, C_w, and D_w are coprime, hence, since $\deg D_w \leqslant 1$ and $\deg C_w = 2$, the class of w is one.

Third step. To finish the proof, it remains to prove that $w = w^{\{\tau,k\}}$. Using (51), one can check that the form w satisfies the functional equation

$$(\phi_w w)' + \psi_w w = 0,$$

with

$$\begin{aligned}\phi_w(x) &= x^3 + \tau^3,\\ \psi_w(x) &= -\phi_w'(x) - C_w(x) = -3(k+2)x^2 + \tau x - \tau^2.\end{aligned}$$

Therefore, this form w is semiclassical of class one and satisfies the same functional equation as that of the form $w^{\{\tau,k\}}$ arising in Proposition 6, and also $\deg \phi_w = \deg \psi_w + 1 = 3$. Moreover, it is easy to verify that $(w)_0 = 1$ and $(w)_1 = 0$, which in fact coincide with the first two moments of the form $w^{\{\tau,k\}}$. As a consequence, using Lemma 2 we conclude that $w = w^{\{\tau,k\}}$. Thus, the proof is complete. □

5. Structure relation and differential equation.

5.1. Structure relation. It is well known that any semiclassical sequence $\{W_n\}_{n\geqslant 0}$ satisfies the following structure relation [28]:

$$\phi(x)W_{n+1}'(x) = \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)W_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)W_n(x),\ n \geqslant 0, \tag{52}$$

where

$$C_{n+1}(x) = -\,C_n(x) + 2(x-\beta_n)D_n(x), \quad n \geqslant 0, \quad \deg C_n \leqslant 2, \tag{53}$$

$$\begin{aligned}\gamma_{n+1}D_{n+1}(x) = &-\phi(x) + \gamma_n D_{n-1}(x) + (x-\beta_n)^2 D_n(x)\\ &-(x-\beta_n)C_n(x), \quad n \geqslant 0, \quad \deg D_n \leqslant 1,\end{aligned} \tag{54}$$

with $D_{-1}(x) = 0$, C_0 and D_0 are given by (8).
Now, we are going to give the elements of the structure relation of the sequence $\{W_n\}_{n\geqslant 0}$ of the first family of Theorem 1.

In general, it is challenging to express the sequences $\{C_n\}_{n\geqslant 0}$ and $\{D_n\}_{n\geqslant 0}$ explicitly using the recurrence relations (52), (53), and (54). However, the cubic decomposition enables us to do so.

Proposition 7. *The elements of the structure relation $\{C_n\}_{n\geqslant 0}$ and $\{D_n\}_{n\geqslant 0}$ of the sequence $\{W_n\}_{n\geqslant 0}$ of the first family of Theorem 1, are given by*

$$C_{3n}(x) = (6n+3\alpha+1)x^2 - \Big(6n+6\alpha+1 - \frac{6(3n+3\alpha+1)(2n+\alpha)}{6n+3\alpha+1}\Big)\tau x$$

$$+\left(6n+6\alpha+1-\frac{6(3n+3\alpha+1)(n+\alpha)}{6n+3\alpha+1}\right)\tau^2, \quad n\geqslant 0,$$
$$D_{3n}(x)=(6n+3\alpha+2)x-(3n+1)\tau, \quad n\geqslant 0,$$
$$C_{3n+1}(x)=(6n+3\alpha+3)x^2-\tau x-(6n+1)\tau^2, \quad n\geqslant 0.$$
$$D_{3n+1}(x)=(6n+3\alpha+4)(x+\tau), \quad n\geqslant 0,$$
$$C_{3n+2}(x)=(6n+3\alpha+5)x^2+\tau x-(6n+6\alpha+7)\tau^2, \quad n\geqslant 0,$$
$$D_{3n+2}(x)=(6n+3\alpha+6)x-(3n+3\alpha+3)\tau, \quad n\geqslant 0.$$

Proof. The proof is similar to that of Proposition 4.2 in [6]. □

5.2. Differential equation. It is a well-known fact that a semiclassical orthogonal polynomial sequence satisfies a second-order differential equation [28]. The purpose of this subsection is to provide an explicit expression for the second-order differential equation that is satisfied by $\{W_n\}_{n\geqslant 0}$. As a reminder, $\{W_n\}_{n\geqslant 0}$ satisfy:

$$J(x;n)W''_{n+1}(x)+K(x;n)W'_{n+1}(x)+L(x;n)W_{n+1}(x)=0, \quad n\geqslant 0, \tag{55}$$

with

$$J(x;n)=\phi(x)D_{n+1}(x), \quad n\geqslant 0,$$
$$K(x;n)=C_0(x)D_{n+1}(x)-W\big(\phi,D_{n+1}\big)(x), \quad n\geqslant 0,$$
$$L(x;n)=W\Big(\frac{1}{2}\big(C_{n+1}-C_0\big),D_{n+1}\Big)(x)-D_{n+1}(x)\sum_{\nu=0}^{n}D_\nu(x), \quad n\geqslant 0,$$

where $W(f,g)=fg'-gf'$ is the Wronskian of f and g.

Proposition 8. *The sequence $\{W_n\}_{n\geqslant 0}$ of the first family of Theorem 1 fulfils (55), where the elements characteristics $J(x;n)$, $K(x;n)$, and $L(x;n)$ are given as follows:*

$$J(x;3n)=(6n+3\alpha+4)(x+\tau)^2(x^2-\tau+x\tau^2), \quad n\geqslant 0,$$
$$K(x;3n)=3(6n+3\alpha+4)(\alpha+1)(x+\tau)x^2, \quad n\geqslant 0,$$
$$L(x;3n)=-(3n+1)(3n+3\alpha+3)(6n+3\alpha+4)(x+\tau)x, \quad n\geqslant 0,$$
$$J(x;3n+1)=(x^3+\tau^3)\big((6n+3\alpha+6)x-(3n+3\alpha+3)\tau\big), \quad n\geqslant 0,$$
$$K(x;3n+1)=9(2n+\alpha+2)(\alpha+1)x^3-3\Big(3(n+\alpha+1)(\alpha+2)-\alpha\Big)x^2$$
$$+3(3n+2\alpha+3)\tau^2x-3(3n+2\alpha+3)\tau^3, \quad n\geqslant 0,$$
$$L(x;3n+1)=-(3n+2)(6n+3\alpha+6)(3n+3\alpha+4)x^2$$

$$\begin{aligned}
&+ (3n+1)(3n+3\alpha+3)(3n+3\alpha+4)\tau x \\
&- (3n+3)(3n+3\alpha+4)\tau^2, \quad n \geqslant 0,
\end{aligned}$$

$$J(x;3n+2) = (x^3+\tau^3)\big((6n+3\alpha+8)x-(3n+4)\tau\big), \quad n \geqslant 0,$$

$$\begin{aligned}
K(x;3n+2) = {}& 3(6n+3\alpha+8)(\alpha+1)x^3 \\
&- \Big((3\alpha+4)(3n+4)+6n+3\alpha+8\Big)\tau x^2 \\
&+ 3(3n+\alpha+4)\tau^2 x - 3(3n+\alpha+4)\tau^3, \quad n \geqslant 0,
\end{aligned}$$

$$\begin{aligned}
L(x;3n+2) = {}& -3(6n+3\alpha+8)(n+1)(3n+3\alpha+5)x^2 \\
&+ (3n+4)(3n+3)(3n+3\alpha+6)\tau x \\
&+ \frac{3(n+\alpha+1)(6n+3\alpha+8)(3n+3)+(3n+3\alpha+4)(3n+4)}{(6n+3\alpha+7)}\tau^2.
\end{aligned}$$

Proof. The proof is similar to that of Proposition 4.3 in [6]. □

Remark 8. *Similarly to the above, the characteristic elements of the structure relation and the second-order differential equation of the other four families can be established.*

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