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THE MAXIMUM OF SOME FUNCTIONAL FOR HOLOMORPHIC AND UNIVALENT FUNCTIONS WITH REAL COEFFICIENTS

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In the paper the maximum of the functional $a_2^k a_3^m(a_3 - \alpha a_2^2)$ in the class S_R of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n = \overline{a_n}$, holomorphic and univalent in the unit disc is obtained for α real and k, m positive integers.

0. We consider a functional

$$H(f) = a_2^k a_3^m \left(a_3 - \alpha a_2^2\right) \tag{1}$$

for $k, m = 1, 2, \ldots$, $\alpha \in \mathbb{R}$, defined on the class S_R of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad a_n = \overline{a_n}, \tag{2}$$

holomorphic and univalent in the unit disc Δ.

In papers [2], [3], [6], the extremal values of functional (1) were determined in the cases:
1^o $m = 0$, $k = 0, 1, 2, \ldots$,
2^o $k = 0$, $m = 1, 2, \ldots$.

In the case $m = 0$ and $k = 0, 1, 2, 3, \ldots$, the maximum of the functional $|H(f)|$ for functions f belonging to the well-known class S was also determined [1], [5], [8].

The aim of the present paper is to obtain the maximum of functional (1) in the class S_R.

1. The functional $H(f)$ is continuous, whereas the class S_R is compact in the topology of locally uniform convergence in the unit disc, therefore there exist functions $f^* \in S_R$, called further extremal ones, for which

$$H(f^*) = \max_{f \in S_R} H(f).$$

It is known [2] that each extremal function from the class S_R is a solution of a differential-functional equation which, in the case of the functional $H(f)$, has the following form:

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{B_1 f(z) + B_2}{f^2(z)} = \frac{B_2 z^4 + B_1 z^3 + B_0 z^2 + B_1 z + B_2}{z^2}, \quad z \in \Delta, \tag{3}$$

where

$$B_1 = a_2^{k-1} a_3^{m-1} \left[(ka_3 + 2ma_2^2)(a_3 - \alpha a_2^2) + 2(1-\alpha)a_2^2 a_3\right], \tag{4}$$

$$B_2 = a_2^k a_3^{m-1} \left[m(a_3 - \alpha a_2^2) + a_3\right], \tag{5}$$

$$B_0 = (2 + 2m + k) a_2^k a_3^m (a_3 - \alpha a_2^2), \tag{6}$$

with that the right-hand side of the equation (3) is non-negative on the circle $|z| = 1$ and has at least one double zero on this circle; besides, the coefficient B_0 is positive.

It is easy to notice that no function (2), for which $a_2 = 0$ or $a_3 = 0$, is an extremal function. Consequently, in our further considerations we assume the coefficients a_2 and a_3 of extremal functions to be different from zero.

At present, we shall succesively consider all the admissible cases of the factorization of the numerator of the right-hand side of equation (3), also taking account of the vanishing or the non-vanishing of the coefficients B_1, B_2.

2. At first, we consider the case when equation (3) has the form

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{B_1 f(z) + B_2}{f^2(z)} = B_2 \frac{(z - \varepsilon_1)^2 (z - \varepsilon_2 r)(z - \varepsilon_2 r^{-1})}{z^2}, \quad z \in \Delta, \quad B_1 \cdot B_2 \neq 0, \tag{a}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, $r \in (0, 1)$.

Comparing the coefficients in the numerators of the right-hand sides of equations (3) and (a), we obtain, among other things,

$$2 + 2\varepsilon \left(r + \frac{1}{r} \right) = \frac{B_0}{B_2} \tag{7}$$

where $\varepsilon = \varepsilon_1 \varepsilon_2 = \pm 1$.

Whereas, integrating equation (a) and comparing the constant terms of the expansions of both sides in Laurent series, we get

$$\log \frac{2 + \left(r + \frac{1}{r}\right)\varepsilon}{(r^{-1} - r)\varepsilon} + \frac{r + r^{-1} + 2\varepsilon}{2 + (r + r^{-1})\varepsilon} \log \frac{1 - r}{1 + r} = \frac{2(a_2 + 2\varepsilon_1)\varepsilon_1}{2 + r + r^{-1}}, \tag{8}$$

whence

$$a_2 = -2\varepsilon_1, \qquad \varepsilon_1 = \pm 1, \tag{9}$$

which, in consequence, yields

$$a_3 = 3\,\varepsilon_1^2 = 3. \tag{10}$$

Taking account of (9) and (10) in (7), from the forms of B_0, B_2 we obtain

$$\frac{3(2 + 2m + k)(3 - 4\alpha)}{3 + 3m - 4\alpha\, m} = 2 + 2\varepsilon(r + \frac{1}{r}) \quad \text{for} \quad \alpha \neq \frac{3m + 3}{4m}. \tag{11}$$

The study of the dependence of α on r described by formula (11) in the cases $\varepsilon = \pm 1$ as well as the examination of the values of the functional $H(f)$ for a_2 and a_3 of forms (9) and (10), where $\varepsilon_1 = \pm 1$, imply

LEMMA 1. *If the extremal function satisfies equation (a), then*

$$H(f) = 2^k 3^m (3 - 4\alpha) \quad \text{for} \quad \alpha < \alpha_1, \quad k, m = 1, 2, \ldots, \tag{12}$$

and

$$H(f) = 2^k 3^m (4\alpha - 3) \qquad \text{for } \alpha \in \left(\alpha_2, \frac{3m+3}{4m} \right) \cup \left(\frac{3m+3}{4m}, +\infty \right), \quad k = 1, 3, \ldots,\ m = 1, 2, \ldots, \tag{13}$$

where $\alpha_1 = \frac{3k}{4k+8}$, $\alpha_2 = \frac{3(8m+3k+8)}{4(8m+3k+6)}$.

Values (12) and (13) are taken by the functional $H(f)$ for the Koebe function only. For $\alpha \in \langle \alpha_1, \alpha_2 \rangle$ and $k, m = 1, 2, \ldots$ as well as for $\alpha > \alpha_2$ and $k = 2, 4, \ldots$, $m = 1, 2, \ldots$, the extremal function does not satisfy equation (a).

3. Let us now consider equation (3) under the assumption that $B_1 \neq 0$ and $B_2 = 0$. After simple calculations we get $B_0 = B_1 a_2$, whence equation (3) takes the form

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1}{f(z)} = \frac{z^2 + a_2 z + 1}{z}, \quad z \in \Delta. \tag{b}$$

It follows from the general properties of equation (3) that

$$a_2^2 = 4$$

and, in consequence,

$$a_3 = \frac{4\alpha m}{m+1}.$$

Summing up, we obtain

LEMMA 2. *If the extremal function satisfies equation (b), then*

$$H(f) = 2^{2m+k+2} m^m \left(\frac{\alpha}{m+1}\right)^{m+1} \tag{14}$$

for $\alpha = \frac{3m+3}{4m}$ *and* $m = 1, 2, 3, \ldots$, $k = 1, 3, 5, \ldots$.

For the remaining values of α, the extremal function fails to satisfy equation (b).

Value (14) is taken by the functional $H(f)$ for the Koebe function only.

4. The successive form of equation (3) to be considered is

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{B_1 f(z) + B_2}{f^2(z)} = B_2 \frac{(z - z_0)^2 (z - \overline{z_0})^2}{z^2}, \quad z \in \Delta, \tag{c}$$

where $z_0 = e^{i\psi}$, $\psi \in \mathbb{R}$, under the condition $B_1 B_2 \neq 0$.

Comparing the coefficients in equations (3) and (c), we get

$$\frac{B_1}{B_2} = -4\cos\psi, \tag{15}$$

$$\frac{B_0}{B_2} = 2 + 4\cos^2\psi. \tag{16}$$

After integrating equation (c) and making use of the fact ([4]), that there exists $x \in \mathbb{R}$ such that $f(e^{ix}) = -\frac{B_2}{B_1}$, as well as of (15) and (16), we obtain

$$a_2 = 2\cos\psi(-1 + \log\cos\psi), \tag{17}$$

$$a_3 = 1 + 2\cos^2\psi[1 + 2(-1 + \log\cos\psi)\log\cos\psi], \tag{18}$$

$$\alpha = \frac{1}{4}\frac{M_1}{M_1 + 2(1 + 2\cos^2\psi)}\frac{1 + 2\cos^2\psi[1 + 2(-1 + \log\cos\psi)\log\cos\psi]}{\cos^2\psi(-1 + \log\cos\psi)^2} \tag{19}$$

where $M_1 = k(1+2\cos^2\psi)+4(2+2m+k)\cos^2\psi(-1+\log\cos\psi)\log\cos\psi$, and $\psi \in (0, \pi/2)$, $k, m = 1, 2, 3, \ldots$.

Consequently, we have proved

LEMMA 3. *If the extremal function satisfies equation (c), then the value of the functional $H(f)$ is given by a_2 and a_3 defined by formulae (17) and (18) where ψ is the function inverse to increasing function (19).*

5. Now, consider the case when equation (3) is of the form

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{B_1 f(z) + B_2}{f^2(z)} = B_2\frac{(z \pm 1)^4}{z^2}, \quad B_1 \cdot B_2 \neq 0, \tag{d}$$

with that

$$\frac{B_1}{B_2} = \mp 4, \tag{20}$$

$$\frac{B_0}{B_2} = 6. \tag{21}$$

Taking account of (20) in equation (d), after integrating we obtain

$$\frac{\sqrt{1 \mp 4f(z)}}{f(z)} \pm 2\log\frac{\mp 4f(z)}{\left(\sqrt{1 \mp 4f(z)} + 1\right)^2} = \frac{1}{z} \pm 2\log z - z + C \tag{22}$$

where C is a constant, $\sqrt{1} = 1$, $\log(-1) = \pi i$.

Comparing the constant terms, we get

$$C = \pm 2 + a_2 \mp 2\log(\mp 1). \tag{23}$$

On the other hand, it is well known that there exists a point $z = e^{ix}$, $x \in \mathbb{R}$, for which $f(e^{ix}) = \pm\frac{1}{4}$. This and (22) imply

$$\operatorname{Re} C = 0. \tag{24}$$

Finally, from (23) and (24) we have

$$a_2 = \mp 2,$$

and so,

$$a_3 = 3.$$

Putting the above values of the coefficients a_2 and a_3 in (20) or (21), we obtain

$$\alpha = \frac{3k}{4k+8}.$$

To sum up, we have shown

LEMMA 4. *If the extremal function satisfies an equation of form (d), then*

$$H(f) = 2^k 3^m (3 - 4\alpha)$$

for $k, m = 1, 2, 3, \ldots$, $\alpha = \frac{3k}{4k+8}$.

For $\alpha \neq \frac{3k}{4k+8}$, $k = 1, 2, 3, \ldots$, the extremal function fails to satisfy equation (d).

6. To finish with, let us consider the case when $B_1 = 0$ and $B_2 \neq 0$ in equation (3). Then this equation is of the form

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1}{f^2(z)} = \frac{(z - z_0)^2 (z - z_1)^2}{z^2} \tag{25}$$

where $|z_0| = |z_1| = 1$ and $z_1 \neq z_0$.

Comparing the coefficients of equations (25) and (3), we get $z_1 = -z_0$ and $z_0^2 = 1$ or $z_0^2 = -1$. However, it turns out that, for $z_0^2 = -1$, the solution of equation (25) is not holomorphic in the disc Δ.

Finally, equation (25) takes the form

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1}{f^2(z)} = \frac{(z-1)^2(z+1)^2}{z^2}. \tag{e}$$

Integrating equation (e) and, next, comparing the constant terms, we obtain

$$a_3 - a_2^2 = -1. \tag{26}$$

Hence and from the condition $B_1 = 0$ we have

$$\begin{cases} a_2^2 = \dfrac{2k+2m+2-(k+2)\alpha+\sqrt{D}}{2(1-\alpha)(k+2m+2)} \\ a_3 = \dfrac{(k+4m+2)\alpha-2(m+1)+\sqrt{D}}{2(1-\alpha)(k+2m+2)} \end{cases} \tag{27}$$

or

$$\begin{cases} a_2^2 = \dfrac{k+2m+(k+2)(1-\alpha)-\sqrt{D}}{2(1-\alpha)(k+2m+2)} \\ a_3 = \dfrac{(k+4m+2)\alpha-2(m+1)-\sqrt{D}}{2(1-\alpha)(k+2m+2)} \end{cases} \tag{28}$$

where $D = [2(m+1)-(k+2)\alpha]^2 + 8\,k\,m\,\alpha,\;\; k,m = 1,2,3,\ldots$.

It follows from the conditions $a_2^2 \le 4$ and $-1 \le a_3 \le 3$ that system (27) has a solution only for $\alpha \in (0, \alpha_3)$ where

$$\alpha_3 = \frac{9k^2+48m^2+42km+42k+96m+48}{12k^2+64m^2+56km+48k+112m+48}, \quad k,m = 1,2,3,\ldots, \tag{29}$$

while system (28) has a solution for $\alpha > 0$, with that, for $\alpha = 1$, $a_2^2 = \frac{k}{k+2m}$ and $a_3 = \frac{-2m}{k+2m}$.

Remark 1 *If (a_2^2, a_3) is a solution of system (27) or (28), then, for any positive integers k, m and for $\alpha > 0$*

$$a_3 - \alpha\, a_2^2 < 0. \tag{30}$$

Remark 2 *If $k, m = 1,2,3,\ldots$ and $\alpha \in (0,1)$, then it follows from system (27) that $a_3 > 0$, whereas for system (28) – that $a_3 < 0$.*

The above considerations imply

Lemma 5. *If $k = 1, 3, \ldots, m = 1, 2, \ldots$ and the extremal function satisfies equation (e), then the value of the functional $H(f)$ is defined by system (27) for $\alpha \in (0, \alpha_3)$ or by system (28) — for $\alpha \in (0, +\infty)$;*

If $k = 2, 4, \ldots, m = 1, 3, \ldots$ and the extremal function satisfies equation (e), then the value of the functional $H(f)$ is defined by system (28) for $\alpha \in (0, +\infty)$;

If $k = 2, 4, \ldots, m = 2, 4, \ldots$, then there exists no extremal function satisfying equation (e).

7. Let us introduce the following notations:

$$M_2(k, m, \psi(\alpha)) = a_2^k a_3^m (a_3 - \alpha a_2^2)$$

where a_2, a_3 are defined by formulae (17), (18), and $\psi(\alpha)$ is the function inverse to the increasing function $\alpha = \alpha(\psi)$, $\psi \in (0, \pi/2)$, given by the formula (19);

$$M_3(k, m, \alpha) = a_2^k a_3^m (a_3 - \alpha a_2^2)$$

where a_2 ans a_3 are defined by equations (27);

$$M_4(k, m, \alpha) = a_2^k a_3^m (a_3 - \alpha a_2^2)$$

where a_2 ans a_3 are defined by equations (28).

Lemmas 1–5 and the continuity of the functional $H(f)$ in the compact class S_R, by proceeding similarly as in papers [2], [6], imply

Theorem 1. *For any function $f \in S_R$:*
1° if $k = 2, 4, \ldots, m = 2, 4, \ldots$, then

$$H(f) \le \begin{cases} 2^k 3^m (3 - 4\alpha) & \text{for} \quad \alpha \le \alpha_1, \\ M_2(k, m, \psi(\alpha)) & \text{for} \quad \alpha > \alpha_1; \end{cases}$$

2° if $k = 1, 3, \ldots, m = 1, 2, \ldots$, then

$$H(f) \le \begin{cases} 2^k 3^m (3 - 4\alpha) & \text{for} \quad \alpha \le \alpha_1, \\ M_2(k, m, \psi(\alpha)) & \text{for} \quad \alpha_1 < \alpha \le \alpha_5, \\ M_3(k, m, \alpha) & \text{for} \quad \alpha_5 < \alpha \le \alpha_2, \\ 2^k 3^m (4\alpha - 3) & \text{for} \quad \alpha > \alpha_2; \end{cases}$$

3° if $k = 2, 4, \ldots$, $m = 1, 3, \ldots$, then

$$H(f) \leq \begin{cases} 2^k 3^m (3 - 4\alpha) & \text{for} \quad \alpha \leq \alpha_1, \\ M_2(k, m, \alpha) & \text{for} \quad \alpha_1 < \alpha \leq \alpha_4, \\ M_4(k, m, \alpha) & \text{for} \quad \alpha > \alpha_4, \end{cases}$$

where $\alpha_1 = \frac{3k}{4k+8}$, $\alpha_2 = \frac{3(8m+3k+8)}{4(8m+3k+6)}$, while α_4, α_5 are the only roots of the equations $M_2(k, m, \psi(\alpha)) = M_4(k, m, \alpha)$ and

$M_2(k, m, \psi(\alpha)) = M_3(k, m, \alpha)$, respectively.

All the estimates given above are exact.

Remark 3 *Theorem 1 (3°) implies the well-known estimate of the functional $a_3^m(a_3 - \alpha a_2^2)$ for $m = 1, 3, 5, \ldots$, $\alpha \in \mathbb{R}$*
[6].

For $m = 2, 4, \ldots$, from Theorem 1 (1°) we obtain the estimate of the functional $a_3^m(a_3 - \alpha a_2^2)$ for $\alpha \leq m + 1$ only. For $\alpha > m + 1$, it is known [6] that the only function extremal with respect to this functional is the function with the coefficients $a_2 = 0$ and $a_3 = 1$. This function, however is not extremal with respect to the functional $H(f)$ investigated in our paper.

From Theorem 1 we do not obtain directly any estimate of the functional $a_2^k(a_3 - \alpha a_2^2)$, either (see [2]), on account of the necessary assumption $m \neq 0$ used in the proof of Lemma 1.

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