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HARMONIC OPERATOR AND CLASSICAL ORTHOGONAL POLYNOMIALS

Abstract. In this paper, we introduce the notion of $\mathfrak{h}_{\xi, \mu, \alpha}$ – classical orthogonal polynomials, where $\mathfrak{h}_{\xi, \mu, \alpha}$ is an operator generalizing the harmonic operator. More precisely, we show that the scaled Hermite polynomial sequence $\{\tilde{H}_n\}_{n \geq 0}$, is actually the only monic orthogonal polynomial sequence that is $\mathfrak{h}_{\xi, \mu, \alpha}$ – classical.

Key words: *harmonic analysis, orthogonal polynomials, polynomial sequence, linear functional*

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1. Introduction. Harmonic operators is a powerful mathematical tool in the realm of the quantum physics; it provides deep insights into a wide range of physical systems. The behavior of a quantum system that oscillates around an equilibrium position can be described by a harmonic operator. This issue is significantly applied to investigating an atom vibrating in a molecule or an electron orbiting a nucleus [9]. It is well known that the harmonic operator is quadratic in both position and momentum, and its eigenfunctions are harmonic oscillator wavefunctions [9]. These wavefunctions and their associated energy levels are crucial for the comprehension of energy quantization in atomic and molecular systems [14]. More studies on harmonic oscillators can be found in [10].

In this paper, we describe all $\mathfrak{h}_{\xi, \mu, \alpha}$ – classical orthogonal polynomial sequences, where ξ, μ, α are nonzero parameters and $\mathfrak{h}_{\xi, \mu, \alpha} = \mathfrak{h}_{\xi} + \mu id + \alpha x \frac{d}{dx}$ is an operator generalizing the so-called harmonic operator denoted here by \mathfrak{h}_{ξ} and given by $\mathfrak{h}_{\xi} := x^2 id + \xi^2 \frac{d^2}{dx^2}$.

Recall that an orthogonal polynomial sequence $\{P_n(x)\}_{n \geq 0}$ is called classical if $\{DP_n(x)\}_{n \geq 0}$, where $D := \frac{d}{dx}$ is the standard derivative, is also orthogonal (Hermite, Laguerre, Bessel or Jacobi). This is called the

Hahn property [7]. In [8], Hahn gave similar characterization theorems for orthogonal polynomials P_n , such that the polynomials ΔP_n or $D_q P_n$ ($n \geq 1$) are again orthogonal. Here ΔP_n is the difference operator and $D_q P_n$ is the q -difference (Jackson) operator.

In a more general setting, let O be a linear operator acting on the space \mathcal{P} of polynomials in one variable, which sends polynomials of degree n to polynomials of degree $n + n_0$ (n_0 is a fixed integer). We call a sequence $\{P_n\}_{n \geq 0}$ of orthogonal polynomials O -classical if there exists a sequence $\{Q_n\}_{n \geq 0}$ of orthogonal polynomials, such that $OP_n = Q_{n+n_0}$ (where $n \geq 0$ if $n_0 \geq 0$ and $n \geq n_0$ if $n_0 < 0$). The concept of O -classical orthogonal polynomials has been studied by many authors, one can see [1], [2], [3], [4].

The paper is organized as follows: Section 2 gives the basic notations and tools that will be used throughout the paper. Section 3 deals with $\mathfrak{h}_{\xi, \mu, \alpha}$ -classical orthogonal polynomial sequence. In Section 4, we give a conclusion.

2. Basic definitions and notation. Let \mathcal{P} denote the vector space of polynomials with coefficients in \mathbb{C} and let \mathcal{P}' be its dual. We denote by $\langle u, p \rangle$ the action of the form or linear functional $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, $(u)_n = \langle u, x^n \rangle$, $n \geq 0$, are called the moments of u . A form u is equivalent to the numerical sequence $\{(u)_n\}_{n \geq 0}$.

In the sequel, we use the term “polynomial sequence” (PS) for any sequence $\{P_n\}_{n \geq 0}$, such that $\deg P_n = n$, $n \geq 0$. We also define a monic polynomial sequence (MPS) as a PS, such that all polynomials have leading coefficient equal to one. Note that if $\langle u, P_n \rangle = 0$, $\forall n \geq 0$, then $u = 0$. Given a MPS $\{P_n\}_{n \geq 0}$, there are complex sequences, $\{\beta_n\}_{n \geq 0}$ and $\{\chi_{n, \nu}\}_{0 \leq \nu \leq n}$, $n \geq 0$, such that

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad (1)$$

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \sum_{\nu=0}^n \chi_{n, \nu} P_{\nu}(x), \quad n \geq 0. \quad (2)$$

This relation is often called the structure relation of $\{P_n\}_{n \geq 0}$, and $\{\beta_n\}_{n \geq 0}$ and $\{\chi_{n, \nu}\}_{0 \leq \nu \leq n, n \geq 0}$ are called the structure coefficients. Moreover, there exists a unique sequence $\{u_n\}_{n \geq 0}$, $u_n \in \mathcal{P}'$, called the dual sequence of $\{P_n\}_{n \geq 0}$, such that

$$\langle u_n, P_m \rangle = \delta_{n, m}, \quad n, m \geq 0,$$

where $\delta_{n, m}$ denotes the Kronecker symbol. Let us remark that if p is a polynomial and $\langle u_n, p \rangle = 0$, $\forall n \geq 0$, then $p = 0$. Besides, it is well

known [12] that

$$\beta_n = \langle u_n, xP_n(x) \rangle, \quad n \geq 0, \quad (3)$$

$$\chi_{n,\nu} = \langle u_\nu, xP_{n+1}(x) \rangle, \quad 0 \leq \nu \leq n. \quad (4)$$

Lemma 1. [12] *For each $u \in \mathcal{P}'$ and each $m \geq 1$, the two following statements are equivalent.*

a) $\langle u, P_{m-1} \rangle \neq 0, \langle u, P_n \rangle = 0, n \geq m.$

b) $\exists \lambda_\nu \in \mathbb{C}, 0 \leq \nu \leq m-1$, such that $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$, with $\lambda_{m-1} \neq 0$. In particular, $\lambda_\nu = \langle u, P_\nu \rangle.$

Given $\varpi \in \mathcal{P}$ and $u \in \mathcal{P}'$, the form ϖu , called the left-multiplication of u by the polynomial ϖ , is defined by

$$\langle \varpi u, p \rangle = \langle u, \varpi p \rangle, \quad \forall p \in \mathcal{P}, \quad (5)$$

and the transpose of the derivative operator on \mathcal{P} defined by $p \rightarrow (Dp)(x) = p'(x)$, is the following (cf. [11]):

$$u \rightarrow Du : \langle Du, p \rangle = -\langle u, p' \rangle, \quad \forall p \in \mathcal{P}, \quad (6)$$

so that we can retain the usual rule of the derivative of a product when applied to the left-multiplication of a form by a polynomial. Indeed, it is easily established that

$$D(pu) = p'u + pD(u). \quad (7)$$

A PS $\{P_n\}_{n \geq 0}$ is regularly orthogonal with respect to the form u if and only if it fulfils

$$\langle u, P_n P_m \rangle = 0, \quad n \neq m, \quad n, m \geq 0, \quad (8)$$

$$\langle u, P_n^2 \rangle \neq 0, \quad n \geq 0. \quad (9)$$

Then the form u is said to be regular (or quasi-definite) and $\{P_n\}_{n \geq 0}$ is an orthogonal polynomial sequence (OPS). The conditions (8) are called the orthogonality conditions and the conditions (9) are called the regularity conditions. We can normalize $\{P_n\}_{n \geq 0}$ so that it becomes monic; then it is unique and we briefly denote it as a MOPS. Considering the corresponding dual sequence $\{u_n\}_{n \geq 0}$, the equality $u = \lambda u_0$ holds, with $\lambda = (u)_0 \neq 0$.

Lemma 2. [13] Let u be a regular form and ϕ be a polynomial, such that $\phi u = 0$. Then $\phi = 0$.

Theorem 1. [12] Let $\{P_n\}_{n \geq 0}$ be a MPS and $\{u_n\}_{n \geq 0}$ its dual sequence. The following statements are equivalent:

- a) The sequence $\{P_n\}_{n \geq 0}$ is orthogonal (with respect to u_0);
- b) $\chi_{n,k} = 0$, $0 \leq k \leq n-1$, $n \geq 1$; $\chi_{n,n} \neq 0$, $n \geq 0$;
- c) $xu_n = u_{n-1} + \beta_n u_n + \chi_{n,n} u_{n+1}$, $\chi_{n,n} \neq 0$, $n \geq 0$, $u_{-1} = 0$;
- d) For each $n \geq 0$, there is a polynomial ϕ_n with $\deg(\phi_n) = n$, such that $u_n = \phi_n u_0$;
- e) $u_n = (\langle u_0, P_n^2 \rangle)^{-1} P_n u_0$, $n \geq 0$;

where β_n and $\chi_{n,k}$ are defined by (3)–(4).

Let $\{P_n\}_{n \geq 0}$ be a MOPS. From statement b) of Theorem 1, the structure relation (2) becomes the following second order recurrence relation:

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad (10)$$

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \quad (11)$$

where $\gamma_{n+1} = \chi_{n,n} \neq 0$, $n \geq 0$, and also by item e) we have:

$$\beta_n = \frac{\langle u_0, xP_n^2(x) \rangle}{\langle u_0, P_n^2(x) \rangle}, \quad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2(x) \rangle}{\langle u_0, P_n^2(x) \rangle}, \quad (12)$$

being the regularity conditions (9) fulfilled if and only if $\gamma_{n+1} \neq 0$, $n \geq 0$.

Note also that $\gamma_1 \dots \gamma_n = \prod_{i=1}^n \gamma_i = \langle u_0, P_n^2(x) \rangle$, $n \geq 1$.

The use of suitable affine transformations requires the use of the following operators on \mathcal{P} [11]:

$$\begin{aligned} p &\rightarrow \tau_b p(x) = p(x - b), \quad b \in \mathbb{C}, \\ p &\rightarrow h_a p(x) = p(ax), \quad a \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

Transposing, we obtain the corresponding operators on \mathcal{P}' .

$$\begin{aligned} u &\rightarrow \tau_b u : \langle \tau_b u, p \rangle = \langle u, \tau_{-b} p \rangle = \langle u, p(x + b) \rangle, \quad \forall p \in \mathcal{P}, \\ u &\rightarrow h_a u : \langle h_a u, p \rangle = \langle u, h_a p \rangle = \langle u, p(ax) \rangle, \quad \forall p \in \mathcal{P}. \end{aligned}$$

Hence, given $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, and a MPS $\{P_n\}_{n \geq 0}$, we may define the outcome of an affine transformation denoted by $\{\tilde{P}_n\}_{n \geq 0}$ as follows:

$$\tilde{P}_n(x) = a^{-n} P_n(ax + b), \quad n \geq 0, \quad (13)$$

with the dual sequence [12]:

$$\tilde{u}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) u_n.$$

In particular, if $\{P_n\}_{n \geq 0}$ is a MOPS, then the MPS defined by (13) is orthogonal and its recurrence coefficients are

$$\tilde{\beta}_n = \frac{\beta_n - b}{a}, \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0. \quad (14)$$

Finally, we recall that a MPS $\{P_n\}_{n \geq 0}$ is called classical, if and only if it satisfies the Hahn's property [7]: the MPS $\{P_n^{[1]}\}_{n \geq 0}$ defined by $P_n^{[1]}(x) := (n+1)^{-1} D P_{n+1}(x)$ is also orthogonal. The classical polynomials are divided into four classes: Hermite, Laguerre, Bessel, and Jacobi [11], and characterized by the functional equation

$$D(\phi u) + \psi u = 0, \quad (15)$$

where ψ and ϕ are two polynomials, such that: $\deg \psi = 1$, $\deg \phi \leq 2$, ϕ is normalized, and $\psi' - \frac{1}{2} \phi'' n \neq 0$, $n \geq 1$ [13]. In fact, since ϕ cannot be identically zero, otherwise u_0 would not be regular, we consider it monic and the same for the form u , that is, $(u)_0 = 1$.

Furthermore, when we apply an affine transformation to a classical MOPS, orthogonal with respect to u_0 , as written in (13), we obtain also a classical MOPS orthogonal with respect to the form \tilde{u}_0 , defined by $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b}) u_0$ and belonging to the same class [11], [13]. In addition, \tilde{u}_0 fulfills $D(\tilde{\phi} u) + \tilde{\psi} u = 0$ where [11]

$$\tilde{\phi}(x) = a^{-t} \phi(ax + b), \quad \tilde{\psi}(x) = a^{1-t} \psi(ax + b), \quad t = \deg(\phi). \quad (16)$$

Note that if $P_n(x) = H_n(x)$ is the monic Hermite polynomial, then we have the following characteristics of Hermite polynomials [5], [11], [13]:

$$\begin{aligned} \beta_n &= 0, \quad n \geq 0, \quad \gamma_{n+1} = \frac{n+1}{2}, \quad n \geq 0; \\ \phi(x) &= 1, \quad \psi(x) = 2x; \end{aligned}$$

$$\langle \mathcal{H}, P_n^2 \rangle = \frac{n!}{2^n}, \quad n \geq 0;$$

$$\text{E.D. } P_{n+1}''(x) - 2xP_{n+1}'(x) + 2(n+1)P_{n+1}(x) = 0, \quad n \geq 0;$$

$$\text{R.S: I. } P_n^{[1]}(x) = P_n(x), \quad n \geq 0;$$

$$\text{R.S: II. } P_n(x) = P_n^{[1]}(x), \quad n \geq 0;$$

$$(\mathcal{H})_{2n} = \frac{(2n)!}{2^{2n}n!}, \quad (\mathcal{H})_{2n+1} = 0, \quad n \geq 0.$$

3. Classical orthogonal polynomials via harmonic and perturbed harmonic operator. Let $\mathfrak{h}_\xi = x^2id + \xi^2D^2$, where id and D are, respectively, the identity and the derivative operator. Our purpose, here, is to find the O -classical orthogonal polynomial sequences, i. e., all MOPS $\{P_n\}_{n \geq 0}$, such that the monic sequence $\{O(P_n)\}_{n \geq 0}$ is also orthogonal, where $O = \mathfrak{h}_\xi$, $O = \mathfrak{h}_{\xi, \mu} := \mathfrak{h}_\xi + \mu id$, $O = \mathfrak{h}_{\xi, \mu, \alpha} := \mathfrak{h}_\xi + \mu id + \alpha xD$, with $(\xi, \mu, \alpha) \in (\mathbb{C} \setminus \{0\})^3$.

Clearly, the operator O raises the degree of any polynomial sequence by two. Denote $Q_{n+2}(x) = O(P_n(x))$, $n \geq 0$, with the initial values $Q_0(x) = 1$, $Q_1(x) = x - c$, $c \in \mathbb{C}$, and suppose that $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1, P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \gamma_{n+1} \neq 0, n \geq 0, \end{cases} \quad (17)$$

$$\begin{cases} Q_0(x) = 1, Q_1(x) = x - c, \\ Q_{n+2}(x) = (x - \tilde{\beta}_{n+1})Q_{n+1}(x) - \tilde{\gamma}_{n+1}Q_n(x), \tilde{\gamma}_{n+1} \neq 0, n \geq 0. \end{cases} \quad (18)$$

3.1. Orthogonality associated to the harmonic operator \mathfrak{h}_ξ .

The following result holds:

Theorem 2. *The orthogonality of any polynomial sequence is not preserved by the harmonic operator.*

Proof. Recall that the operator \mathfrak{h}_ξ is given, in the space \mathcal{P} , by

$$\begin{aligned} \mathfrak{h}_\xi : \mathcal{P} &\longrightarrow \mathcal{P} \\ f &\longmapsto x^2f + \xi^2f'', \quad \xi \neq 0. \end{aligned}$$

In particular, we have

$$\mathfrak{h}_\xi(x^n) = x^{n+2} + n(n-1)\xi^2x^{n-2}, \quad n \geq 0.$$

In this case, the orthogonality of polynomial sequence is not preserved. Indeed, we have

$$\mathfrak{h}_\xi(P_0(x)) = \mathfrak{h}_\xi(1) = x^2 = Q_2(x).$$

Then $Q_2(0) = Q_2'(0) = 0$: a contradiction. \square

3.2. Orthogonality associated to the perturbed harmonic operator $\mathfrak{h}_{\xi, \mu}$. The same result is the following:

Theorem 3. *The orthogonality of any polynomial sequence is not preserved by $\mathfrak{h}_{\xi, \mu}$.*

Proof. Indeed, recall that the operator $\mathfrak{h}_{\xi, \mu}$ can be written as follows:

$$\begin{aligned} \mathfrak{h}_{\xi, \mu} : \mathcal{P} &\longrightarrow \mathcal{P} \\ f &\longmapsto (x^2 + \mu)f + \xi^2 f'', \quad \mu \neq 0. \end{aligned}$$

With another reasoning, the orthogonality of any polynomial sequence is not preserved. Indeed, differentiating (17), we obtain

$$P''_{n+2}(x) = 2P'_{n+1}(x) + (x - \beta_{n+1})P''_{n+1}(x) - \gamma_{n+1}P''_n(x), \quad n \geq 0. \quad (19)$$

Multiplying (17) and (19), respectively, by $(x^2 + \mu)$ and ξ^2 , take the sum of the two resulting equations to get

$$Q_{n+4}(x) = (x - \beta_{n+1})Q_{n+3}(x) - \gamma_{n+1}Q_{n+2}(x) + \xi^2 P'_{n+1}(x), \quad n \geq 0.$$

From (18), we obtain

$$(\beta_{n+1} - \tilde{\beta}_{n+1})Q_{n+3}(x) + (\gamma_{n+1} - \tilde{\gamma}_{n+1})Q_{n+2}(x) = 2\xi^2 P'_{n+1}(x), \quad n \geq 0.$$

Note that $P'_{n+1}(x) = 0$, $n \geq 0$; then $P_{n+1}(x) = c_{n+1}$, $n \geq 0$: a contradiction. \square

3.3. The $\mathfrak{h}_{\xi, \mu, \alpha}$ - classical orthogonal polynomials. Recall that the operator $\mathfrak{h}_{\xi, \mu, \alpha}$ is defined by

$$\begin{aligned} \mathfrak{h}_{\xi, \mu, \alpha} : \mathcal{P} &\longrightarrow \mathcal{P} \\ f &\longmapsto (x^2 + \mu)f + \xi^2 f'' + \alpha x f'. \end{aligned}$$

For $n \geq 0$, we have

$$\mathfrak{h}_{\xi, \mu, \alpha}(x^n) = x^{n+2} + (\alpha n + \mu)x^n + n(n-1)\xi^2 x^{n-2}.$$

By transposition of the operator $\mathfrak{h}_{\xi, \mu, \alpha}$, we get

$${}^t\mathfrak{h}_{\mu, \xi, \alpha} = (x^2 + \mu)f + \xi^2 f'' - \alpha x f' \quad (20)$$

$$= \mathfrak{h}_{\xi, \mu, -\alpha}. \quad (21)$$

Now, denote by $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ the dual basis in \mathcal{P}' corresponding to $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, respectively. Then, according to Lemma 1 and (20), we obtain

$$(x^2 + \mu)v_{n+2} - \alpha x v'_{n+2} + \xi^2 v''_{n+2} = u_n, \quad n \geq 0. \quad (22)$$

Our next goal is to describe all $\mathfrak{h}_{\xi, \mu, \alpha}$ -classical orthogonal polynomials, i.e., the MOPS $\{P_n\}_{n \geq 0}$, such that the monic sequence $\{Q_n\}_{n \geq 0}$, where

$$Q_{n+2}(x) := (x^2 + \mu)P_n(x) + \xi^2 P''_n(x) + \alpha x P'_n(x), \quad n \geq 0, \quad (23)$$

with $Q_0(x) = 1$, $Q_1(x) = x - c$, $c \in \mathbb{C}$, is also orthogonal.

A first result will be deduced as a consequence of the relations (23), (17), and (18).

Lemma 3. *The sequences $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ are related as follows:*

$$\alpha Q_{n+2}(x) = 2\xi^2 P'_{n+1}(x) + \alpha x P_{n+1}(x), \quad n \geq 0. \quad (24)$$

where

$$\tilde{\beta}_{n+3} = \beta_{n+1}, \quad n \geq 0, \quad (25)$$

$$\tilde{\gamma}_{n+3} = \gamma_{n+1} - \alpha, \quad n \geq 0. \quad (26)$$

Proof. Differentiating (17), we obtain

$$P'_{n+2}(x) = P_{n+1}(x) + (x - \beta_{n+1})P'_{n+1}(x) - \gamma_{n+1}P'_n(x), \quad n \geq 0, \quad (27)$$

$$P''_{n+2}(x) = 2P'_{n+1}(x) + (x - \beta_{n+1})P''_{n+1}(x) - \gamma_{n+1}P''_n(x), \quad n \geq 0. \quad (28)$$

Multiply the last equation by αx , the relation (28) by ξ^2 , and (17) by $x^2 + \mu$. Take the sum of the three resulting equations:

$$\begin{aligned} & (x^2 + \mu)P_{n+1}(x) + \xi^2 P'_{n+2}(x) + \alpha x P'_{n+2}(x) = \\ & = (x - \beta_{n+1})(x^2 + \mu)P_{n+1} - (x^2 + \mu)\gamma_{n+1}P_n(x) + \\ & + \alpha x P_{n+1} + \alpha x(x - \beta_{n+1})P'_{n+1}(x) - \alpha x \gamma_{n+1}P'_n(x) + \\ & + 2\xi^2 P'_{n+1}(x) + \xi^2(x - \beta_{n+1})P''_{n+1}(x) - \xi^2 \gamma_{n+1}P''_n(x), \quad n \geq 0. \end{aligned}$$

According to (23), we obtain

$$\begin{aligned} Q_{n+4}(x) - xQ_{n+3}(x) + \beta_{n+1}Q_{n+3}(x) + \gamma_{n+1}Q_{n+2}(x) &= \\ &= \alpha xP_{n+1}(x) + 2\xi^2 P'_{n+1}(x), \quad n \geq 0. \end{aligned}$$

Or, equivalently,

$$\begin{aligned} Q_{n+4}(x) - Q_{n+4}(x) - \tilde{\beta}_{n+3}Q_{n+3}(x) - \tilde{\gamma}_{n+3}Q_{n+2}(x) + \beta_{n+1}(x)Q_{n+3}(x) + \\ + \gamma_{n+1}Q_{n+2}(x) = \alpha xP_{n+1}(x) + 2\xi^2 P'_{n+1}(x), \quad n \geq 0. \end{aligned}$$

Then, for $n \geq 0$, it follows that

$$(\beta_{n+1} - \tilde{\beta}_{n+3})Q_{n+3}(x) + (\gamma_{n+1} - \tilde{\gamma}_{n+3})Q_{n+2}(x) = \alpha xP_{n+1} + 2\xi^2 P'_{n+1}(x). \quad (29)$$

By comparing the degrees in the last equation, we obtain

$$\begin{aligned} \beta_{n+1} &= \tilde{\beta}_{n+3}, \quad n \geq 0, \\ \gamma_{n+1} - \alpha &= \tilde{\gamma}_{n+3}, \quad n \geq 0, \end{aligned}$$

and we get the desired result. \square

If n is replaced by $n - 1$ in (24), we obtain the following result:

Lemma 4. *The following fundamental relation holds:*

$$Q_{n+1}(x) = xP_n(x) + \frac{2\xi^2}{\alpha} P'_n(x), \quad n \geq 0. \quad (30)$$

Proof. Indeed, we prove that $Q_1(x) = x$. Just show that $c = 0$. Firstly, based on the orthogonality of $\{Q_n\}_{n \geq 0}$, with respect to the form v_0 , we have

$$\langle v_0, Q_1 Q_2 \rangle = 0. \quad (31)$$

On the other hand, taking $n = 0$ in (23), (24) and (18) we, respectively, obtain

$$Q_2(x) = x^2 + \mu \quad (32)$$

$$= x^2 - \beta_0 x + \frac{2\xi^2}{\alpha} \quad (33)$$

$$= x^2 - (c + \tilde{\beta}_1)x + \tilde{\beta}_1 c - \tilde{\gamma}_1. \quad (34)$$

This gives the following system:

$$\begin{cases} \beta_0 = 0, \\ c + \tilde{\beta}_1 = 0, \\ \tilde{\beta}_1 c - \tilde{\gamma}_1 = \frac{2\xi^2}{\alpha} = \mu. \end{cases} \quad (35)$$

Using (31) and (32), we obtain

$$\langle v_0, x^3 - cx^2 + \mu x - c\mu \rangle = 0. \quad (36)$$

Using (23), we have

$$Q_3(x) = x^3 + (\mu + \alpha)x. \quad (37)$$

Substituting (32) and (37) in (36), this gives

$$\langle v_0, Q_3 \rangle - c\langle v_0, Q_2 \rangle + \alpha\langle v_0, x \rangle = 0.$$

Equivalently,

$$\langle v_0, x \rangle = 0, \quad (38)$$

since $\alpha \neq 0$. Then, using the fact that $Q_1(x) = x - c$ and the orthogonality of $\{Q_n\}_{n \geq 0}$, we get $c\langle v_0 \rangle = 0$, which gives $c = 0$; then $Q_1(x) = x$. Hence, (30) is valid. \square

Based on Proposition 4, we can state the following principal result:

Theorem 4. *The scaled Hermite polynomial sequence $\{a^{-n}H_n(ax)\}_{n \geq 0}$, where $a^2 = -\frac{\alpha}{4\xi^2} = -\frac{\mu}{2\xi^2}$, is actually the only MOPS that is $\mathfrak{h}_{\xi, \mu, \alpha}$ -classical. More precisely, we have*

$$\begin{aligned} P_n(x) &= a^{-n}H_n(ax), \quad n \geq 0, \\ Q_n(x) &= a^{-n}H_n(ax), \quad n \geq 0, \end{aligned}$$

where $a^2 = -\frac{\alpha}{4\xi^2} = -\frac{\mu}{2\xi^2}$.

Proof. Applying the form v_0 in (30), we obtain, for $n \geq 0$:

$$\langle v_0, Q_{n+1} \rangle = \left\langle v_0, xP_{n+1} + \frac{2\xi^2}{\alpha}P'_n \right\rangle = 0.$$

But the right-hand hand side may be read as

$$\left\langle xv_0 - \frac{2\xi^2}{\alpha}v'_0, P_n \right\rangle = 0, \quad n \geq 0.$$

Hence, we get for all polynomials P , by expanding P in the basis $\{P_n\}_{n \geq 0}$, the following relation:

$$\left\langle xv_0 - \frac{2\xi^2}{\alpha}v'_0, P \right\rangle = 0.$$

Then we finally obtain the following functional equation:

$$v'_0 - \frac{\alpha}{2\xi^2}xv_0 = 0. \tag{39}$$

This implies that v_0 is the Hermite functional according to the corresponding PE (16), i.e., $Q_n(x) = a^{-n}H_n(ax)$, where $a^2 = -\frac{\alpha}{4\xi^2}$, with $\{a^{-n}H_n(ax)\}_{n \geq 0}$ is the scaled Hermite polynomial sequence.

We can, also, obtain the sequence $\{P_n(x)\}_{n \geq 0}$. Indeed, by according (35), (25), and (26), we obtain $\beta_n = 0$, $n \geq 0$ and $\gamma_{n+1} = \tilde{\gamma}_{n+3} + \alpha = \frac{n+3}{2a^2} + \alpha$. By (35), we have

$$\mu = -\tilde{\gamma}_1 = \frac{2\xi^2}{\alpha} = -\frac{1}{2a^2}. \tag{40}$$

On the other hand, taking $n = 1$ in (23), we get

$$Q_3(x) = x^3 - \beta_1x^2 + \left(\frac{4\xi^2}{\alpha} - \gamma_1\right)x - \frac{2\xi^2}{\alpha}\beta_1,$$

which gives, after identification with (37), the following:

$$\frac{4\xi^2}{\alpha} - \gamma_1 = \mu + \alpha \text{ and } \beta_1 = 0.$$

Equivalently, by using (40), we obtain

$$2\mu - \tilde{\gamma}_3 - \alpha = \mu + \alpha.$$

By using the fact that $\tilde{\gamma}_3 = \frac{3}{2a^2}$, this gives $\alpha = 2\mu$. Hence, $\gamma_{n+1} = \frac{n+1}{2a^2}$, $n \geq 0$, and then $P_n(x) = a^{-n}H_n(ax)$, $n \geq 0$: the scaled Hermite polynomials. \square

Remark. As a consequence, we have

$$\mathfrak{h}_{\xi, \mu, \alpha}(\tilde{H}_n(x)) = \tilde{H}_{n+2}(x), \quad n \geq 0,$$

where $\tilde{H}_n(x)$, $n \geq 0$, is the scaled Hermite polynomials. In particular, for $n = 0$ and $n = 1$, we, respectively, obtain

$$\mathfrak{h}_{\xi, \mu, \alpha}(1) = \tilde{H}_2(x),$$

$$\mathfrak{h}_{\xi, \mu, \alpha}(x) = \tilde{H}_3(x).$$

Which, finally, gives the following:

$$\mathfrak{h}_{\xi, \mu, \alpha}^n(1) = \tilde{H}_{2n}(x), \quad n \geq 1,$$

$$\mathfrak{h}_{\xi, \mu, \alpha}^n(x) = \tilde{H}_{2n+1}(x), \quad n \geq 1.$$

Here $\mathfrak{h}_{\xi, \mu, \alpha}^n := \mathfrak{h}_{\xi, \mu, \alpha} \circ \dots \circ \mathfrak{h}_{\xi, \mu, \alpha}$.

4. Conclusion. We have described the $\mathfrak{h}_{\xi, \mu, \alpha}$ – classical orthogonal polynomials using the Pearson equation that the corresponding linear functionals satisfy. Indeed, we have proved that the Hermite polynomial sequence $\{a^{-n} H_n(ax)\}_{n \geq 0}$, where $a^2 = -\frac{\alpha}{4\xi^2} = -\frac{\mu}{2\xi^2}$, is the unique $\mathfrak{h}_{\xi, \mu, \alpha}$ – classical orthogonal polynomial sequence.

At the same time, we have highlighted certain formulas:

$$\mathfrak{h}_{\xi, \mu, \alpha}^n(1 = \tilde{H}_0(x)) = \tilde{H}_{2n}(x), \quad n \geq 1,$$

$$\mathfrak{h}_{\xi, \mu, \alpha}^n(x = \tilde{H}_1(x)) = \tilde{H}_{2n+1}(x), \quad n \geq 1,$$

where $\tilde{H}_n(x)$, $n \geq 0$, is the scaled Hermite polynomials and $\mathfrak{h}_{\xi, \mu, \alpha}^n := \mathfrak{h}_{\xi, \mu, \alpha} \circ \dots \circ \mathfrak{h}_{\xi, \mu, \alpha}$.

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