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# HARMONIC OPERATOR AND CLASSICAL ORTHOGONAL POLYNOMIALS

**Abstract.** In this paper, we introduce the notion of $\mathfrak{h}_{\xi,\,\mu,\,\alpha}$ – classical orthogonal polynomials, where $\mathfrak{h}_{\xi,\,\mu,\,\alpha}$ is an operator generalizing the harmonic operator. More precisely, we show that the scaled Hermite polynomial sequence $\{\widetilde{H}_n\}_{n\geqslant 0}$, is actually the only monic orthogonal polynomial sequence that is $\mathfrak{h}_{\xi,\,\mu,\,\alpha}$ – classical.

**Key words:** *Harmonic analysis, Orthogonal polynomials, Polynomial sequence, linear functional*

**2020 Mathematical Subject Classification:** *33C45, 42C05*

**1. Introduction.** Harmonic operators is a powerful mathematical tool in the realm of the quantum physics; it provides deep insights into a wide range of physical systems. The behavior of a quantum system that oscillates around an equilibrium position can be described by a harmonic operator. This issue is significantly applied to investigating an atom vibrating in a molecule or an electron orbiting a nucleus [9]. It is well known that the harmonic operator is quadratic in both position and momentum, and its eigenfunctions are harmonic oscillator wavefunctions [9]. These wavefunctions and their associated energy levels are crucial for the comprehension of energy quantization in atomic and molecular systems [14]. More studies on harmonic oscillators can be found in [10].

In this paper, we describe all $\mathfrak{h}_{\xi,\,\mu,\,\alpha}$ – classical orthogonal polynomial sequences, where $\xi$, $\mu$, $\alpha$ are nonzero parameters and $\mathfrak{h}_{\xi,\,\mu,\,\alpha} = \mathfrak{h}_\xi + \mu id + \alpha x \frac{d}{dx}$ is an operator generalizing the so-called harmonic operator denoted here by $\mathfrak{h}_\xi$ and given by $\mathfrak{h}_\xi := x^2 id + \xi^2 \dfrac{d^2}{dx^2}$.

Recall that an orthogonal polynomial sequence $\{P_n(x)\}_{n\geqslant 0}$ is called classical if $\{DP_n(x)\}_{n\geqslant 0}$, where $D := \dfrac{d}{dx}$ is the standard derivative, is also orthogonal (Hermite, Laguerre, Bessel or Jacobi). This is called the *Hahn property* [7]. In [8], Hahn gave similar characterization theorems for orthogonal polynomials $P_n$, such that the polynomials $\Delta P_n$ or $D_q P_n$ $(n \geqslant 1)$ are again orthogonal. Here $\Delta P_n$ is the difference operator and $D_q P_n$ is the $q$-difference (Jackson) operator.

In a more general setting, let $O$ be a linear operator acting on the space $\mathcal{P}$ of polynomials in one variable, which sends polynomials of degree $n$ to polynomials of degree $n + n_0$ ($n_0$ is a fixed integer). We call a sequence $\{P_n\}_{n\geqslant 0}$ of orthogonal polynomials $O$-classical if there exists a sequence $\{Q_n\}_{n\geqslant 0}$ of orthogonal polynomials, such that $OP_n = Q_{n+n_0}$ (where $n \geqslant 0$ if $n_0 \geqslant 0$ and $n \geqslant n_0$ if $n_0 < 0$). The concept of $O$-classical orthogonal polynomials has been studied by many authors, one can see [1], [2], [3], [4].

The paper is organized as follows: Section 2 gives the basic notations and tools that will be used throughout the paper. Section 3 deals with $\mathfrak{h}_{\xi,\,\mu,\,\alpha}$ - classical orthogonal polynomial sequence. In Section 4, we give a conclusion.

**2. Basic definitions and notation.** Let $\mathcal{P}$ denote the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, p\rangle$ the action of the form or linear functional $u \in \mathcal{P}'$ on $p \in \mathcal{P}$.
In particular, $(u)_n = \langle u, x^n\rangle$, $n \geqslant 0$, are called the moments of $u$. A form $u$ is equivalent to the numerical sequence $\{(u)_n\}_{n\geqslant 0}$.

In the sequel, we use the term "polynomial sequence" (PS) for any sequence $\{P_n\}_{n\geqslant 0}$, such that $\deg P_n = n,\ n \geqslant 0$. We also define a monic polynomial sequence (MPS) as a PS, such that all polynomials have leading coefficient equal to one. Note that if $\langle u, P_n\rangle = 0,\ \forall n \geqslant 0$, then $u = 0$. Given a MPS $\{P_n\}_{n\geqslant 0}$, there are complex sequences, $\{\beta_n\}_{n\geqslant 0}$ and $\{\chi_{n,\,\nu}\}_{0\leqslant \nu\leqslant n}$, $n \geqslant 0$, such that

$$P_0(x) = 1,\ \ P_1(x) = x - \beta_0, \tag{1}$$

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\,\nu} P_\nu(x),\ \ n \geqslant 0. \tag{2}$$

This relation is often called the structure relation of $\{P_n\}_{n\geqslant 0}$, and $\{\beta_n\}_{n\geqslant 0}$ and $\{\chi_{n,\,\nu}\}_{0\leqslant \nu\leqslant n,\, n\geqslant 0}$ are called the structure coefficients. Moreover, there exists a unique sequence $\{u_n\}_{n\geqslant 0}$, $u_n \in \mathcal{P}'$, called the dual sequence of $\{P_n\}_{n\geqslant 0}$, such that

$$\langle u_n, P_m\rangle = \delta_{n,\,m},\ \ n, m \geqslant 0,$$

where $\delta_{n,\,m}$ denotes the Kronecker symbol. Let us remark that if $p$ is a polynomial and $\langle u_n, p\rangle = 0$, $\forall n \geqslant 0$, then $p = 0$. Besides, it is well known

[12] that

$$\beta_n = \langle u_n, xP_n(x)\rangle,\ n \geqslant 0, \tag{3}$$

$$\chi_{n,\nu} = \langle u_\nu, xP_{n+1}(x)\rangle,\ 0 \leqslant \nu \leqslant n,\ n \geqslant 0. \tag{4}$$

**Lemma 1.** [12] *For each $u \in \mathcal{P}'$ and each $m \geqslant 1$, the two following statements are equivalent.*

**a)** $\langle u, P_{m-1}\rangle \neq 0,\ \langle u, P_n\rangle = 0, n \geqslant m.$

**b)** $\exists \lambda_\nu \in \mathbb{C},\ 0 \leqslant \nu \leqslant m-1$, *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$, *with* $\lambda_{m-1} \neq 0$. *In particular,* $\lambda_\nu = \langle u, P_\nu\rangle$.

Given $\varpi \in \mathcal{P}$ and $u \in \mathcal{P}'$, the form $\varpi u$, called the left-multiplication of $u$ by the polynomial $\varpi$, is defined by

$$\langle \varpi u, p\rangle = \langle u, \varpi p\rangle,\ \ \forall p \in \mathcal{P}, \tag{5}$$

and the transpose of the derivative operator on $\mathcal{P}$ defined by $p \to (Dp)(x) = p'(x)$, is the following (cf. [11]):

$$u \to Du :\ \ \langle Du, p\rangle = -\langle u, p'\rangle,\ \ \forall p \in \mathcal{P}, \tag{6}$$

so that we can retain the usual rule of the derivative of a product when applied to the left-multiplication of a form by a polynomial. Indeed, it is easily established that

$$D(pu) = p'u + pD(u). \tag{7}$$

A PS $\{P_n\}_{n\geqslant 0}$ is regularly orthogonal with respect to the form $u$ if and only if it fulfils

$$\langle u, P_nP_m\rangle = 0,\ n \neq m,\quad n, m \geqslant 0, \tag{8}$$

$$\langle u, P_n^2\rangle \neq 0,\ n \geqslant 0. \tag{9}$$

Then the form $u$ is said to be regular (or quasi-definite) and $\{P_n\}_{n\geqslant 0}$ is an orthogonal polynomial sequence (OPS). The conditions (8) are called the orthogonality conditions and the conditions (9) are called the regularity conditions. We can normalize $\{P_n\}_{n\geqslant 0}$ so that it becomes monic; then it is unique and we briefly denote it as a MOPS. Considering the corresponding dual sequence $\{u_n\}_{n\geqslant 0}$, the equality $u = \lambda u_0$ holds, with $\lambda = (u)_0 \neq 0$.

**Lemma 2.** [13] *Let $u$ be a regular form and $\phi$ be a polynomial, such that $\phi u = 0$. Then $\phi = 0$.*

**Theorem 1.** [12] *Let $\{P_n\}_{n\geqslant 0}$ be a MPS and $\{u_n\}_{n\geqslant 0}$ its dual sequence. The following statements are equivalent:*

**a)** *The sequence $\{P_n\}_{n\geqslant 0}$ is orthogonal (with respect to $u_0$);*

**b)** $\chi_{n,k} = 0,\ 0 \leqslant k \leqslant n-1, \quad n \geqslant 1; \quad \chi_{n,n} \neq 0,\ n \geqslant 0$;

**c)** $xu_n = u_{n-1} + \beta_n u_n + \chi_{n,n} u_{n+1},\ \chi_{n,n} \neq 0, \quad n \geqslant 0,\ u_{-1} = 0$;

**d)** *For each $n \geqslant 0$, there is a polynomial $\phi_n$ with $\deg(\phi_n) = n$, such that $u_n = \phi_n u_0$;*

**e)** $u_n = \big(< u_0, P_n^2 >\big)^{-1} P_n u_0,\ n \geqslant 0$;

*where $\beta_n$ and $\chi_{n,k}$ are defined by (3)–(4).*

Let $\{P_n\}_{n\geqslant 0}$ be a MOPS. From statement $b)$ of Theorem 1, the structure relation (2) becomes the following second order recurrence relation:

$$P_0(x) = 1,\ \ P_1(x) = x - \beta_0, \tag{10}$$
$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geqslant 0, \tag{11}$$

where $\gamma_{n+1} = \chi_{n,n} \neq 0,\ n \geqslant 0$, and also by item $e)$ we have:

$$\beta_n = \frac{\langle u_0, xP_n^2(x)\rangle}{\langle u_0, P_n^2(x)\rangle}, \qquad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2(x)\rangle}{\langle u_0, P_n^2(x)\rangle}, \tag{12}$$

being the regularity conditions (9) fulfilled if and only if $\gamma_{n+1} \neq 0,\ n \geqslant 0$. Note also that $\gamma_1 \dots \gamma_n = \prod_{i=1}^{n} \gamma_i = \langle u_0, P_n^2(x)\rangle,\ n \geqslant 1$.

The use of suitable affine transformations requires the use of the following operators on $\mathcal{P}$ [11]:

$$p \to \tau_b\, p(x) = p(x-b),\ \ b \in \mathbb{C},$$
$$p \to h_a\, p(x) = p(ax),\ \ a \in \mathbb{C}\backslash\{0\}.$$

Transposing, we obtain the corresponding operators on $\mathcal{P}'$.

$$u \to \tau_b u : \ < \tau_b u, p > = < u, \tau_{-b}\, p > = < u, p(x+b) > , \quad \forall p \in \mathcal{P},$$
$$u \to h_a u : \ < h_a u, p > = < u, h_a\, p > = < u, p(ax) > , \quad \forall p \in \mathcal{P}.$$

Hence, given $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, and a MPS $\{P_n\}_{n\geqslant 0}$, we may define the outcome of an affine transformation denoted by $\{\tilde{P}_n\}_{n\geqslant 0}$ as follows:

$$\tilde{P}_n(x) = a^{-n} P_n(ax+b), \;\; n \geqslant 0, \tag{13}$$

with the dual sequence [12]:

$$\tilde{u}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) u_n.$$

In particular, if $\{P_n\}_{n\geqslant 0}$ is a MOPS, then the MPS defined by (13) is orthogonal and its recurrence coefficients are

$$\widetilde{\beta}_n = \frac{\beta_n - b}{a}, \;\; \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geqslant 0. \tag{14}$$

Finally, we recall that a MPS $\{P_n\}_{n\geqslant 0}$ is called classical, if and only if it satisfies the Hahn´s property [7]: the MPS $\{P_n^{[1]}\}_{n\geqslant 0}$ defined by $P_n^{[1]}(x) := (n+1)^{-1} D P_{n+1}(x)$ is also orthogonal. The classical polynomials are divided into four classes: Hermite, Laguerre, Bessel, and Jacobi [6], and characterized by the functional equation

$$D(\phi u) + \psi u = 0, \tag{15}$$

where $\psi$ and $\phi$ are two polynomials, such that: $\deg \psi = 1$, $\deg \phi \leqslant 2$, $\phi$ is normalized, and $\psi' - \frac{1}{2}\phi'' n \neq 0, n \geqslant 1$ [13]. In fact, since $\phi$ cannot be identically zero, otherwise $u_0$ would not be regular, we consider it monic and the same for the form $u$, that is, $(u)_0 = 1$.

Furthermore, when we apply an affine transformation to a classical MOPS, orthogonal with respect to $u_0$, as written in (13), we obtain also a classical MOPS orthogonal with respect to the form $\tilde{u}_0$, defined by $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b}) u_0$ and belonging to the same class [11], [13]. In addition, $\tilde{u}_0$ fulfills $D(\tilde{\phi} u) + \tilde{\psi} u = 0$ where [11]

$$\tilde{\phi}(x) = a^{-t} \phi\left(ax+b\right), \quad \tilde{\psi}(x) = a^{1-t} \psi\left(ax+b\right), \quad t = \deg\left(\phi\right). \tag{16}$$

Note that if $P_n(x) = H_n(x)$ is the monic Hermite polynomial, then we have the following characteristics of Hermite polynomials [5], [11], [13]:

$$\beta_n = 0, \; n \geqslant 0, \quad \gamma_{n+1} = \frac{n+1}{2}, \; n \geqslant 0;$$
$$\phi(x) = 1, \quad \psi(x) = 2x;$$

$\langle \mathcal{H}, P_n^2 \rangle = \dfrac{n!}{2^n},\ n \geqslant 0;$

**E.D.** $P''_{n+1}(x) - 2xP'_{n+1}(x) + 2(n+1)P_{n+1}(x) =\ 0,\ n \geqslant 0;$

**R.S: I.** $P_n^{[1]}(x) = P_n(x),\ n \geqslant 0;$

**R.S: II.** $P_n(x) = P_n^{[1]}(x),\ n \geqslant 0;$

$(\mathcal{H})_{2n} = \dfrac{(2n)!}{2^{2n}n!}, \quad (\mathcal{H})_{2n+1} = 0,\ n \geqslant 0.$

**3. Classical orthogonal polynomials via harmonic and perturbed harmonic operator.** Let $\mathfrak{h}_\xi = x^2 id + \xi^2 D^2$, where $id$ and $D$ are, respectively, the identity and the derivative operator. Our purpose, here, is to find the $O$-classical orthogonal polynomial sequences, i. e., all MOPS $\{P_n\}_{n\geqslant 0}$, such that the monic sequence $\{O(P_n)\}_{n\geqslant 0}$ is also orthogonal, where $O = \mathfrak{h}_\xi$, $O = \mathfrak{h}_{\xi,\mu} := \mathfrak{h}_\xi + \mu id$, $O = \mathfrak{h}_{\xi,\mu,\alpha} := \mathfrak{h}_\xi + \mu id + \alpha x D$, with $(\xi, \mu, \alpha) \in \big(\mathbb{C}\backslash\{0\}\big)^3$.

Clearly, the operator $O$ raises the degree of any polynomial sequence by two. Denote $Q_{n+2}(x) = O\big(P_n(x)\big),\ n \geqslant 0$, with the initial values $Q_0(x) = 1$, $Q_1(x) = x - c$, $c \in \mathbb{C}$, and suppose that $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ \gamma_{n+1} \neq 0,\ n \geqslant 0, \end{cases} \tag{17}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - c, \\ Q_{n+2}(x) = (x - \tilde{\beta}_{n+1})Q_{n+1}(x) - \tilde{\gamma}_{n+1}Q_n(x),\ \tilde{\gamma}_{n+1} \neq 0,\ n \geqslant 0. \end{cases} \tag{18}$$

**3.1. Orthogonality associated to the harmonic operator $\mathfrak{h}_\xi$.** The following result holds:

**Theorem 2**. *The orthogonality of any polynomial sequence is not preserved by the harmonic operator.*

**Proof.** Recall that the operator $\mathfrak{h}_\xi$ is given, in the space $\mathcal{P}$, by

$$\begin{aligned} \mathfrak{h}_\xi : \mathcal{P} &\longrightarrow \mathcal{P} \\ f &\longmapsto x^2 f + \xi^2 f'',\ \xi \neq 0. \end{aligned}$$

In particular, we have

$$\mathfrak{h}_\xi(x^n) = x^{n+2} + n(n-1)\xi^2 x^{n-2},\ n \geqslant 0.$$

In this case, the orthogonality of polynomial sequence is not preserved. Indeed, we have

$$\begin{aligned}\mathfrak{h}_\xi\big(P_0(x)\big) &= \mathfrak{h}_\xi(1)\\ &= x^2\\ &= Q_2(x).\end{aligned}$$

Then $Q_2(0) = Q_2'(0) = 0$: a contradiction. $\square$

**3.2. Orthogonality associated to the perturbed harmonic operator $\mathfrak{h}_{\xi,\mu}$.** The same result is the following:

**Theorem 3**. *The orthogonality of any polynomial sequence is not preserved by $\mathfrak{h}_{\xi,\mu}$.*

**Proof.** Indeed, recall that the operator $\mathfrak{h}_{\xi,\mu}$ can be written as follows:

$$\begin{aligned}\mathfrak{h}_{\xi,\mu} : \mathcal{P} &\longrightarrow \mathcal{P}\\ f &\longmapsto (x^2+\mu)f + \xi^2 f'',\ \mu \neq 0.\end{aligned}$$

With another reasoning, the orthogonality of any polynomial sequence is not preserved. Indeed, differentiating (17), we obtain

$$P''_{n+2}(x) = 2P'_{n+1}(x) + (x-\beta_{n+1})P''_{n+1}(x) - \gamma_{n+1}P''_n(x),\, n \geqslant 0. \tag{19}$$

Multiplying (17) and (19), respectively, by $(x^2+\mu)$ and $\xi^2$, take the sum of the two resulting equations to get

$$Q_{n+4}(x) = (x-\beta_{n+1})Q_{n+3}(x) - \gamma_{n+1}Q_{n+2}(x) + \xi^2 P'_{n+1}(x),\ n \geqslant 0.$$

From (18), we obtain

$$(\beta_{n+1}-\tilde{\beta}_{n+1})Q_{n+3}(x) + (\gamma_{n+1}-\tilde{\gamma}_{n+1})Q_{n+2}(x) = 2\xi^2 P'_{n+1}(x),\ n \geqslant 0.$$

Note that $P'_{n+1}(x) = 0,\ n \geqslant 0$; then $P_{n+1}(x) = c_{n+1},\ n \geqslant 0$: a contradiction. $\square$

**3.3. The $\mathfrak{h}_{\xi,\mu,\alpha}$ - classical orthogonal polynomials.** Recall that the operator $\mathfrak{h}_{\xi,\mu,\alpha}$ is defined by

$$\begin{aligned}\mathfrak{h}_{\xi,\mu,\alpha} : \mathcal{P} &\longrightarrow \mathcal{P}\\ f &\longmapsto (x^2+\mu)f + \xi^2 f'' + \alpha x f'.\end{aligned}$$

For $n \geqslant 0$, we have

$$\mathfrak{h}_{\xi,\,\mu,\,\alpha}(x^n) = x^{n+2} + (\alpha n + \mu)x^n + n(n-1)\xi^2 x^{n-2}.$$

By transposition of the operator $\mathfrak{h}_{\xi,\,\mu,\,\alpha}$, we get

$$\begin{aligned} {}^{t}\mathfrak{h}_{\mu,\,\xi,\,\alpha} &= (x^2+\mu)f + \xi^2 f'' - \alpha x f', && (20)\\ &= \mathfrak{h}_{\xi,\,\mu,\,-\alpha}. && (21) \end{aligned}$$

Now, denote by $\{u_n\}_{n\geqslant 0}$ and $\{v_n\}_{n\geqslant 0}$ the dual basis in $\mathcal{P}'$ corresponding to $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively. Then, according to Lemma 1 and (20), we obtain

$$(x^2+\mu)v_{n+2} - \alpha x v'_{n+2} + \xi^2 v''_{n+2} = u_n,\ n \geqslant 0. \tag{22}$$

Our next goal is to describe all $\mathfrak{h}_{\xi,\,\mu,\,\alpha}$-classical orthogonal polynomials, i. e., the MOPS $\{P_n\}_{n\geqslant 0}$, such that the monic sequence $\{Q_n\}_{n\geqslant 0}$, where

$$Q_{n+2}(x) := (x^2+\mu)P_n(x) + \xi^2 P''_n(x) + \alpha x P'_n(x), \quad n \geqslant 0, \tag{23}$$

with $Q_0(x) = 1$, $Q_1(x) = x - c$, $c \in \mathbb{C}$, is also orthogonal.

A first result will be deduced as a consequence of the relations (23), (17), and (18).

**Lemma 3**. *The sequences $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ are related as follows:*

$$\alpha Q_{n+2}(x) = 2\xi^2 P'_{n+1}(x) + \alpha x P_{n+1}(x),\ n \geqslant 0. \tag{24}$$

*where*

$$\begin{aligned} \tilde{\beta}_{n+3} &= \beta_{n+1},\ n \geqslant 0. && (25)\\ \tilde{\gamma}_{n+3} &= \gamma_{n+1} - \alpha,\ n \geqslant 0. && (26) \end{aligned}$$

**Proof.** Differentiating (17), we obtain

$$P'_{n+2}(x) = P_{n+1}(x) + (x-\beta_{n+1})P'_{n+1}(x) - \gamma_{n+1}P'_n(x),\ n \geqslant 0, \tag{27}$$

$$P''_{n+2}(x) = 2P'_{n+1}(x) + (x-\beta_{n+1})P''_{n+1}(x) - \gamma_{n+1}P''_n(x),\ n \geqslant 0. \tag{28}$$

Multiply the last equation by $\alpha x$, the relation (28) by $\xi^2$, and (17) by $x^2+\mu$. Take the sum of the three resulting equations:

$$(x^2+\mu)P_{n+1}(x) + \xi^2 P'_{n+2}(x) + \alpha x P'_{n+2}(x) =$$

$$
\begin{aligned}
&= (x-\beta_{n+1})(x^2+\mu)P_{n+1} - (x^2+\mu)\gamma_{n+1}P_n(x) + \\
&+\alpha x P_{n+1} + \alpha x(x-\beta_{n+1})P'_{n+1}(x) - \alpha x\gamma_{n+1}P'_n(x) + \\
&+2\xi^2 P'_{n+1}(x) + \xi^2(x-\beta_{n+1})P''_{n+1}(x) - \xi^2\gamma_{n+1}P''_n(x),\ n \geqslant 0.
\end{aligned}
$$

According to (23), we obtain

$$
\begin{aligned}
Q_{n+4}(x) - xQ_{n+3}(x) + \beta_{n+1}Q_{n+3}(x) + \gamma_{n+1}Q_{n+2}(x) &= \\
&= \alpha x P_{n+1}(x) + 2\xi^2 P'_{n+1}(x),\ n \geqslant 0.
\end{aligned}
$$

Or, equivalently,

$$
\begin{aligned}
Q_{n+4}(x) - Q_{n+4}(x) - \tilde{\beta}_{n+3}Q_{n+3}(x) - \tilde{\gamma}_{n+3}Q_{n+2}(x) + \beta_{n+1}(x)Q_{n+3}(x) + \\
+ \gamma_{n+1}Q_{n+2}(x) = \alpha x P_{n+1}(x) + 2\xi^2 P'_{n+1}(x),\ n \geqslant 0.
\end{aligned}
$$

Then, for $n \geqslant 0$, it follows that

$$
(\beta_{n+1}-\tilde{\beta}_{n+3})Q_{n+3}(x)+(\gamma_{n+1}-\tilde{\gamma}_{n+3})Q_{n+2}(x) = \alpha x P_{n+1} + 2\xi^2 P'_{n+1}(x). \quad (29)
$$

By comparing the degrees in the last equation, we obtain

$$
\begin{aligned}
\beta_{n+1} &= \tilde{\beta}_{n+3},\ n \geqslant 0, \\
\gamma_{n+1} - \alpha &= \tilde{\gamma}_{n+3},\ n \geqslant 0,
\end{aligned}
$$

and we get the desired result. □

If $n$ is replaced by $n-1$ in (24), we obtain the following result:

**Lemma 4.** *The following fundamental relation holds:*

$$
Q_{n+1}(x) = xP_n(x) + \frac{2\xi^2}{\alpha}P'_n(x),\ n \geqslant 0. \quad (30)
$$

**Proof.** Indeed, we prove that $Q_1(x) = x$. Just show that $c = 0$. Firstly, based on the orthogonality of $\{Q_n\}_{n\geqslant 0}$, with respect to the form $v_0$, we have

$$
\langle v_0, Q_1Q_2 \rangle = 0. \quad (31)
$$

On the other hand, taking $n = 0$ in (23), (24) and (18) we, respectively, obtain

$$
Q_2(x) \quad = \quad x^2 + \mu, \quad (32)
$$

$$= x^2 - \beta_0 x + \frac{2\xi^2}{\alpha}, \quad (33)$$

$$= x^2 - (c + \tilde{\beta}_1)x + \tilde{\beta}_1 c - \tilde{\gamma}_1. \quad (34)$$

This gives the following system:

$$\begin{cases} \beta_0 = 0, \\ c + \tilde{\beta}_1 = 0, \\ \tilde{\beta}_1 c - \tilde{\gamma}_1 = \frac{2\xi^2}{\alpha} = \mu. \end{cases} \quad (35)$$

Using (31) and (32), we obtain

$$\langle v_0, x^3 - cx^2 + \mu x - c\mu \rangle = 0. \quad (36)$$

Using (23), we have

$$Q_3(x) = x^3 + (\mu + \alpha)x. \quad (37)$$

Substituting (32) and (37) in (36), this gives

$$\langle v_0, Q_3 \rangle - c\langle v_0, Q_2 \rangle + \alpha\langle v_0, x \rangle = 0.$$

Equivalently,

$$\langle v_0, x \rangle = 0, \quad (38)$$

since $\alpha \neq 0$. Then, using the fact that $Q_1(x) = x - c$ and the orthogonality of $\{Q_n\}_{n\geqslant 0}$, we get $c(v_0)_0 = 0$, which gives $c = 0$; then $Q_1(x) = x$. Hence, (30) is valid. □

Based on Proposition 4, we can state the following principal result:

**Theorem 4.** *The scaled Hermite polynomial sequence $\{a^{-n}H_n(ax)\}_{n\geqslant 0}$, where $a^2 = -\frac{\alpha}{4\xi^2} = -\frac{\mu}{2\xi^2}$, is actually the only MOPS that is $\mathfrak{h}_{\xi,\mu,\alpha}$ – classical. More precisely, we have*

$$P_n(x) = a^{-n}H_n(ax), \; n \geqslant 0,$$
$$Q_n(x) = a^{-n}H_n(ax), \; n \geqslant 0,$$

*where $a^2 = -\frac{\alpha}{4\xi^2} = -\frac{\mu}{2\xi^2}$.*

**Proof.** Applying the form $v_0$ in (30), we obtain, for $n \geqslant 0$:

$$\langle v_0, Q_{n+1} \rangle = \left\langle v_0, xP_{n+1} + \frac{2\xi^2}{\alpha}P'_n \right\rangle = 0.$$

But the right-hand hand side may be read as

$$\left\langle xv_0 - \frac{2\xi^2}{\alpha}v_0', P_n \right\rangle = 0, \ n \geqslant 0.$$

Hence, we get for all polynomials $P$, by expanding $P$ in the basis $\{P_n\}_{n\geqslant 0}$, the following relation:

$$\left\langle xv_0 - \frac{2\xi^2}{\alpha}v_0', P \right\rangle = 0.$$

Then we finally obtain the following functional equation:

$$v_0' - \frac{\alpha}{2\xi^2}xv_0 = 0. \tag{39}$$

This implies that $v_0$ is the Hermite functional according to the corresponding PE (16), i. e., $Q_n(x) = a^{-n}H_n(ax)$, where $a^2 = -\frac{\alpha}{4\xi^2}$, with $\{a^{-n}H_n(ax)\}_{n\geqslant 0}$ is the scaled Hermite polynomial sequence.

We can, also, obtain the sequence $\{P_n(x)\}_{n\geqslant 0}$. Indeed, by according (35), (25), and (26), we obtain $\beta_n = 0$, $n \geqslant 0$ and $\gamma_{n+1} = \tilde{\gamma}_{n+3} + \alpha = = \frac{n+3}{2a^2} + \alpha$. By (35), we have

$$\begin{aligned} \mu &= -\tilde{\gamma}_1 \\ &= \frac{2\xi^2}{\alpha} \\ &= -\frac{1}{2a^2}. \end{aligned} \tag{40}$$

On the other hand, taking $n = 1$ in (23), we get

$$Q_3(x) = x^3 - \beta_1 x^2 + \left(\frac{4\xi^2}{\alpha} - \gamma_1\right)x - \frac{2\xi^2}{\alpha}\beta_1,$$

which gives, after identification with (37), the following:

$$\frac{4\xi^2}{\alpha} - \gamma_1 = \mu + \alpha \text{ and } \beta_1 = 0.$$

Equivalently, by using (40), we obtain

$$2\mu - \tilde{\gamma}_3 - \alpha = \mu + \alpha.$$

By using the fact that $\tilde{\gamma}_3 = \dfrac{3}{2a^2}$, this gives $\alpha = 2\mu$. Hence, $\gamma_{n+1} = \dfrac{n+1}{2a^2}$, $n \geqslant 0$, and then $P_n(x) = a^{-n} H_n(ax)$, $n \geqslant 0$: the scaled Hermite polynomials. □

**Remark.** *As a consequence, we have*

$$\mathfrak{h}_{\xi,\mu,\alpha}\big(\tilde{H}_n(x)\big) = \tilde{H}_{n+2}(x),\ n \geqslant 0,$$

*where* $\tilde{H}_n(x)$, $n \geqslant 0$, *is the scaled Hermite polynomials. In particular, for* $n = 0$ *and* $n = 1$, *we, respectively, obtain*

$$\mathfrak{h}_{\xi,\mu,\alpha}(1) = \tilde{H}_2(x),$$

$$\mathfrak{h}_{\xi,\mu,\alpha}(x) = \tilde{H}_3(x).$$

*Which, finally, gives the following:*

$$\mathfrak{h}^n_{\xi,\mu,\alpha}(1) = \tilde{H}_{2n}(x),\ n \geqslant 1,$$

$$\mathfrak{h}^n_{\xi,\mu,\alpha}(x) = \tilde{H}_{2n+1}(x),\ n \geqslant 1.$$

*Here* $\mathfrak{h}^n_{\xi,\mu,\alpha} := \mathfrak{h}_{\xi,\mu,\alpha} \circ \ldots \circ \mathfrak{h}_{\xi,\mu,\alpha}$.

**4. Conclusion.** We have described the $\mathfrak{h}_{\xi,\mu,\alpha}$ – classical orthogonal polynomials using the Pearson equation that the corresponding linear functionals satisfy. Indeed, we have proved that the Hermite polynomial sequence $\{a^{-n} H_n(ax)\}_{n \geqslant 0}$, where $a^2 = -\dfrac{\alpha}{4\xi^2} = -\dfrac{\mu}{2\xi^2}$, is the unique $\mathfrak{h}_{\xi,\mu,\alpha}$ – classical orthogonal polynomial sequence.

At the same time, we have highlighted certain formulas:

$$\mathfrak{h}^n_{\xi,\mu,\alpha}\big(1 = \tilde{H}_0(x)\big) = \tilde{H}_{2n}(x),\ n \geqslant 1,$$

$$\mathfrak{h}^n_{\xi,\mu,\alpha}\big(x = \tilde{H}_1(x)\big) = \tilde{H}_{2n+1}(x),\ n \geqslant 1,$$

where $\tilde{H}_n(x)$, $n \geqslant 0$, is the scaled Hermite polynomials and $\mathfrak{h}^n_{\xi,\mu,\alpha} := \mathfrak{h}_{\xi,\mu,\alpha} \circ \ldots \circ \mathfrak{h}_{\xi,\mu,\alpha}$.

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