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ON REMOVING RESTRICTIONS IN THE BERNSTEIN THEOREM AND ITS MODIFICATIONS

Abstract. In 1930, S. N. Bernstein proved the following theorem: *Let f and F be complex polynomials such that 1) $\deg f \leqslant \deg F = n$; 2) F has all its zeros in closure of the disc $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$; 3) $|f(z)| \leqslant |F(z)|$ for $|z| = 1$. Then $|f'(z)| \leqslant |F'(z)|$ in $\mathbb{C} \backslash \Delta$.* In a huge number of papers that appeared after 1930 and related to this theorem, the restrictions on the geometry of domains and conditions 1) and 2) of the theorem usually remained unchanged. In this article, we consistently remove these restrictions and find out how this will affect the final inequality $|f'(z)| \leqslant |F'(z)|$ of the Bernstein theorem and many of its modifications, generalizations, and consequences.

Key words: *differential inequalities for polynomials, differential operator, L^p inequalities, convex sets*

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1. Introduction. In 1887, the scientific studies [19, ch. IV, § 86] (see also [20, ch. IV, § 86]) led the famous chemist D. I. Mendeleev to the following problem:

Let $f(x)$ be a real polynomial and suppose that $|f(x)| \leqslant M$ for all $x \in [a, b]$. Give the best estimate for $|f'(x)|$ on $[a, b]$.

This problem was solved in 1890 by A. A. Markov [16] (see also [17, p. 51 – 75]). In 1892, his brother V. A. Markov found the analogous estimate for the k-th order derivative of $f(x)$ [18].

Further, the "Mendeleev problem" was considered and solved for trigonometric and complex polynomials. For more details, see introduction of [12]. The main achievement of research conducted before 1930 in the case of complex polynomials is probably the following theorem due to S. N. Bernstein. Let Δ stand for the open unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$.

Theorem A. [6] (see also [4, p. 497], [23, p. 510]). *Consider polynomials $f(z)$ and $F(z)$ such that:*

$$(*)\begin{cases} 1)\ \deg f \leqslant \deg F = n, \\ 2)\ \text{F has all its zeros in } \overline{\Delta}, \\ 3)\ |f(z)| \leqslant |F(z)| \text{ for } z \in \partial\Delta. \end{cases}$$

Then for $|z| \geqslant 1$ we have

$$|f'(z)| \leqslant |F'(z)|. \tag{1}$$

For $z \in \mathbb{C}\backslash\overline{\Delta}$, in (1) equality holds only if $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

In 1964, in the book of V. I. Smirnov and N. A. Lebedev [27, ch. V, § 1, 3°, Theorem 1, p. 356], Theorem A was generalized. The differential operator $f(z) \longrightarrow f'(z)$ from Theorem A was changed by the differential operator $S_\alpha\colon f(z) \longrightarrow (zf'(z) - \alpha f(z))$, where α is a complex number. The parameter α gave additional opportunities to obtain new differential inequalities for polynomials.

For a fixed $\rho \geqslant 1$ and $n \in \mathbb{N}$, denote by $D_{\rho,n}$ the image of the disc $\{t \in \mathbb{C}\colon |t| < \rho\}$ under the mapping $\phi(t) = \dfrac{nt}{t+1}$.

Theorem B. [27, ch. V, § 1, 3°, Theorem 1, p. 356]. *Let $\rho \geqslant 1$ be a fixed number. Let $f(z)$ and $F(z)$ be polynomials satisfying conditions $(*)$. Then for $|z| \geqslant \rho$*

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F](z)| \tag{2}$$

for all $\alpha \in \overline{D_{\rho,n}}$.

For $|z| \geqslant \rho > 1$ and $\alpha \in D_{\rho,n}$, in (2) equality holds iff $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Inequality (2) with $\alpha = 0$ gives inequality (1).

In fact, in [27], for $\alpha \in \partial D_{\rho,n}$ or $z \in \partial\Delta$ a more complex study of the equality sign in (2) was also conducted.

After Theorem A was proved, a large number of articles devoted to differential inequalities for polynomials were published, see, for example, [23], [15], [27], [10], [9], [1] and the references therein. For most of these results, conditions $(*)$ or their modifications are the key point.

In the article we consistently remove restrictions in the theorems of S. N. Bernstein and V. I. Smirnov, exploring the effect of this removing of restrictions on the final inequalities.

2. Removing conditions 1) and 2) of $(*)$**.** Naturally, the following question arises: how important are conditions 1) and 2) of $(*)$ from Theorem A for proving inequality (1) or its analogues? If 1) or 2) in $(*)$ are not true, is it possible to obtain an analogue of inequality (1), preserving the sharpness in this analogue? We pose the same question with respect to Theorem B.

These questions are not new. Probably, the first result in this direction was obtained by S. N. Bernstein [5, p. 168]. He considered the case when in Theorem A conditions 2) and 3) of $(*)$ are true, but condition 1) is not satisfied, i. e. $\deg f = m > \deg F = n$. He noted that in this case the following inequality takes place

$$|f'(z)| \leqslant |(z^{m-n}F(z))'|, \ |z| \geqslant 1.$$

In [13], [14], an analogue of inequality (2) for the Smirnov operator was obtained in the case, when condition 2) of $(*)$ is waived. This means that the polynomial F can have zeros in $\mathbb{C}\backslash\overline{\Delta}$. Here the set of variation of the parameter α in a rather complicated way depends on location of the zeros of F, lying in $\mathbb{C}\backslash\overline{\Delta}$.

Theorem C. *Let* $\rho \geqslant 1$. *Let* $f(z)$ *and* $F(z)$ *be polynomials, such that*
1) $\deg f \leqslant \deg F = n$,
2) $z_1, \ldots, z_k$ *are all the zeros of* F *lying in* $\mathbb{C}\backslash\overline{\Delta}$, $d_1, \ldots, d_k$ *are their orders, correspondingly,* $1 \leqslant d = d_1 + \cdots + d_k \leqslant n - 1$,
3) $|f(z)| \leqslant |F(z)|$, *for* $z \in \mathbb{C}\backslash\Delta$.
Then

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F](z)| \tag{3}$$

for $|z| = \rho$ *and for* $\alpha \in D(\rho, k, n)$, *where* $D(\rho, k, n)$ *is one of the sets:*
a) complement to the disc

$$\{\alpha \in \mathbb{C} \colon |\alpha - c| < r\},$$

where

$$c = (n-d)\frac{\rho^2}{\rho^2 - 1} + \rho^2 \sum_{j=1}^{k} \frac{d_j}{\rho^2 - |z_j|^2},$$

$$r = (n-d)\frac{\rho}{\rho^2 - 1} + \rho \sum_{j=1}^{k} \frac{d_j |z_j|}{|\rho^2 - |z_j|^2|},$$

if $\rho > 1$ and all the zeros $z_1, \dots, z_k$ do not belong to the circle $|z| = \rho$;
b) complement to the strip

$$\left\{\alpha \in \mathbb{C}\colon |\operatorname{Re}\alpha - \omega| < (n-d)\frac{\rho}{\rho^2-1}\right\},$$

where

$$\omega = (n-d)\frac{\rho^2}{\rho^2-1} + \frac{d}{2},$$

if $\rho > 1$ and $|z_1| = \dots = |z_k| = \rho$;
c) complement to the strip

$$\{\alpha \in \mathbb{C}\colon |\operatorname{Re}\alpha - x| < y\},$$

where

$$x = (n-d)\frac{\rho^2}{\rho^2-1} + \rho^2\sum_{j=1}^{s}\frac{d_j}{\rho^2-|z_j|^2} + \frac{1}{2}\sum_{j=s+1}^{k} d_j,$$

$$y = (n-d)\frac{\rho^2}{\rho^2-1} + \rho\sum_{j=1}^{s}\frac{d_j|z_j|}{|\rho^2-|z_j|^2|},$$

if $\rho > 1$ and $|z_1|, \dots, |z_s| \neq \rho$, $|z_{s+1}| = \dots = |z_k| = \rho$, for some $s \in \mathbb{N}$, $s \leqslant k-1$, $1 \leqslant d_1 + \dots + d_s \leqslant n-2$;
d) the half-plane

$$\left\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha \leqslant \sum_{j=1}^{k}\frac{d_j}{1-|z_j|} + \frac{n}{2}\left(1-\frac{d}{n}\right)\right\},$$

if $\rho = 1$.

For $z = z_j$ for one of $j = 1, \dots, k$ and $d_j > 1$, in (3) equality holds for every $\alpha \in \mathbb{C}$ and all pairs of f and F, satisfying the theorem conditions.

If $z = z_j$ for one of $j = 1, \dots k$, $d_j = 1$, $\alpha \in \mathbb{C}$, or $z \in \mathbb{C}\backslash\overline{\Delta}$, $z \neq z_j$, $j = 1, \dots, k$, $\alpha \in$ int $D(\rho, k, n)$, then equality in (3) takes place only if $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Let us note that condition 3) in Theorem C is much stronger than condition 3) in Theorems A and B. However, if for a pairs of polynomials $\{f, F\}$ conditions $(*)$ are satisfied, then, by the maximum modulus principle, conditions 3) from Theorem C and Theorem A are equivalent.

In the following theorem, we remove restriction 2) of $(*)$ on localization of zeros of the polynomial F.

Theorem 1. *Let $\rho \geqslant 1$ be a fixed number. Let $f(z)$ and $F(z)$ be polynomials, such that:*
1) $\deg f \leqslant \deg F = n$;
2) $z_1, \dots, z_k$, $1 \leqslant k \leqslant n$, are all zeros of F, lying in $\mathbb{C}\backslash\overline{\Delta}$, with their multiplicity taking into account;
3) $|f(z)| \leqslant |F(z)|$ on $\partial\Delta$.
Then for $|z| \geqslant \rho$

$$|S_\alpha[f](z)| \leqslant |S_\alpha[Fq](z)|, \tag{4}$$

for all $\alpha \in \overline{D_{\rho,n}}$, where S_α is the Smirnov operator, $D_{\rho,n}$ is the set from Theorem B, $q(z) = \prod_{j=1}^{k} \dfrac{1 - \overline{z_j} z}{z - z_j}$.

For $|z| \geqslant \rho > 1$ and $\alpha \in D_{\rho,n}$, equality in (4) holds only if $f(z) = e^{i\gamma} F(z) q(z)$, $\gamma \in \mathbb{R}$.

The advantage of Theorem 1 in comparison with Theorem C is not only the laconism of its statement and the universality of the set of variation of the parameter α. This set in Theorem 1 does not depend on mutual location of the points z and z_j, $j = 1, \dots, k$. The main advantage lies in the conditions 3): $|f(z)| \leqslant |F(z)|$. In Theorem C, this inequality should be true in $\mathbb{C}\backslash\Delta$. In Theorem 1, this should be fulfilled on the circle $|z| = 1$, as in the classical cases of Theorem A and Theorem B. This fact substantially extends the possibility of applying Theorem 1 in comparison with Theorem C.

Let us compare inequalities (3) and (4). As it follows from Theorem C and Theorem 1, both inequalities are sharp. To compare these inequalities, we should take an ordered pair of polynomials $\{f, F\}$ satisfying the conditions of Theorem C. We need to find out which of the inequalities

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F](z)| \quad \text{or} \quad |S_\alpha[f](z)| \leqslant |S_\alpha[Fq](z)|$$

gives better result for the same $\alpha \in U := \operatorname{int}[D_{\rho,n} \cap D(\rho, k, n)]$, $\rho = |z|$, for the fixed $\rho > 1$. Note that $U \neq \varnothing$.

Denote $F_1(z) = F(z)q(z)$. All the zeros of F_1 belong to $\overline{\Delta}$. On $\partial\Delta$, we have $|F_1(z)| = |F(z)|$. Therefore, applying Theorem B to the pair of polynomials $\{F, F_1\}$, we obtain

$$|S_\alpha[F](z)| \leqslant |S_\alpha[F_1](z)|, \quad |z| = \rho. \tag{5}$$

In (5), the equality does not hold. Indeed, assuming the converse, by Theorem B, we have $F(z) = e^{i\gamma}F_1(z)$, $\gamma \in \mathbb{R}$. The last equality contradicts the fact $|q(z)| \neq 1$ for $|z| = \rho > 1$. Consequently,

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F](z)| < |S_\alpha[Fq](z)|.$$

So, under the conditions of Theorem C, for the pair of polynomials $\{f, F\}$, inequality (3) is better than inequality (4). However, the set of all pairs of polynomials $\{f, F\}$, such that 3) from Theorem C is true, is very poor in comparison with the set of polynomials, that satisfy conditions of Theorem 1.

Proof of Theorem 1. Let

$$F_1(z) = F(z)q(z) = F(z)\prod_{j=1}^{k}\left(\frac{z - \frac{1}{\overline{z_j}}}{1 - \frac{z}{z_j}} \cdot \frac{\overline{z_j}}{z_j}\right) = \sigma F(z)\prod_{j=1}^{k}\frac{z - a_j}{1 - \overline{a_j}z},$$

where $\sigma = \prod_{j=1}^{k}\frac{\overline{z_j}}{z_j}$, $|\sigma| = 1$, $a_j = \frac{1}{\overline{z_j}} \in \Delta$, $j = 1, \ldots, k$. Then, by condition 2) of Theorem 1, it follows that F_1 is a polynomial of degree n, having all zeros in $\overline{\Delta}$, and $|f(z)| \leqslant |F_1(z)|$ on $\partial\Delta$. Hence, by Theorem B, for all $\alpha \in \overline{D_{\rho,n}}$ we have:

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F_1](z)|, \quad |z| \geqslant \rho. \tag{6}$$

Thus, we obtain (4). By Theorem B, for $|z| \geqslant \rho > 1$ and $\alpha \in D_{\rho,n}$, in (6) equality holds only if $f = e^{i\gamma}F_1 = e^{i\gamma}Fq$, $\gamma \in \mathbb{R}$. $\square$

There is a large number of differential inequalities for polynomials, using the condition: all zeros of polynomials lie in $\overline{\Delta}$ or in $\mathbb{C}\backslash\Delta$. Using the simple technique, given in the proof of Theorem 1, it is possible to remove the condition on the zeros of polynomials in such inequalities and prove sharp generalizations.

As an application, consider several examples.

1. In [8, Theorem 1], for a polynomial $F(z)$ of degree n, having all zeros in Δ, the following inequality is proved:

$$\min_{|z|=1}\left|zF'(z) + \frac{n\beta}{2}F(z)\right| \geqslant n\left|1 + \frac{\beta}{2}\right|\min_{|z|=1}|F(z)| \tag{7}$$

for all $\beta \in \overline{\Delta}$. The inequality is sharp, the equality holds for $F(z) = \mu z^n$, $\mu \in \mathbb{C}$.

Remark 1. *Every polynomial F, with all its zeros in $\overline{\Delta}$ can be uniformly approximated in $\overline{\Delta}$ by polynomials F_j, $\deg F_j = \deg F$, having zeros in Δ. Hence, (7) is also true if all zeros of a polynomial F belong to $\overline{\Delta}$.*

Removing the condition on zeros of a polynomial F from (7), we get

Proposition 1. *Let F be a polynomial, $\deg F = n$. By Λ denote the set $\{z_1, \dots, z_k\} \neq \varnothing$ of all zeros of F, lying in $\mathbb{C}\backslash\Delta$, taking into account their multiplicity; $q(z) = \prod_{z_j \in \Lambda} \dfrac{1 - \overline{z_j}z}{z - z_j}$. Then for all $\beta \in \overline{\Delta}$*

$$\min_{|z|=1} \Big| z(F(z)q(z))' + \frac{n\beta}{2} F(z)q(z) \Big| \geqslant n \Big| 1 + \frac{\beta}{2} \Big| \min_{|z|=1} |F(z)|. \tag{8}$$

In (8), equality holds for $F(z) = c(z - e^{i\gamma})^n$, $\gamma \in \mathbb{R}$, $c \in \mathbb{C}\backslash\{0\}$, all $\beta \in \overline{\Delta}$ for $n \geqslant 2$ or $\beta = -1$ for $n = 1$.

Proof. Under conditions of Proposition 1, $F_1(z) = F(z)q(z)$ is a polynomial having all its zeros in $\overline{\Delta}$. Taking into account Remark 1, apply inequality (7) to F_1. Since $|q(z)| \equiv 1$ on $\partial\Delta$, we have

$$\min_{|z|=1} \Big| zF_1'(z) + \frac{n\beta}{2} F_1(z) \Big| \geqslant n \Big| 1 + \frac{\beta}{2} \Big| \min_{|z|=1} |F_1(z)| = n \Big| 1 + \frac{\beta}{2} \Big| \min_{|z|=1} |F(z)|.$$

So, we have proved (8).

The statement about the equality sign is checked by direct calculation. □

2. Also in [8, Theorem 2], for a polynomial $F(z)$ of degree n, having all its zeros in $\mathbb{C}\backslash\Delta$, the following sharp inequality was obtained: for all $\beta \in \overline{\Delta}$ and $|z| = 1$

$$\Big| zF'(z) + \frac{n\beta}{2} F(z) \Big| \leqslant \tag{9}$$

$$\leqslant \frac{n}{2} \Big[\Big(\Big| 1 + \frac{\beta}{2} \Big| + \Big| \frac{\beta}{2} \Big| \Big) \max_{|z|=1} |F(z)| - \Big(\Big| 1 + \frac{\beta}{2} \Big| - \Big| \frac{\beta}{2} \Big| \Big) \min_{|z|=1} |F(z)| \Big].$$

In (9), equality holds for the polynomials $F(z) = a + bz^n$, where $|a| = |b| = 1/2$. In particular, with the same conditions on a polynomial $F(z)$ as in (8) and $\beta = 0$, from (8), the following known result by Aziz and Dawood [2] holds:

$$|F'(z)| \leqslant \frac{n}{2} \big(\max_{|z|=1} |F(z)| - \min_{|z|=1} |F(z)| \big), \quad \text{for all } \ z \in \partial\Delta. \tag{10}$$

Using still the same method from the proof of Theorem 1, we obtain a corresponding statement for an arbitrary polynomial F.

Proposition 2. *Let F be a polynomial of degree n. By Λ_- denote the set of all zeros of F, lying in Δ, taking into account their multiplicity. Let $s \in [0, n]$ be order of the zero $z = 0$ of F. If $\Lambda_- = \{z_1, \dots, z_k\} \neq \varnothing$, denote $q(z) = \prod_{z_j \in \Lambda_-} \frac{1 - \overline{z_j} z}{z - z_j}$. If $\Lambda_- = \varnothing$, put $q(z) \equiv 1$. Then for all $\beta \in \overline{\Delta}$ and $|z| = 1$*

$$\left| z(F(z)q(z))' + \frac{(n-s)\beta}{2} F(z)q(z) \right| \leqslant \tag{11}$$

$$\leqslant \frac{n-s}{2} \left[\left(\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right| \right) \max_{|z|=1} |F(z)| - \left(\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right| \right) \min_{|z|=1} |F(z)| \right].$$

Inequality (11) is sharp. For $\Lambda_- \neq \varnothing$, equality holds for $F(z) = cz^n$, $c \in \mathbb{C} \setminus \{0\}$.

Proof. In the case $\Lambda_- = \varnothing$ inequality (11) is identical to (9). If $\Lambda_- \neq \varnothing$, then $F_1(z) = F(z)q(z)$ is a polynomial of degree $(n-s)$. All its zeros lie in $\mathbb{C} \setminus \Delta$. Therefore, applying (9) to F_1, we obtain (11), taking into account $|q(z)| \equiv 1$ on $\partial\Delta$. It remains to be noted that for all $\beta \in \overline{\Delta}$ the polynomial $F(z) = cz^n$ transforms inequality (11) to the identity for $z \in \partial\Delta$. $\square$

Corollary 1. *Using the notations of Proposition 2, from (11) with $\beta = 0$, we obtain an analogue of (10):*

$$\left| (F(z)q(z))' \right| \leqslant \frac{(n-s)}{2} \left(\max_{|z|=1} |F(z)| - \min_{|z|=1} |F(z)| \right)$$

for $|z| = 1$ and for arbitrary polynomial F.

3. For a polynomial $F(z)$ and $0 < p < \infty$, denote

$$||F||_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |F(e^{it})|^p dt \right)^{\frac{1}{p}}, \quad ||F||_0 = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{it})| dt \right).$$

R. P. Boas and Q. I. Rahman [7] proved the following inequality for a polynomial F of degree n with all zeros in $\mathbb{C} \setminus \Delta$:

$$||F(Rz)||_p \leqslant \frac{||z + R^n||_p}{||z + 1||_p} ||F(z)||_p \quad \text{for} \quad R > 1 \quad \text{and} \quad p \geqslant 1. \tag{12}$$

Inequality (12) is sharp. The equality holds for $F(z) = az^n + b$, $|a| = |b|$. Later, Q. I. Rahman and G. Schmeisser [24] proved that (12) is also true for $p \in [0, 1)$. About L^p inequalities for polynomials, see also [25].

We again use the method from the proof of Theorem 1. We obtain an analogue of (12) for arbitrary polynomial F without restrictions on the location of the zeros of F.

Proposition 3. *For a polynomial F, denote by Λ_- the set of all zeros of F lying in Δ, taking into account their multiplicity. Let $s \in [0, n]$ be order of the zero $z = 0$ of F. If $\Lambda_- = \{z_1, \ldots, z_k\} \neq \varnothing$, denote $q(z) = \prod_{z_j \in \Lambda_-} \dfrac{1 - \overline{z_j} z}{z - z_j}$. If $\Lambda_- = \varnothing$, put $q(z) \equiv 1$. Then for $p \geqslant 0$ and $R > 1$*

$$||F(Rz)q(Rz)||_p \leqslant \frac{||z + R^{n-s}||_p}{||z + 1||_p} ||F(z)||_p. \tag{13}$$

Inequality (13) is sharp. The equality holds for $F(z) = cz^n$, $c \in \mathbb{C} \backslash \{0\}$.

Proof. If $\Lambda_- = \varnothing$, then inequalities (13) and (12) are the same. If $\Lambda_- \neq \varnothing$, then $F_1(z) = F(z)q(z)$ is a polynomial of degree $(n - s)$, having all zeros in $\mathbb{C} \backslash \Delta$. Apply (12) to F_1 with $p \in [0, \infty)$:

$$||F_1(Rz)||_p = ||F(Rz)q(Rz)||_p \leqslant$$

$$\leqslant \frac{||z + R^{n-s}||_p}{||z + 1||_p} ||F(z)q(z)||_p = \frac{||z + R^{n-s}||_p}{||z + 1||_p} ||F(z)||_p, \quad R > 1.$$

We have proved (13). □

In this paper, we do not set an impossible goal to reformulate all known statements about differential inequalities for complex polynomials, removing restriction on the zeros of polynomials. There are too many such results. We just want to point out a way to achieve this goal.

Theorem 2. *Let $\rho \geqslant 1$ be a fixed number. Let $f(z)$ and $F(z)$ be polynomials, $\deg f = m$, $\deg F = n$, such that $|f(z)| \leqslant |F(z)|$ on $\partial\Delta$. Let the set $D_{\rho,n}$ of variation of the parameter α be as in Theorem 1. Let $\Lambda = \{z_1, \ldots, z_k\}$ denote the set of all zeros of F, lying in $\mathbb{C} \backslash \overline{\Delta}$, taking into account their multiplicity. Let $q(z)$ be as Theorem 1, if $\Lambda = \varnothing$ we put $q(z) \equiv 1$. Then for $|z| \geqslant \rho$*

$$|zf'(z) - \alpha f(z)| \leqslant |z|^l |z(F(z)q(z))' - (\alpha - l)F(z)q(z)| \tag{14}$$

for $\alpha \in \overline{D_{\rho,m}}$, *if* $l = m - n > 0$;

$$|zf'(z) - \alpha f(z)| \leqslant |z(F(z)q(z))' - \alpha F(z)q(z)| \tag{15}$$

for $\alpha \in \overline{D_{\rho,n}}$, *if* $m \leqslant n$.

For $|z| \geqslant \rho > 1$ *and* $\alpha \in D_{\rho,m}$, *equality in (14) holds only for polynomial* $f(z) = e^{i\gamma} z^l F(z)q(z)$. *For* $|z| \geqslant \rho > 1$ *and* $\alpha \in D_{\rho,n}$, *equality in (15) holds only for* $f(z) = e^{i\gamma}F(z)q(z)$, $\gamma \in \mathbb{R}$.

Proof. In the case $m \leqslant n$ and $\Lambda = \varnothing$, the statement of Theorem 2 equals to the statement of Theorem B. If $m \leqslant n$ and $\Lambda \neq \varnothing$, then condition of Theorem 1, including the statement about the equality sign, are fulfilled. So, we obtain (15).

Now let $m > n$. Consider new polynomial $F_2(z) = z^l F(z)$, $\deg F_2 = m$ and $|f(z)| \leqslant |F_2(z)|$ on $\partial\Delta$. Then we can apply Theorem 1 to the pair of polynomials $\{f, F_2\}$. Consequently, for $|z| \geqslant \rho \geqslant 1$ and all $\alpha \in \overline{D_{\rho,m}}$ we get

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F_2 q](z)|,$$

i. e.

$$\begin{aligned}|zf'(z) - \alpha f(z)| &\leqslant |z(z^l F(z)q(z))' - \alpha z^l F(z)q(z)| = \\ &= |z|^l |z(F(z)q(z))' - (\alpha - l)F(z)q(z)|.\end{aligned}$$

Moreover, from Theorem 1, for $|z| \geqslant \rho > 1$ and $\alpha \in D_{\rho,m}$, in (14) equality holds only for $f(z) = e^{i\gamma}F_2(z)q(z)$, i. e. $f(z) = e^{i\gamma}z^l F(z)q(z)$, $\gamma \in \mathbb{R}$. □

3. On geometry of domains in the theorems of S. N. Bernstain and V. I. Smirnov. In most classical results on differential inequalities for polynomials, condition 2) of (∗) is often used: it is required that a polynomial F has all its zeros in the *unit disc* $\overline{\Delta}$. However, there are a number of studies, where the condition that the zeros of a polynomial F belong to $\overline{\Delta}$ (or $\mathbb{C}\backslash\Delta$) is replaced by the condition that the zeros are localized in a compact set with certain restrictions on its geometry, see, for example [22], [26], [27, ch. V, § 1, 2°, 3°, p. 351, 352, 365, 366]. In this section, we continue the research of these authors.

In [27, ch. V, § 1, 3°, Theorem 2, p. 362], Theorem B was generalized. Here, instead of Δ (in Theorem B), an arbitrary open disc $\Delta(w, R)$ of center w and radius R was considered, or exterior of this disc.

Further in this section we will denote by $[\xi, \eta]$ the segment in $\mathbb{C}$ with endpoints ξ and η. For $n \in \mathbb{N}$, $0 < \rho < 1$, by $D^*_{\rho,n}$ denote the complement to the closed disc with diameter $\left[-\frac{n\rho}{1-\rho}, \frac{n\rho}{1+\rho}\right]$.

Theorem D. *[27, ch. V, § 1, 3°, Theorem 2, p. 362]. Let $f(z)$ and $F(z)$ be polynomials such that:*
1) $\deg f \leqslant \deg F = n$,
2) *F has all its zeros in a disc $\overline{\Delta(w,R)}$ (or in $\mathbb{C}\backslash\Delta(w,R)$),*
3) *$|f(z)| \leqslant |F(z)|$ on $\partial\Delta(w,R)$.*
Suppose ρ, $\rho \geqslant 1$ (or $0 < \rho < 1$, correspondingly) be a fixed number. Then for z, $|z-w| \geqslant R\rho$ (or $|z-w| \leqslant R\rho$, correspondingly), we have

$$|(z-w)f'(z) - \alpha_1 f(z)| \leqslant |(z-w)F'(z) - \alpha_1 F(z)| \tag{16}$$

for $\alpha_1 \in \overline{D_{\rho,n}}$ (or $\alpha_1 \in \overline{D^*_{\rho,n}}$, correspondingly).

For z, $|z-w| \geqslant R\rho$, $\rho > 1$, and $\alpha_1 \in D_{\rho,n}$ (or z, $|z-w| \leqslant R\rho$, $\rho < 1$, and $\alpha_1 \in D^*_{\rho,n}$, correspondingly), in (16) equality holds only for $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

In [27, ch. V, § 1, 3°, p. 366], an analogue of Theorem B was obtained for a convex set. In particular, the following theorem was proved.

Theorem E. *Let $B \subset \mathbb{C}$ be a bounded strictly convex domain. For a fixed point $\zeta \in \partial B$, let $\overline{\Delta(w_\zeta, R_\zeta)}$ be a closed disc of minimal radius, containing B, such that $\zeta \in \partial\Delta(w_\zeta, R_\zeta)$. Fix $w_0 \in B$, $w_0 \neq w_\zeta$. Denote $\rho_* = \min\limits_{\zeta\in\partial B}\left|\dfrac{\zeta - w_0}{\zeta - w_\zeta}\right|$. For the function $\psi(\zeta) = \arg\dfrac{\zeta - w_0}{\zeta - w_\zeta}$ continuous on ∂B, denote $\theta_1 = \min\limits_{\zeta\in\partial B}\psi(\zeta)$, $\theta_2 = \max\limits_{\zeta\in\partial B}\psi(\zeta)$.*

Consider the polynomials $f(z)$ and $F(z)$ such that:
1) $\deg f \leqslant \deg F = n$,
2) *F has all its zeros in B,*
3) *$|f(z)| \leqslant |F(z)|$ on ∂B.*
Then for $z \notin B$

$$|(z-w_0)f'(z) - \beta f(z)| \leqslant |(z-w_0)F'(z) - \beta F(z)| \tag{17}$$

for all β from the closet set $\Omega \ni 0$, bounded by the smooth curve, which consists of the arc of a circle

$$\Gamma = \left\{\frac{n}{2}\rho_* e^{i\theta} : \theta_1 \leqslant \theta \leqslant \theta_2\right\}$$

and two rays, passing from the endpoints of the curve Γ.

The following theorem complements Theorem E.

Theorem 3. *Let $E \subset \mathbb{C}$ be a compact set, $B = \operatorname{conv} E$ be the convex hull of E. Consider polynomials $f(z)$ and $F(z)$, such that:*
1) $\deg f \leqslant \deg F = n$,
2) F has all its zeros in E,
3) $|f(z)| \leqslant |F(z)|$ on ∂E.
Fix a point $z \notin B$. Consider a supporting line l to B, $z \notin l$, separating B and z. Take a disc $\Delta(v_z, R_z)$, $v_z \neq z$, such that $\Delta(v_z, R_z) \cap l = \varnothing$, $z \in \Delta(v_z, R_z)$, and the segment $[z, v_z]$ that is orthogonal to l. Then

$$|(z - v_z)f'(z) - \alpha_1 f(z)| \leqslant |(z - v_z)F'(z) - \alpha_1 F(z)|, \tag{18}$$

for all $\alpha_1 \in \overline{D^*_{\rho_1, n}}$, where $\rho_1 = \dfrac{|z - v_z|}{R_z}$.

For $\alpha_1 \in D^*_{\rho_1, n}$, in (18) equality holds only if $f = e^{i\gamma} F$, $\gamma \in \mathbb{R}$.

Remark 2. *For all pairs of polynomials $\{f, F\}$, from Theorem 3, for fixed z, v_z, R_z from Theorem 3 and w_0 from Theorem E, neither (17) nor (18) is a consequence of the other. This means that there is no $\lambda \in \mathbb{C}$, such that*

$$\lambda(z - w_0) = z - v_z \quad \text{and} \quad \lambda\Omega \subset \overline{D^*_{\rho_1, n}} \tag{19}$$

(where Ω is the set from Theorem E) and there is no $\mu \in \mathbb{C}$ such that

$$\mu(z - v_z) = z - w_0 \quad \text{and} \quad \mu\overline{D^*_{\rho_1, n}} \subset \Omega. \tag{20}$$

Indeed, the inclusion $\lambda\Omega \subset \overline{D^*_{\rho_1, n}}$ is not fulfilled, because $0 \in \Omega$, but $0 \notin \overline{D^*_{\rho_1, n}}$, Hence, for all λ there exists a neighborhood $\mathcal{U}_\lambda$ of the origin such that $\mathcal{U}_\lambda \not\subset \overline{D^*_{\rho_1, n}}$. Thus, (19) does not take place for all $\lambda \in \mathbb{C}$.

By conditions of Theorem E, $B \ni w_0 \neq z \notin B$. Therefore, if (20) is true, then $\mu \neq 0$. Consider the set

$$\mathbb{C} \backslash \Delta\Big(0, \frac{n\rho_1}{1 - \rho_1}\Big) = \bigcup_{\theta \in [\theta_1, \theta_1 + 2\pi]} l_\theta, \quad \text{where} \quad l_\theta = \Big\{ s e^{i\theta}, s \in \Big[\frac{n\rho_1}{1 - \rho_1}, \infty\Big)\Big\},$$

θ_1 is a constant from Theorem E. Obviously, $\mathbb{C} \backslash \Delta\big(0, \frac{n\rho_1}{1-\rho_1}\big) \subset \overline{D^*_{\rho_1, n}}$. Since Ω does not contain the rays $L_\theta = \big\{ s e^{i\theta} \colon s \in \big(\frac{n\rho_*}{2}, +\infty\big)\big\}$ for $\theta \in (\theta_1, \theta_2)$, then Ω does not contain the domain $\bigcup_{\theta \in (\theta_1, \theta_2)} L_\theta$. Consequently, for all $\mu \neq 0$ and sufficiently large $T \geqslant 1$ the set

$$\{\alpha_1 \in \mathbb{C} \colon |\alpha_1| > T|\mu| \frac{n\rho_*}{2}, \arg \alpha_1 \in (\theta_1, \theta_2)\}$$

is a subdomain of $\mu D^*_{\rho_1,n}$, but does not intersect Ω.

Therefore, Theorem E and Theorem 3 complement each other.

Proof of Theorem 3. By condition 2) of Theorem 3, all zeros of F lie in $\mathbb{C}\backslash\Delta(v_z, R_z)$. By conditions 1), 3), and the maximum modulus principle, we have $|f(z)| \leqslant |F(z)|$ for all $z \in \mathbb{C}\backslash B$. In particular, the last inequality takes place on $\partial\Delta(v_z, R_z)$. Then we apply Theorem D to f and F, inequality (16) takes the form

$$|(\zeta - v_z)f'(z) - \alpha_1 f(z)| \leqslant |(\zeta - v_z)F'(z) - \alpha_1 F(z)|, \tag{21}$$

for all $\zeta \in \overline{\Delta(v_z, R_z)}$, such that $|\zeta - v_z| \leqslant R_z\rho_1$ for the fixed $\rho_1 \in (0,1)$ and all $\alpha_1 \in \overline{D^*_{\rho_1,n}}$. In particular, (21) takes place for $\zeta = z$, where z is the point from the statement of Theorem 3, with $\rho_1 = \dfrac{|z - v_z|}{R_z}$.

By Theorem D, for $\zeta = z$, $|z - v_z| = R_z\rho_1$, $\rho_1 < 1$, and $\alpha_1 \in D^*_{\rho_1,n}$, in (21) equality holds only in the case $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$. $\square$

Corollary 2. *Let conditions of Theorem 3 be satisfied, $B = \operatorname{conv} E$, $z \notin B$ be a fixed point. Then*

$$|f'(z) - \beta f(z)| \leqslant |F'(z) - \beta F(z)| \tag{22}$$

for all β, $|\beta| \geqslant \dfrac{n}{\rho(z,B)}$.

For β, $|\beta| > \dfrac{n}{\rho(z,B)}$, in (22) equality holds only if $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Proof. Let c be the projection of the point z onto B, i. e. c is the nearest point to z, lying on ∂B. By h denote the straight line, passing through z and c. Let l be the straight line, passing through c orthogonal to h. Then (see [21, p. 81, Theorem 1.9.1]) l is a supporting line to B, separating B and z. Therefore we can write inequality (18) from Theorem 3 for the polynomials f and F. Here we take the open disc $\Delta(v_z, R_z)$, such that $c \in \partial\Delta(v_z, R_z)$, $R_z = \rho(z,B) + |z - v_z|$. Rewrite (18) in the form (22), where $\beta = \dfrac{\alpha_1}{z - v_z}$, $\alpha_1 \in \overline{D^*_{\rho_1,n}}$, $\rho_1 = \dfrac{|z - v_z|}{R_z}$. Note that

$$\Big\{\alpha_1 \in \mathbb{C}\colon |\alpha_1| \geqslant \frac{n\rho_1}{1-\rho_1}\Big\} \subset \overline{D^*_{\rho_1,n}}.$$

Consequently, in (22) we can take β, such that

$$|\beta| \geqslant \frac{n\rho_1}{(1-\rho_1)|z - v_z|} = \frac{n}{R_z - |z - v_z|} = \frac{n}{\rho(z,B)}.$$

Consider the question about equality in (22) for β, $|\beta| > \dfrac{n}{\rho(z,B)}$. By Theorem 3, for $\alpha_1 \in D^*_{\rho_1,n}$, in (18) equality holds iff $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$. Taking α_1, $|\alpha_1| > \dfrac{n\rho_1}{1-\rho_1}$, we have $\alpha_1 \in D^*_{\rho_1,n}$ and β, $|\beta| > \dfrac{n}{\rho(z,B)}$. Therefore, for the point z and such β, in (22) equality holds only in the case $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$. □

Corollary 2 gives us an opportunity to take β from (22) outside a disc centered at the origin. The following theorem states that we can also take β inside some disc centered at the origin.

Theorem 4. *Let E be a compact set in $\mathbb{C}$, $B = \operatorname{conv} E$ be a strictly convex set, f, F, z be as in Theorem 3. Then*

$$|f'(z) - \beta f(z)| \leqslant |F'(z) - \beta F(z)| \tag{23}$$

for all β, $|\beta| \leqslant \dfrac{n}{\rho(z,B)+2R}$, where R is radius of the smallest disc $\overline{\Delta(w,R)}$, containing B, such that $\partial\Delta(w,R)$ contains the projection c of z onto B, w belongs to the straight line, passing through z and c.

For β, $|\beta| < \dfrac{n}{\rho(z,B)+2R}$, in (23) equality holds only if $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Proof. Consider the disc $\overline{\Delta(w,R)}$ from the statement of Theorem 4. This disc always exists, because B is a strictly convex set, i. e. ∂B does not contain segments. By h denote the straight line, passing through the points z, c, and w. The supporting line l to B, passing through the point c orthogonally to the straight line h, separates B and z (see [21, p. 81, Theorem 1.9.1]). Since $c \in \partial\Delta(w,R)$ and l is a tangent to $\partial\Delta(w,R)$, we have $z \notin \overline{\Delta(w,R)}$.

All zeros of F belong to $\overline{\Delta(w,R)}$ by condition 2) of Theorem 3. By 1) and 3), and the maximum modulus principle, on $\partial\Delta(w,R)$ we have $|f(z)| \leqslant |F(z)|$. Applying Theorem D, we obtain the inequality

$$|(z-w)f'(z) - \alpha_1 f(z)| \leqslant |(z-w)F'(z) - \alpha_1 F(z)|, \tag{24}$$

where $|z-w| = R\rho_1, \rho_1 > 1$, $\alpha_1 \in \overline{D_{\rho_1,n}}$. Rewriting (24) we have (23), where $\beta = \dfrac{\alpha_1}{z-w}$. Take all $\alpha_1 \in \overline{D_{\rho_1,n}}$ such that $|\alpha_1| \leqslant \dfrac{n\rho_1}{\rho_1+1}$. Hence, in (23) we can use all β, satisfying the condition

$$|\beta| = \frac{|\alpha_1|}{|z-w|} \leqslant \frac{n}{(\rho_1+1)R}.$$

Since $|z-w| = |z-c| + |c-w| = \rho(z,B) + R$, we have $(\rho_1+1)R = = |z-w| + R = \rho(z,B) + 2R$. Hence, in (23), it is possible to take all β, $|\beta| \leqslant \dfrac{n}{\rho(z,B)+2R}$.

The statement about equality sign follows from Theorem D, because to obtain β, $|\beta| < \dfrac{n}{\rho(z,B)+2R}$, we can take $\alpha_1 \in D_{\rho_1,n}$, $|\alpha_1| < \dfrac{n\rho_1}{1+\rho_1}$. $\square$

Multiplying both sides of (22) and (23) by $|z|$ and taking $\tau = z\beta$, we obtain the following corollary.

Corollary 3. *Let E, B, f, F, z be as in Theorem 4. Then*

$$|S_\tau[f](z)| \leqslant |S_\tau[F](z)| \tag{25}$$

for all τ, $|\tau| \geqslant \dfrac{n|z|}{\rho(z,B)}$.

For τ, $|\tau| > \dfrac{n|z|}{\rho(z,B)}$, in (25) equality holds only if $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

If, in addition, B is a strictly convex set, then (25) takes place for all τ from the complement to the annulus

$$A = \left\{\tau\colon \frac{n|z|}{\rho(z,B)+2R} < |\tau| < \frac{n|z|}{\rho(z,B)}\right\},$$

where R is the constant from Theorem 4.

For $\tau \in \mathbb{C}\backslash\overline{A}$, in (25) equality holds only in the case $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Remark 3. *If in Corollary 3 we take $E = \overline{\Delta}$, then $\rho(z,B) = |z|-1$, the constant R from Theorem 4 equals 1. Therefore, by Corollary 3, (25) takes place for τ from the complement $\Sigma_{|z|}$ of the annulus*

$$\left\{\tau\colon \frac{n|z|}{|z|+1} < |\tau| < \frac{n|z|}{|z|-1}\right\}.$$

The set $\Sigma_{|z|}$ is contained into the set $\overline{D_{|z|,n}}$ from Theorem B. However, $\Sigma_{|z|} \neq \overline{D_{|z|,n}}$. This is rather expected, because in Corollary 3 we consider much more wider class of sets E compared with Theorem B, where $\overline{\Delta}$ is taken. In addition, it is interesting to note that boundary circles of $\Sigma_{|z|}$ are tangent to $\partial D_{|z|,n}$.

The Jung theorem [11] states that every compact set $E \subset \mathbb{C}$ can be placed into a closed disc of radius $\dfrac{\operatorname{diam} E}{\sqrt{3}}$.

Definition 1. [21, p. 297] For a compact set $E \subset \mathbb{C}$, $R \geqslant \dfrac{\operatorname{diam} E}{\sqrt{3}}$, the *$R$-strictly convex hull of E* is the intersection of all closed discs of radius R, containing E. This set is denoted by $\operatorname{strco}_R E$.

The following corollary allows us to remove the restriction on the compact set E in Theorem 4: $\operatorname{conv} E$ is a strictly convex set, and to obtain an inequality of type (23).

Corollary 4. *Let $E \subset \mathbb{C}$ be a compact set, $B = \operatorname{conv} E$, $z \notin B$. Suppose $f(z)$ and $F(z)$ be polynomials such that:*
1) $\deg f \leqslant \deg F = n$,
2) *F has all its zeros in E,*
3) $|f(z)| \leqslant |F(z)|$ *on* ∂E.
Then

$$|f'(z) - \beta f(z)| \leqslant |F'(z) - \beta F(z)|, \tag{26}$$

for all β,

$$|\beta| \leqslant \frac{n}{\rho(z, strco_{R^*} E) + 2R^*},$$

where

$$R^* = \max\left\{\frac{diam\, E}{\sqrt{3}}, \frac{diam^2\, E}{\rho(z, B)}\right\}.$$

For β, $|\beta| < \dfrac{n}{\rho(z, strco_{R^*} E) + 2R^*}$ in (26) equality holds iff $f = e^{i\gamma} F$, $\gamma \in \mathbb{R}$.

Proof. Fix $R > R^*$ and construct the R-strictly convex hull of E. It is known [21, p. 306] that the Hausdorff distance h between B and $\operatorname{strco}_R E$ satisfies the inequality

$$h(B, \operatorname{strco}_R E) \leqslant \frac{\operatorname{diam}^2 E}{R}. \tag{27}$$

Inequalities $R > \dfrac{\operatorname{diam}^2 E}{\rho(z, B)}$ and (27) imply the inequality $h(B, \operatorname{strco}_R E) < < \rho(z, B)$. Therefore, $z \notin \operatorname{strco}_R E$.

Apply Theorem 4 to the polynomials f and F, point z, and the strictly convex compact set $\operatorname{strco}_R E$. We obtain (26) for all β such that

$$|\beta| \leqslant \frac{n}{\rho(z, \operatorname{strco}_R E) + 2R}.$$

From [21, p. 359, Theorem 4.4.5], (see also [3]), for $R > R^*$, the following inequality holds

$$h(\mathrm{strco}_R E, \mathrm{strco}_{R^*} E) \leqslant \Big(\sqrt{\frac{R+R^*}{R-R^*}} - 1\Big)(R - R^*).$$

Therefore, $h(\mathrm{strco}_R E, \mathrm{strco}_{R^*} E) \underset{R\to R^*}{\longrightarrow} 0$. Since (see [21, p. 34])

$$|\rho(z, \mathrm{strco}_R E) - \rho(z, \mathrm{strco}^*_R E)| \leqslant h(\mathrm{strco}_R E, \mathrm{strco}_{R^*} E),$$

we have $\rho(z, \mathrm{strco}_R E) \underset{R\to R^*}{\longrightarrow} \rho(z, \mathrm{strco}_{R^*} E)$, and the function $\rho(z, \mathrm{strco}_R E)$ is continuous at the point $R = R^*$. Hence, in (26), it is possible to take all β such that

$$|\beta| < \frac{n}{\rho(z, \mathrm{strco}_{R^*} E) + 2R^*}.$$

For β, $|\beta| = \dfrac{n}{\rho(z, \mathrm{strco}_{R^*} E) + 2R^*}$, we obtain (26) by passing to the limit by β.

Take β, $|\beta| < \dfrac{n}{\rho(z, \mathrm{strco}_{R^*} E) + 2R^*}$. Choose $R > R^*$, R is close to R^*, such that

$$|\beta| < \frac{n}{\rho(z, \mathrm{strco}_R E) + 2R}.$$

By Theorem 4, for this β, in (26) equality holds iff $f = e^{i\gamma} F$, $\gamma \in \mathbb{R}$. $\square$

Remark 4. *Note that Theorem 4 and Corollary 4 complement each other even in the case of a strictly convex compact set E.*

1) Take $E = \Delta$. The set of variation of the parameter β from Theorem 4 is the disc $\overline{\Delta\Big(0, \frac{n}{|z|+1}\Big)}$.

Note that $\mathrm{strco}_{R^*} E = \Delta$, where the constant R^* from Corollary 4 equals $R^* = \max\left\{\frac{2}{\sqrt{3}}, \frac{4}{|z|-1}\right\}$.

For $|z| \geqslant 2\sqrt{3} + 1$, we have $R^* = \frac{2}{\sqrt{3}}$, and, by Corollary 4, we can take $\beta \in \overline{\Delta\Big(0, \frac{n}{|z|-1+\frac{4}{\sqrt{3}}}\Big)}$. This disc is contained in $\overline{\Delta\Big(0, \frac{n}{|z|+1}\Big)}$. For $|z| < 2\sqrt{3} + 1$, we obtain $R^* = \frac{4}{|z|-1}$. In (26), it is possible to take β from the disc $\overline{\Delta\Big(0, \frac{n}{|z|-1+\frac{8}{|z|-1}}\Big)}$. And again, this disc is contained in $\overline{\Delta\Big(0, \dfrac{n}{|z|+1}\Big)}$. Therefore, for $E = \Delta$, Theorem 4 gives a wider set of variation of parameter β in comparison with Corollary 4.

2) Consider the compact set E, bounded by the arc γ of the circle $\partial\Delta$ with endpoints $(1-\varepsilon, \pm\sqrt{2\varepsilon-\varepsilon^2})$, $0 < \varepsilon < 1$, $1 \in \gamma$, and the arc γ', symmetric to γ with respect to the straight line $\mathrm{Re}z = 1-\varepsilon$. Take $z > 1$.

In Theorem 4, projection of z onto E is the point $c = 1$, the disc $\overline{\Delta(w, R)} = \Delta$. So, for all $0 < \varepsilon < 1$, in Theorem 4 we obtain $\beta \in \overline{\Delta\Big(0, \dfrac{n}{z+1}\Big)}$.

In Corollary 4,

$$R^* = \max\left\{\frac{2\sqrt{2\varepsilon-\varepsilon^2}}{\sqrt{3}}, \frac{8\varepsilon-4\varepsilon^2}{z-1}\right\} \xrightarrow[\varepsilon\to 0]{} 0.$$

Here we can take

$$\beta \in \overline{\Delta\Big(0, \frac{n}{\rho(z, \mathrm{strco}_{R^*}E) + 2R^*}\Big)}.$$

By Definition 1, we have $E \subset \mathrm{strco}_{R^*}E$. Therefore, $\rho(z, \mathrm{strco}_{R^*}E) \leqslant \leqslant \rho(z, E) = z-1$. Also we note that $R^* < 1$ for sufficiently small ε. Consequently, $\rho(z, \mathrm{strco}_{R^*}E) + 2R^* < z+1$. Hence, in the case of such compact set, it is better to use Corollary 4 (not Theorem 4), because the set of variation of the parameter β in Corollary 4 contains $\overline{\Delta\Big(0, \dfrac{n}{z+1}\Big)}$.

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