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J-SYMMETRICAL FUNCTIONS AND SERIES IN THE COMPLEX PLANE

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In the recent paper [1, 2, 3] the authors have introduced the notion of (j,k)-symmetrical functions, proved several properties of these functions and given their different applications. In the present paper the authors extend the considerations onto the j-symmetrical functions. They deduce the general form of j-symmetrical functions (thm.1) and show some criteria of expandability of a function into series with j-symmetrical components (thm.3 and thm.4).

§ 1. Introduction

By $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{C}$ let us denote the set of all integers, the set of all positive integers and the set of all complex numbers, respectively. Let $k \in \mathbb{N}$ be arbitrarily fixed and let $\varepsilon_k = \exp(\frac{2\pi i}{k})$. A nonempty subset U of the complex plane $\mathbb{C}$ will be called $k-$symmetrical if $\varepsilon_k U = U$. The family of all functions $f\colon U \to \mathbb{C}$ will be denoted by $\mathcal{F}(U)$. For every $j \in \mathbb{Z}$ a function $f \in \mathcal{F}(U)$ will be called $(j,k)-$symmetrical if for each $z \in U$ $f(\varepsilon_k z) = \varepsilon_k^j f(z)$. The class of all (j,k)-symmetrical functions will be denoted by $\mathcal{F}_k^j(U)$. Let us notice that $\mathcal{F}_2^0(U)$ and $\mathcal{F}_2^1(U)$ are well known families of even functions and odd functions, respectively. Of course, the set $\mathcal{F}(U)$, with common operations, is a complex linear space and all $\mathcal{F}_k^j(U)$ are its linear subspaces.

Now we define the operators $G_k^j\colon \mathcal{F}(U) \to \mathcal{F}(U)$, $j \in \mathbb{Z}$, such that for every $f \in \mathcal{F}(U)$ and $z \in U$

$$G_k^j f(z) = k^{-1} \sum_{l=0}^{k-1} \varepsilon_k^{-jl} f(\varepsilon_k^l z). \tag{1}$$

In the paper [1] has been shown that G_k^j are linear operators and $G_k^j(\mathcal{F}(U)) = \mathcal{F}_k^j(U)$.

In the next we will use the following result from [1].

LEMMA 1. *Let $U \subset \mathbb{C}$ be a k-symmetrical set. Every function $f \in \mathcal{F}(U)$ can be written in the form*

$$f = \sum_{j=0}^{k-1} G_k^j f \tag{2}$$

and this partition is unique in the following sense : if $f = \sum_{j=0}^{k-1} f_k^j$, where $f_k^j \in \mathcal{F}_k^j(U)$ for $j = 0, 1, \ldots, k-1$, then $f_k^j = G_k^j f$.

From this lemma it follows that the space $\mathcal{F}(U)$ is the simple sum of the subspaces $\mathcal{F}_k^j(U)$ $j = 0, 1, \ldots, k-1$.

§ 2. The J-symmetrical functions

For $r > 0$ let us denote by C_r the positively oriented circle $\{z = r\exp(it): t \in \langle 0, 2\pi\rangle\}$. A set $U \subset \mathbb{C}$ will be called circular if for each $z \in U - \{0\}$ the circle $C_{|z|}$ is included in U. Of course, every circular set U is a k-symmetrical set for every $k \in \mathbb{N}$. Unless stated otherwise, the letter U will represent an arbitrarily fixed nonempty circular subset in the complex plane $\mathbb{C}$.

By $\mathcal{P}(U)$ we will denote the class of all functions $f: U \to \mathbb{C}$ such that for every circle $C_r \subset U$ the function $f|C_r$ is continuous. Of course, the set $\mathcal{P}(U)$ with common operations is a complex linear space.

For every $j \in \mathbb{Z}$ a function $f \in \mathcal{P}(U)$ will be called j-symmetrical if it is (j,k)-symmetrical for each $k \in \mathbb{N}$. The family of all j-symmetrical functions from $\mathcal{P}(U)$ will be denoted by $\mathcal{P}^j(U)$. Let $\mathcal{P}_k^j(U) = \mathcal{F}_k^j(U) \cap \mathcal{P}(U)$. Then $\mathcal{P}^j(U) = \bigcap_{k\in\mathbb{N}} \mathcal{P}_k^j(U)$ and $\mathcal{P}^j(U)$ is a linear subspace of $\mathcal{P}(U)$.

The following theorem gives the general form of the elements of the space $\mathcal{P}^j(U)$.

THEOREM 1. *Let $j \in \mathbb{Z}$. Every $f \in \mathcal{P}^j(U)$ has the form*

$$f(z) = z^j a_j(z), \quad 0 \neq z \in U, \tag{3}$$

where a_j are some functions which are constant on the circles $C_{|z|}$, $z \in U$. If $0 \in U$, then

$$f(0) = \begin{cases} c & for \quad j = 0, \\ 0 & for \quad j \neq 0, \end{cases}$$

where c is a complex number.

PROOF. Let us take any function $f \in \mathcal{P}^j(U)$. Then $f \in \mathcal{P}^j_k(U)$ for every $k \in \mathbb{N}$ and, in view of Lemma 1,

$$f(z) = G^j_k f(z), \quad 0 \neq z \in U.$$

Therefore

$$\begin{aligned} k(\varepsilon_k - 1)z^{-j} f(z) &= k(\varepsilon_k - 1)z^{-j} G^j_k f(z) \\ &= \sum_{l=0}^{k-1} f(\varepsilon^l_k z)(\varepsilon^l_k z)^{-j-1} \varepsilon^l_k (\varepsilon_k - 1) z := \sigma_k(z). \end{aligned} \tag{4}$$

Now, let us observe that the points $z_l := \varepsilon^l_k z$ belong to the circle $C_{|z|}$ and if k tends to infinity, then

$$z_{l+1} - z_l = \varepsilon^l_k (\varepsilon_k - 1) z \to 0.$$

From the continuity of the function $f|C_{|z|}$ we have

$$\lim_{k \to \infty} \sigma_k(z) = \int_{C_{|z|}} f(w) w^{-j-1} dw,$$

because $\sigma_k(z)$ are the integral sums of the above integral.

On the other hand $\lim_{k\to\infty} k(\varepsilon_k - 1) = 2\pi i$, so from (4) we obtain

$$f(z) = z^j (2\pi i)^{-1} \int_{C_{|z|}} f(w) w^{-j-1} dw.$$

Putting

$$a_j(z) = (2\pi i)^{-1} \int_{C_{|z|}} f(w) w^{-j-1} dw$$

we have (3) and the functions a_j are constant on the circles $C_{|z|}$. If $0 \in U$, then we can put $c = f(0)$.

The proof is complete. □

For every $j \in \mathbb{Z}$ let us define the operators $G^j\colon \mathcal{P}(U) \to \mathcal{P}(U)$, such that for every $f \in \mathcal{P}(U)$ and $z \in U - \{0\}$

$$G^j f(z) = z^j (2\pi i)^{-1} \int_{C_{|z|}} f(w) w^{-j-1} dw.$$

If $0 \in U$, then

$$G^j f(0) = \begin{cases} f(0) & for \quad j = 0, \\ 0 & for \quad j \neq 0. \end{cases}$$

From Theorem 1 it follows:

THEOREM 2. *For every $j \in \mathbb{Z}$ the operator G^j is a linear surjection of the space $\mathcal{P}(U)$ onto $\mathcal{P}^j(U)$; that is $G^j(\mathcal{P}(U)) = \mathcal{P}^j(U)$*

REMARK. *From the proof of Theorem 1 it follows that for every $j \in \mathbb{Z}$ and $f \in \mathcal{P}(U)$*

$$\lim_{k\to\infty} G^j_k f(z) = G^j f(z).$$

§ 3. The series of j-symmetrical functions

Since for every circular set $U \subset \mathbb{C}$ and every $k \in \mathbb{N}$ and $j \in \mathbb{Z}$ we have $\mathcal{P}(U) \subset \mathcal{F}(\mathcal{U})$ and $\mathcal{P}^j(U) \subset \mathcal{F}^j_k(U)$, so by Lemma 1, every function $f \in \mathcal{P}(U)$ can be uniquely presented as the sum (2) of (j,k)-symmetrical functions. There arises a natural question: is it possble to construct a partition of every function $f \in \mathcal{P}(U)$ onto a series of j-symmetrical functions, corresponding to the partition (2). More precisely, we will consider the problem of the possibility of the presentation of the functions $f \in \mathcal{P}(U)$ in the form

$$f = \sum_{n\in Z} G^n f. \tag{5}$$

We will understand the convergence of the series (5) as the convergence of the sequence

$$h_k = \sum_{n=-k}^{k} G^n f$$

in every point $z \in U$.

Let $f \in \mathcal{P}(U)$ and $z = r\exp(it) \in U$. By g_r let us denote the function, which is defined on the interval $\langle 0, 2\pi\rangle$ by the formula

$$g_r(t) = f(r\exp(it)), \quad t \in \langle 0, 2\pi\rangle.$$

If there exists the differential $g_r'(t)$ of g_r at the point t, then we will call it the circular differential of f at the point $z = r\exp(it) \in U$ and we will denote it by $f'^c(z)$.

THEOREM 3. *Let $f \in \mathcal{P}(U)$. If there exists the finite circular differential $f'^c(z)$ in a point $z \in U$, then*

$$f(z) = \sum_{n\in\mathbb{Z}} G^n f(z).$$

Moreover, if the function f'^c is bounded on U, then the expansion (5) *holds on U and the series $\sum_{n\in\mathbb{Z}} G^n f$ converges uniformly on every circle $C_r \subset U$.*

PROOF. Let $z = r\exp(it) \in U$. Then

$$\sum_{n\in\mathbb{Z}} G^n f(z) = \sum_{n\in\mathbb{Z}} \exp(int)(2\pi)^{-1} \int_0^{2\pi} f(r\exp(is))\exp(-ins)ds.$$

Of course, the above series is the Fourier series of the function g_r at the point t. For $k \in \mathbb{N}$ let us denote

$$S_k(t) = \sum_{n=-k}^{k} \exp(int)(2\pi)^{-1} \int_0^{2\pi} g_r(s) exp(-ins)ds.$$

Then we obtain

$$S_k(t) = (2\pi)^{-1} \int_0^{2\pi} g_r(s) \sum_{n=-k}^{k} \exp(in(t-s))ds$$
$$= (2\pi)^{-1} \int_0^{2\pi} g_r(s) D_k(t-s)ds, \tag{6}$$

where

$$D_k(t-s) = \left(\sin\frac{t-s}{2}\right)^{-1} \sin\left(\left(k+\frac{1}{2}\right)(t-s)\right).$$

Since f has the finite circular differential $f'^c(z)$ at the point $z = r\exp(it) \in U$, so the function g_r fulfils, in the point t, the Lipschitz condition

$$|g_r(t+h) - g_r(t)| \le L|h|,$$

with $L = 2|f'^c(z)|$ and $|h|$ sufficiently small. It is obvious that the functions $\operatorname{Re} g_r$ and $\operatorname{Im} g_r$ fulfil the above condition, too.

From (6) it follows that

$$\operatorname{Re} S_k(t) = (2\pi)^{-1} \int_0^{2\pi} \operatorname{Re} g_r(s) D_k(t-s)ds, \tag{7}$$

$$\operatorname{Im} S_k(t) = (2\pi)^{-1} \int_0^{2\pi} \operatorname{Im} g_r(s) D_k(t-s)ds. \tag{8}$$

Of course, the integrals (7) and (8) are the k-th sums of Fourier series of the functions $\operatorname{Re} g_r$ and $\operatorname{Im} g_r$ at the point t.

If k tends to infinity, then the integrals (7) and (8) tend to the values $\operatorname{Re} g_r(t)$ and $\operatorname{Im} g_r(t)$, respectively, because the functions $\operatorname{Re} g_r$ and $\operatorname{Im} g_r$ fulfil the Lipschitz condition at the point t. From this we obtain

$$\lim_{k\to\infty} \operatorname{Re} S_k(t) = \operatorname{Re} g_r(t), \lim_{k\to\infty} \operatorname{Im} S_k(t) = \operatorname{Im} g_r(t),$$

so

$$\lim_{k\to\infty} S_k(t) = g_r(t).$$

This completes the proof of the first part of the theorem.

Now let us assume that f'^c is a bounded function on U. Then for every r, such that $z \in U$ for $|z| = r$, the function g_r fulfils the Lipschitz condition with the constant $L = 2\sup\{|f'^c(z)| : z \in C_r\}$ in every point $t \in \langle 0, 2\pi\rangle$. Therefore for every circle $C_r \subset U$ the Fourier series of the functions $\operatorname{Re} g_r$, $\operatorname{Im} g_r$ converge uniformly to these functions.

This completes the proof. □

From the considerations in the proof it follows more general result.

THEOREM 4. *Let $f \in \mathcal{P}(U)$. If for every point $z \in U$ the function $g_{|z|}$ satisfies the Lipschitz condition at every point $t \in \langle 0, 2\pi\rangle$, then expansion (5) holds and the series $\sum_{n\in\mathbb{Z}} G^n f$ converges uniformly on every circle $C_r \subset U$.*

Now let us consider some series $\sum_{n\in\mathbb{Z}} g_n$ of n-symmetrical functions ($g_n \in \mathcal{P}^n(U)$). Let us assume that this series converges to a function g. From Theorem 1 it follows that

$$g_n(z) = z^n a_n(z), \quad 0 \neq z \in U.$$

Therefore for every $z \in U$

$$g(z) = \sum_{n\in\mathbb{Z}} z^n a_n(z), \tag{9}$$

where a_n are constant functions on the circles $C_{|z|}$, with $z \neq 0$ and

$$a_n(0) = \begin{cases} g(0) & for \quad n = 0, \\ 0 & for \quad n \neq 0. \end{cases}$$

In general, of course, the sum g of the series (9) not belongs to the space $\mathcal{P}(U)$, but it is true the following result.

THEOREM 5. *If series (9) converges on U uniformly on every circle $C_r \subset U$, then $g \in \mathcal{P}(U)$ and $g_n(z) = G^n g(z)$ for every $n \in \mathbb{Z}$.*

PROOF. The relation $g \in \mathcal{P}(U)$ is obvious. Let $0 \neq z \in U$. Then for every $j \in \mathbb{Z}$

$$G^j g(z) = \sum_{n\in\mathbb{Z}} G^j(z^n a_n(z)) = \sum_{n\in\mathbb{Z}} z^j (2\pi i)^{-1} \int_{C_{|z|}} w^n a_n(w) w^{-j-1} dw$$

$$\sum_{n\in\mathbb{Z}} z^j a_n(z)(2\pi)^{-1} \int_0^{2\pi} exp((n-j)it)dt = z^j a_j(z) = g_j(z).$$

This completes the proof. □

THEOREM 6. *Let $f \in \mathcal{P}(U)$. If f can be presented in the form (5) and the series $\sum_{n\in\mathbb{Z}} G^n f$ converges uniformly on every circle $C_r \subset U$, then partition (5) is unique in the following sense: if $f = \sum_{n\in\mathbb{Z}} f_n$, where $f_n \in \mathcal{P}^n(U)$ for every $n \in \mathbb{Z}$, then $f_n = G^n f$.*

From Theorem 3 and Theorem 6 we obtain.

COROLLARY . *Let $f \in \mathcal{P}(U)$. If f has the circular differential f'^c bounded on U, then f can be presented in form (5) and this partition is unique.*

Литература

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