Труды Петрозаводского государственного университета
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COEFFICIENT PROBLEM FOR SOME CLASS OF FUNCTIONS NONVANISHING IN THE UNIT DISK

W. Majchrzak, A. Szwankowski

In this paper we determine the bounds of the functional $b_2 - \alpha b_1$, α-real, for holomorphic univalent and bounded functions, symmetric with respect to the real axis, nonvanishing in the unit disk. The result generalizes the estimates of b_1 and b_2 for these functions, obtained by Śladkowska [4].

1. Introduction. Let $\mathcal{B}_0^R(b)$, $0 < |b| < 1$, denote the class of all functions $f(z) = b + \sum_{n=1}^{\infty} b_n z^n$, $b_n = \overline{b_n}$, that are holomorphic univalent in the unit disk Δ and satisfy the conditions

$$f(\Delta) \subset \Delta, \quad 0 \notin f(\Delta).$$

$\mathcal{B}_0^R(b)$ is a normal but not compact family in the topology of locally uniform convergence in Δ. However, it becomes compact if the function $f(z) = b$, $z \in \Delta$, belongs, in addition, to $\mathcal{B}_0^R(b)$.

The main aim of the present paper is to obtain the bounds of the functional

$$H(f) = b_2 - \alpha b_1, \quad f \in \mathcal{B}_0^R(b), \tag{1}$$

where α is real.

To solve this problem, we shall use the variational technique developed by Śladkowska [4] for the class $\mathcal{B}_0^R(b)$.

The problem posed here is connected with a coefficient problem for the class $\mathcal{B}_0(b) \supset \mathcal{B}_0^R(b)$ of functions holomorphic and univalent in Δ satisfying the conditions $f(0) = b$, $f(\Delta) \subset \Delta$ and $0 \notin f(\Delta)$ (cf. [2], [1], [3]).

2. Estimation of $H(f)$. From Schiffer's equation [4] we get that each extremal function with respect to $H(f)$ belonging to $\mathcal{B}_0^R(b)$ satisfies the following differential-functional equation

$$\frac{\zeta^2 w'^2}{2w} \frac{P(w)}{b_1 (b-w)^3 (1-bw)^3} = -2b_2 + \alpha b_1 - b_1 \left(\zeta + \frac{1}{\zeta}\right) \tag{2}$$

where

$$\begin{aligned} P(w) &= Kw^4 + Lw^3 + Mw^2 + Lw + K, \\ K &= -2b^5 b_1 + 6b^5 b_3 - 4b^4 b_1 b_2 + 2b^3 b_1 - 6b^3 b_3 - 2b^3 b_1^3 \\ &\quad -\alpha\left(4b^5 b_2 - 4b^4 b_1^2 - 4b^3 b_2\right), \\ L &= 4b^6 b_1 - 12b^6 b_3 + 6b^5 b_1 b_2 + 12b^3 b_1 b_2 - 4b^2 b_1 + 12b^2 b_3 \\ &\quad +6b^2 b_1^3 - 2bb_1 b_2 + 2b_1^3 - \alpha(-8b^6 b_2 + 6b^5 b_1^2 + 12b^3 b_1^2 \\ &\quad +8b^2 b_2 - 2bb_1^2), \\ M &= -2b^7 b_1 + 6b^7 b_3 - 2b^6 b_1 b_2 - 6b^5 b_1 + 18b^5 b_3 - 18b^4 b_1 b_2 \\ &\quad -18b^3 b_3 + 6b^3 b_1 - 6b^2 b_1 b_2 + 2bb_1 - 6bb_3 - 12bb_1^3 + 2b_1 b_2 \\ &\quad -\alpha(4b^7 b_2 - 2b^6 b_1^2 + 12b^5 b_2 - 18b^4 b_1^2 - 12b^3 b_2 - 6b^2 b_1^2 \\ &\quad -4bb_2 + 2b_1^2). \end{aligned}$$

The studying of solutions of equation (2) will consist of two cases: $1^\circ \quad K = 0, \quad 2^\circ \quad K \neq 0.$

If $b \in (0,1)$ and $K = 0$, then $L \neq 0$ and equation (2) has the following form:

$$\frac{L(w+1)^2 \zeta^2 w'^2}{2b_1 (b-w)^3 (1-bw)^3} = -b_1 \frac{(\zeta+1)^2}{\zeta} \tag{3}$$

or

$$\frac{L(w+1)^2 \zeta^2 w'^2}{2b_1 (b-w)^3 (1-bw)^3} = -b_1 \frac{(\zeta-1)^2}{\zeta}. \tag{4}$$

From the condition $K = 0$ we have

$$b_3 = \frac{1}{3(b^2-1)} \left[b^2 b_1 + 2bb_1 b_2 - b_1 + b_1^3 + \alpha\left(2b^2 b_2 - 2bb_1^2 - 2b_2\right)\right]. \tag{5}$$

Comparing the coefficients of the right-hand sides of (3) and (4) with the analogous coefficients of (2), we obtain, respectively,

$$b_2 = \left(1 + \frac{\alpha}{2}\right) b_1 \quad \text{or} \quad b_2 = \left(\frac{\alpha}{2} - 1\right) b_1. \tag{6}$$

Since, in the case under consideration, $M = 2L$, by (5) and (6) we get

$$b_1 = \frac{\left(1-b^2\right)(2-\alpha)}{2\,(b+2)} \quad \text{or} \quad b_1 = -\frac{\left(1-b^2\right)(2+\alpha)}{2\,(b+2)}. \tag{7}$$

In consequence,

$$b_2 - \alpha b_1 = \frac{\left(1-b^2\right)(2-\alpha)^2}{4\,(b+2)} \quad \text{or} \quad b_2 - \alpha b_1 = \frac{\left(1-b^2\right)(2+\alpha)^2}{4\,(b+2)}. \tag{8}$$

Equations (3) and (4) can be integrated and their solutions with the initial condition $f\,(0) = b$ satisfy the relation

$$\frac{b_1\,(1+b)}{1-b}\,\frac{(w-1)^2}{(w-b)\,(1-bw)} = \zeta + \frac{1}{\zeta} \mp 2. \tag{9}$$

In the case when $b \in (0,1)$ and $K \neq 0$, equation (2) can take one of the following forms:

$$\frac{\zeta^2 K\,(w+1)^2\,(w-1)^2\,w'^2}{2b_1 w\,(b-w)^3\,(1-bw)^3} = -2b_2 + \alpha b_1 - b_1\left(\zeta + \frac{1}{\zeta}\right) \tag{10}$$

or

$$\frac{\zeta^2 K\,(w+1)^2\,(w-c)\left(w-\frac{1}{c}\right)w'^2}{2b_1 w\,(b-w)^3\,(1-bw)^3} = -2b_2 + \alpha b_1 - b_1\left(\zeta + \frac{1}{\zeta}\right) \tag{11}$$

where $c \in (0,1)$.

Comparing the coefficients of equation (10) with those of equation (2), we get

$$b_2 - \alpha b_1 = \left(1 - \frac{\alpha}{2}\right) b_1 \quad \text{and} \quad b_1 = \frac{1}{2}\,(\alpha-2)\,b\left(1-b^2\right) \tag{12}$$

or

$$b_2 - \alpha b_1 = \left(1 + \frac{\alpha}{2}\right) b_1 \quad \text{and} \quad b_1 = \frac{1}{2}\,(\alpha+2)\,b\left(1-b^2\right). \tag{13}$$

In consequence, we have

$$b_2 - \alpha b_1 = -\frac{1}{4}\,(2-\alpha)^2\,b\left(1-b^2\right) \tag{14}$$

or

$$b_2 - \alpha b_1 = -\frac{1}{4}(2+\alpha)^2 b\left(1-b^2\right). \tag{15}$$

If an extremal function satisfies (11), it must map Δ onto $\Delta \setminus [-1, 0]$. The function that transforms Δ onto this set is of the form

$$\frac{\left(\sqrt{w}-\sqrt{b}\right)\left(1+\sqrt{b}\sqrt{w}\right)}{\left(\sqrt{w}+\sqrt{b}\right)\left(1-\sqrt{b}\sqrt{w}\right)} = \pm\zeta. \tag{16}$$

From (16) and from the fact that the right-hand side of (11) has two distinct roots we obtain

$$b_2 - \alpha b_1 = \frac{-8b(1-b)\left(b^2+2b-1\right)}{(1+b)^3} - \alpha\frac{4b(1-b)}{1+b} \tag{17}$$

where α satisfies the inequality

$$\left[\alpha - \frac{6-4b-2b^2}{(1+b)^2}\right]\left[\alpha - \frac{2-12b-6b^2}{(1+b)^2}\right] > 0 \tag{18}$$

or

$$b_2 - \alpha b_1 = \frac{-8b(1-b)\left(b^2+2b-1\right)}{(1+b)^3} + \alpha\frac{4b(1-b)}{1+b}, \tag{19}$$

where α satisfies

$$\left[\alpha - \frac{2b^2+4b-6}{(1+b)^2}\right]\left[\alpha - \frac{6b^2+12b-2}{(1+b)^2}\right] > 0.$$

Assume now that $\alpha \in (-\infty, 0)$. Note that, for such an α, the coefficient b_1 of a function maximizing the functional under consideration is positive and, for a minimizing function, b_1 is negative.

Hence and by taking account the well-known estimate of the coefficient b_1 [4], after comparing relations (8), (14), (15), (17) and (19) we obtain the following results:

THEOREM 1. *If $f \in \mathcal{B}_0^R(b)$, then*

$$b_2 - \alpha b_1 \le \begin{cases} \dfrac{-8b(1-b)\left(b^2+2b-1\right)}{(1+b)^3} - \alpha\dfrac{4b(1-b)}{1+b}, & \\ & (b,\alpha) \in D_1 \cup D_2, \\ \dfrac{\left(1-b^2\right)(2-\alpha)^2}{4(b+2)}, & (b,\alpha) \in D_3 \end{cases} \tag{20}$$

where

$$\begin{array}{lll} D_1 = \{(b,\alpha): & 0 < b \le b^*, & \alpha \le 0\}, \\ D_2 = \{(b,\alpha): & b^* \le b < 1, & \alpha \le \alpha_0(b)\}, \\ D_3 = \{(b,\alpha): & b^* \le b < 1, & \alpha_0(b) \le \alpha \le 0\} \end{array}$$

and

$$b^* = \frac{2\sqrt{3}}{3} - 1, \quad \alpha_0(b) = \left(2 - 12b - 6b^2\right) / (1+b)^2.$$

Estimate (20) *is sharp. The equality for* $(b,\alpha) \in D_1 \cup D_2$ *is realized by some function described in* (16) *and, for* $(b,\alpha) \in D_3$*, by some function defined in* (9)*.*

THEOREM 2. *If* $f \in \mathcal{B}_0^R(b)$*, then, for* $b \in (0,1)$

$$b_2 - \alpha b_1 \ge \begin{cases} -\frac{1}{4}(2-\alpha)^2 b\left(1-b^2\right), & \alpha \in [\alpha_1(b); 0], \\ \dfrac{-8b(1-b)\left(b^2+2b-1\right)}{(1+b)^3} + \alpha \dfrac{4b(1-b)}{1+b}, & \\ & \alpha \in (-\infty, \alpha_1(b)], \end{cases} \tag{21}$$

where $\alpha_1(b) = \left(2b^2 + 4b - 6\right) / (1+b)^2$.

Estimate (21) *is sharp. For* $\alpha \in [\alpha_1(b), 0]$*, the minimum is realized by a function satisfying equation* (10)*, while, for* $\alpha \in (-\infty, \alpha_1(b)]$*, by some function described in* (16)*.*

Note next that, for $\alpha \in (0, +\infty)$, the coefficient b_1 of a function maximizing the functional $H(f)$ is negative and it is positive for a minimizing function. Moreover, if $f(z) \in \mathcal{B}_0^R(b)$, then $f(-z) \in \mathcal{B}_0^R(b)$.

Taking account of these properties and proceeding similarly as in the case included in Theorems 1 and 2, we get the estimates of $H(f)$ for $\alpha \in [0, +\infty)$ and $b \in (0,1)$.

It is also well known that if $f \in \mathcal{B}_0^R(b)$, $b \in (0,1)$ then, $(-f) \in \mathcal{B}_0^R(b)$, $b \in (-1,0)$. Hence, from Theorems 1 and 2 we can directly derive the estimates of $H(f)$ for $b \in (-1,0)$.

Finally, remark that from (20) and (21) we can immediately get the well-known estimates of the coefficients b_1 and b_2 [4] for the class $\mathcal{B}_0^R(b)$.

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Dept. of Special Functions Lodz University
90-238 S. Banacha, Lodz, Poland