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M. Y. Mir, G. M. Sofi, S. L. Wali, W. M. Shah

SOME REFINEMENTS OF BERNSTEIN TYPE INEQUALITIES FOR RATIONAL FUNCTIONS WITH ZEROS IN A HALF PLANE

Abstract. Sofi and Shah proved some Bernstein-type inequalities for the class of rational functions $r(z)$ with all poles in a half-plane. In this paper, we consider all zeros of $r(z)$ to lie in the right half-plane and prove several refinements of these results, which, in turn, generalize some previously known results.

Key words: *polynomial, rational function, half plane, Bernstein-type inequality*

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1. Introduction. Let $p(z) := \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n with complex coefficients, and let $p'(z)$ be its derivative. Concerning the maximum modulus of $p(z)$ and $p'(z)$ on unit disk, Bernstein [3] proved the following inequality that bears his name:

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|. \tag{1}$$

Since a polynomial is entirely determined by its zeros (up to multiplication by a constant), it is natural to investigate the inequality (1) under some restrictions on the zeros of p. Two significant results in this direction are the following:

Theorem 1. *If $p(z) := \sum_{j=0}^{n} c_j z^j$ has all zeros in $|z| \geqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

Theorem 2. *If $p(z) := \sum_{j=0}^{n} c_j z^j$ has all zeros in $|z| \leqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{3}$$

Inequality (2) was conjectured by Erdös and later verified by Lax [6], whereas inequality (3) is due to Turán [12]. There have been many generalizations and refinements to (1), (2), and (3): for details, see [2], [4] [5], [7], and [13].

Li et al. [8] extended several of these Bernstein-type inequalities to the rational functions with prescribed poles. It turns out that analogues of (1), (2), and (3) hold for rational functions: they are obtained by replacing n with the Blaschke product. For instance, the analogue of Bernstein's inequality (1) is given by the following theorem.

Theorem 3. *If $r(z) = \dfrac{p(z)}{\prod_{j=0}^{n}(z - a_j)}$, where $p(z)$ is a polynomial of degree n, then*

$$|r'(z)| \leqslant |\mathcal{B}'(z)| \max_{|z|=1} |r(z)| \tag{4}$$

for $|z| = 1$, where $\mathcal{B}(z) = \prod_{j=1}^{n} \left(\dfrac{1 - \overline{a_j} z}{z - a_j}\right)$ denotes the Blaschke product and $|a_j| > 1$, $j = 1, 2, 3, \ldots n$.

All the work done so far on Bernstein-type inequalities for rational functions (with one exception [11]) involves the rational functions with poles inside/outside a disk with an estimation of bounds on the boundary of that disk (for details, see [1], [14]). In [11], Sofi and Shah proved Bernstein-type inequalities for rational functions having poles in the half-plane instead of inside or outside a disk. One key difference in the estimation of the bounds on the boundary of the half-plane is that the boundary of the half-plane (unlike that of a disk) is not a compact set.

In this paper, we prove some inequalities for rational functions having poles in a half-plane under some restrictions on their zeros. We will use the following notations and definitions:

$$\mathbb{I} := \{z \colon \operatorname{Re}(z) = 0\}.$$

$$\mathbb{I}^+ := \{z \colon \operatorname{Re}(z) > 0\}.$$

$$\mathbb{I}^- := \{z\colon\ \mathrm{Re}(z) < 0\}.$$

Further, for $p(z) = \sum_{j=0}^{n} c_j z^j$ and $z \in \mathbb{I}$, we also define conjugate transpose p^* of p as

$$p^*(z) := (-1)^n \overline{p(-\overline{z})} = \overline{c_n} z^n - \overline{c_{n-1}} z^{n-1} + \ldots (-1)^n \overline{c_0}.$$

Also, we denote the set of rational functions $\mathcal{R}_n$ by

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) := \Big\{ \frac{p(z)}{w(z)}, a_j \in \mathbb{I}^+, j = 1, 2, \ldots, n \Big\},$$

where $w(z) = \prod_{j=1}^{n} (z - a_j)$, and a Blaschke product for the half-plane by

$$B(z) := \prod_{j=1}^{n} \Big(\frac{z + \overline{a_j}}{z - a_j} \Big).$$

It is clear that $|B(z)| = 1$ on $\mathbb{I}$ and $B(z) = \dfrac{w^*(z)}{w(z)}$, with $w^*(z) := (-1)^n \overline{w(-\overline{z})}$.

Now, for $r(z) = \dfrac{p(z)}{w(z)}$, define $r^*(z) := B(z)\overline{r(-\overline{z})}$. Therefore, if $r \in \mathcal{R}_n$, then $r^*(z) = \dfrac{p^*(z)}{w(z)} \in \mathcal{R}_n$.

We are also going to need the idea of the polar derivative of a polynomial $p(z)$ of degree n with respect to a complex number α, which is defined as

$$D_\alpha p(z) := np(z) + (\alpha - z)p'(z).$$

Clearly $D_\alpha p(z)$, like the derivative, is a polynomial of degree $n - 1$, and

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

With regards to the polar derivative, it was E. Laguerre in 1874, who first recognized its importance in locating the zeros of polynomials, but he used the term 'emanant' for it. Later G. Polya and G. Szego called $D_\alpha p(z)$ the derivative with respect to α. It was Marden, who proposed the name polar derivative (for details see [9] and [10]).

We need the following lemmas.

Lemma 1. *Let i be the imaginary unit. The function $B(z) = i$ has exactly n simple roots and they all lie on imaginary axis $\mathbb{I}$, and for $r \in \mathcal{R}_n$ and $z \in \mathbb{I}$:*

$$r'(z)(B(z) - i) - B'(z)r(z) = (B(z) - i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z - t_k|^2}, \tag{5}$$

where $t_1, t_2, \ldots, t_n$ are the n simple roots of $B(z) = i$ and

$$\frac{1}{u_k} = B'(t_k) = i \sum_{j=1}^{n} \frac{2\,\mathrm{Re}(a_j)}{|t_k - a_j|^2}, \quad 1 \leqslant k \leqslant n. \tag{6}$$

Moreover, for $z \in \mathbb{I}$:

$$\frac{B'(z)}{B(z)} = \sum_{j=1}^{n} \frac{2\,\mathrm{Re}(a_j)}{|z - a_j|^2}. \tag{7}$$

Lemma 2. *Let $r \in \mathcal{R}_n$ be such that all zeros of r lie in $\mathbb{I} \cup \mathbb{I}^+$; then, for $z \in \mathbb{I}$:*

$$\mathrm{Re}\left(\frac{r'(z)}{r(z)}\right) \leqslant \frac{1}{2}|B'(z)|. \tag{8}$$

Lemma 3. *$z \in \mathbb{I}$ if and only if $|B(z)| = 1$.*

The above lemmas are due to Sofi et al. [11].

With the above notations, we state our main results as follows:

Theorem 4. *Let t_1, $t_2, \ldots, t_n$ be the n simple roots of $B(z) = i$ and s_1, $s_2, \ldots, s_n$ be those of $B(z) = -i$. If $r \in \mathcal{R}_n$ and $z \in \mathbb{I}$, then*

$$|r'(z)|^2 + |(r^*(z))'|^2 \leqslant \frac{1}{2}|B'(z)|^2 \left\{ \left(\max_{1 \leqslant k \leqslant n} |r(t_k)| \right)^2 + \left(\max_{1 \leqslant k \leqslant n} |r(s_k)| \right)^2 \right\}. \tag{9}$$

The result is sharp and equality holds for $r(z) = \lambda B(z)$ with $|\lambda| = 1$.

Proof. From equation (5), we have

$$\begin{aligned} |r'(z)B(z) - r(z)B'(z) - ir'(z)| &= \left| (B(z) - i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z - t_k|^2} \right| \\ &\leqslant |B(z) - i|^2 \sum_{k=1}^{n} \frac{|u_k||r(t_k)|}{|z - t_k|^2} \end{aligned}$$

$$\leqslant |B(z)-i|^2 \sum_{k=1}^{n} \frac{|u_k|}{|z-t_k|^2} \max_{1\leqslant k\leqslant n} |r(t_k)|. \quad (10)$$

Now, using Lemma 1 and noting that $s_1, s_2, \ldots, s_n$ are the zeros of $B(z)+i$, we get

$$r'(z)(B(z)+i) - B'(z)r(z) = (B(z)+i)^2 \sum_{k=1}^{n} \frac{v_k r(s_k)}{|z-s_k|^2}, \quad (11)$$

where

$$\frac{1}{v_k} = B'(s_k) = -i \sum_{j=1}^{n} \frac{2\,\mathrm{Re}(a_j)}{|s_k-a_j|^2}, \quad 1 \leqslant k \leqslant n.$$

(For reference, see [11]) Hence,

$$|r'(z)B(z) - r(z)B'(z) + ir'(z)| \leqslant |B(z)+i|^2 \sum_{k=1}^{n} \frac{|v_k|}{|z-s_k|^2} \max_{1\leqslant k\leqslant n} |r(s_k)|. \quad (12)$$

Now, taking $r(z) \equiv 1$ in (5) and using (11), we get:

$$|B'(z)| = |B(z)-i|^2 \Big| \sum_{k=1}^{n} \frac{|u_k|}{|z-t_k|^2} \Big|. \quad (13)$$

$$|B'(z)| = |B(z)+i|^2 \Big| \sum_{k=1}^{n} \frac{|v_k|}{|z-s_k|^2} \Big|. \quad (14)$$

Using (13) and (14) in (10) and (12), respectively, we get:

$$|r'(z)B(z) - r(z)B'(z) - ir'(z)| \leqslant |B'(z)| \max_{1\leqslant k\leqslant n} |r(t_k)|. \quad (15)$$

$$|r'(z)B(z) - r(z)B'(z) + ir'(z)| \leqslant |B'(z)| \max_{1\leqslant k\leqslant n} |r(s_k)|. \quad (16)$$

Squaring and adding (15) and (16), we get:

$$|r'(z)B(z) - r(z)B'(z) - ir'(z)|^2 + |r'(z)B(z) - r(z)B'(z) + ir'(z)|^2$$
$$\leqslant |B'(z)|^2 \Big\{ \big(\max_{1\leqslant k\leqslant n} |r(t_k)| \big)^2 + \big(\max_{1\leqslant k\leqslant n} |r(s_k)| \big)^2 \Big\}. \quad (17)$$

Now, using the identity

$$|A+B|^2 + |A-B|^2 = 2|A|^2 + 2|B|^2$$

in (17) with $A = r'(z)B(z) - r(z)B'(z)$ and $B = ir'(z)$, we get:

$$2|r'(z)B(z) - r(z)B'(z)|^2 + 2|r'(z)|^2 \\ \leqslant |B'(z)|^2\Big\{\Big(\max_{1\leqslant k\leqslant n}|r(t_k)|\Big)^2 + \Big(\max_{1\leqslant k\leqslant n}|r(s_k)|\Big)^2\Big\}. \quad (18)$$

Also, since $r^*(z) := B(z)\overline{r(-\overline{z})}$, we have

$$(r^*(z))' = B'(z)\overline{r(-\overline{z})} - B(z)\overline{r'(-\overline{z})}.$$

This gives, for $z \in \mathbb{I}$,

$$\begin{aligned} |(r^*(z))'| &= |B'(z)\overline{r(z)} - B(z)\overline{r'(z)}| = \frac{1}{|B(z)|}|B'(z)\overline{r(z)} - B(z)\overline{r'(z)}| \\ &= \left|\frac{B'(z)}{B(z)}\overline{r(z)} - \overline{r'(z)}\right| \\ &= \left|\overline{\frac{B'(z)}{B(z)}r(z) - r'(z)}\right| \qquad \text{since } \frac{B'(z)}{B(z)} \text{ is a real number by (7)} \\ &= \left|\frac{B'(z)}{B(z)}r(z) - r'(z)\right| = |B'(z)r(z) - B(z)r'(z)| \end{aligned}$$

Hence, from (18), we get:

$$2|r'(z)|^2 + 2|(r^*(z))'|^2 \leqslant |B'(z)|^2\Big\{\Big(\max_{1\leqslant k\leqslant n}|r(t_k)|\Big)^2 + \Big(\max_{1\leqslant k\leqslant n}|r(s_k)|\Big)^2\Big\}.$$

Equivalently, for $z \in \mathbb{I}$:

$$|r'(z)|^2 + |(r^*(z))'|^2 \leqslant \frac{1}{2}|B'(z)|^2\Big\{\Big(\max_{1\leqslant k\leqslant n}|r(t_k)|\Big)^2 + \Big(\max_{1\leqslant k\leqslant n}|r(s_k)|\Big)^2\Big\}.$$

Now, if we take $r(z) = \lambda B(z)$ with $|\lambda| = 1$, then

$$r^*(z) = B(z)\overline{\lambda B(\overline{-z})} = \overline{\lambda},$$

which gives

$$(r^*(z))' = 0,$$

and, since

$$B(t_k) = i \ \text{ and } \ B(s_k) = -i \ \text{ for all } \ k = 1, 2, \dots, n,$$

we have:

$$|r(t_k)| = |r(s_k)| = 1 \text{ for all } k = 1, 2, \ldots, n,$$

and we get the equality in (9). This completes the proof of Theorem 4. $\square$

Remark. *The conclusion of Theorem 4 is not only an improvement over the Corollary 1 of [11], but is also sharp, unlike the result in [11].*

Using Theorem 2.5, we get an estimate of the modulus of the polar derivative $D_\alpha(p(z))$ on the imaginary axis, when α lies in the right half-plane, and to that effect we have the following corollary:

Corollary 1. *Suppose $p(z)$ is a polynomial of degree n and $\alpha \in \mathbb{I}^+$. Then, for $z \in \mathbb{I}$:*

$$|D_\alpha(p(z))|^2 + |D_\alpha(p^*(z))|^2 \leqslant 2n \operatorname{Re}(\alpha) \frac{|z-\alpha|^{n-1}}{|z_0-\alpha|^n} |p(z_0)|^2,$$

where $z_0 \in \{t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n\}$ with

$$|r(z_0)| = \max\{|r(t_1)|, |r(t_2)|, \ldots, |r(t_n)|, |r(s_1)|, |r(s_2)|, \ldots, |r(s_n)|\},$$

$r(z) = \dfrac{p(z)}{(z-\alpha)^n}$. *The inequality is sharp and equality holds for $p(z) = \lambda(z+\overline{\alpha})^n$ with $|\lambda| = 1$.*

Proof. If we assume that $a_1 = \cdots = a_n = \alpha$, that is, $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$, then (9) becomes

$$|r'(z)|^2 + |(r^*(z))'|^2 \leqslant |B'(z)||r(z_0)|^2. \tag{19}$$

Now,

$$r'(z) = \left(\frac{p(z)}{(z-\alpha)^n}\right)' = \frac{(z-\alpha)^n p'(z) - np(z)(z-\alpha)^{n-1}}{(z-\alpha)^{2n}} = \frac{-D_\alpha(p(z))}{(z-\alpha)^{n+1}}.$$

Similarly, $(r^*(z))' = \dfrac{-D_\alpha(p^*(z))}{(z-\alpha)^{n+1}}$. Also, from (7) for $z \in \mathbb{I}$, $|B'(z)| = \dfrac{2n\operatorname{Re}(\alpha)}{|z-\alpha|^2}$. Using all these values in (19), we get

$$\left|\frac{D_\alpha(p(z))}{(z-\alpha)^{n+1}}\right|^2 + \left|\frac{D_\alpha(p^*(z))}{(z-\alpha)^{n+1}}\right|^2 \leqslant \frac{2n\operatorname{Re}(\alpha)}{|z-\alpha|^2}\left|\frac{p(z_0)}{(z_0-\alpha)^n}\right|^2$$

$$\leqslant \frac{2n\,\mathrm{Re}(\alpha - z)}{|z-\alpha|^2}\left|\frac{p(z_0)}{(z_0-\alpha)^n}\right|^2, \quad \text{for} \quad z \in \mathbb{I}.$$

This implies

$$|D_\alpha(p(z))|^2 + |D_\alpha(p^*(z))|^2 \leqslant 2n\,\mathrm{Re}(\alpha - z)\frac{|z-\alpha|^{n-1}}{|z_0-\alpha|^n}|p(z_0)|^2 \quad \text{for} \quad z \in \mathbb{I}.$$

The case for equality follows from the case for equality of Theorem 4. □

Remark. *The proofs of both Theorem 4 and Corollary 1 improve the proofs of Corollary 1 and Theorem 7 of [11] and lead to better and sharp inequalities.*

We next consider the case when the zeros of $r(z)$ are restricted to the closed right half-plane and prove the following:

Theorem 5. *Let $t_1, t_2, \ldots, t_n$ be the n simple zeros of $B(z) - i$ and $s_1, s_2, \ldots, s_n$ be the n simple zeros of $B(z) + i$. If $r \in \mathcal{R}_n$ has all zeros in $\mathbb{I} \cup \mathbb{I}^+$, then, for $z \in \mathbb{I}$:*

$$|r'(z)| \leqslant \frac{1}{2}|B'(z)|\left\{\left(\max_{1\leqslant k\leqslant n}|r(t_k)|\right)^2 + \left(\max_{1\leqslant k\leqslant n}|r(s_k)|\right)^2\right\}^{\frac{1}{2}}. \tag{20}$$

The result is sharp and equality holds for $r(z) = B(z) \pm i$.

Proof. By the assumption, all zeros of $r(z)$ lie in $\mathbb{I} \cup \mathbb{I}^+$; therefore, by Lemma 2,

$$\mathrm{Re}\left(\frac{r'(z)}{r(z)}\right) \leqslant \frac{1}{2}|B'(z)|$$

for $z \in \mathbb{I}$, which are not zeros of $r(z)$. Since, by virtue of (7), $|B'(z)| > 0$ for $z \in \mathbb{I}$, we get:

$$\mathrm{Re}\left(\frac{r'(z)}{r(z)|B'(z)|}\right) \leqslant \frac{1}{2}.$$

This gives, for $z \in \mathbb{I}$:

$$\left|\frac{r'(z)}{r(z)|B'(z)|}\right| \leqslant \left|\frac{r'(z)}{r(z)|B'(z)|} - 1\right|.$$

Equivalently,

$$|r'(z)| \leqslant |r'(z) - |B'(z)|r(z)|.$$

Again, by (7), $\dfrac{B'(z)}{B(z)} > 0$ for $z \in \mathbb{I}$. By Lemma 3 $|B(z)| = 1$ for $z \in \mathbb{I}$, and we have:

$$\frac{B'(z)}{B(z)} = \left|\frac{B'(z)}{B(z)}\right| = |B'(z)| \quad \text{for} \quad z \in \mathbb{I}$$

$$|B(z)r'(z)| \leqslant |B(z)r'(z) - B'(z)r(z)| \quad \text{for} \quad z \in \mathbb{I}.$$

That is,

$$|r'(z)| \leqslant |B(z)r'(z) - B'(z)r(z)| \quad \text{for} \quad z \in \mathbb{I}.$$

But

$$|B(z)r'(z) - B'(z)r(z)| = |(r^*(z))'| \quad \text{for} \quad z \in \mathbb{I},$$

which gives

$$|r'(z)| \leqslant |(r^*(z))'| \quad \text{for} \quad z \in \mathbb{I}.$$

Therefore, from (9) we get:

$$|r'(z)| \leqslant \frac{1}{2}|B'(z)|\left\{\Big(\max_{1\leqslant k\leqslant n}|r(t_k)|\Big)^2 + \Big(\max_{1\leqslant k\leqslant n}|r(s_k)|\Big)^2\right\}^{\frac{1}{2}}.$$

To prove the case for equality, take $r(z) = B(z) + i$, so that the left-hand side of inequality (20) reduces to $|B'(z)|$. Also, t_k are zeros of $B(z) + i$ and s_k are zeros of $B(z) - i$, and, hence, $r(t_k) = B(t_k) + i = 0$ and $r(s_k) = B(s_k) + i = B(s_k) - i + 2i = 2i$. Hence, using these values, the right-hand side of (20) also reduces to $|B'(z)|$. This completes the proof of Theorem 5. □

Using Theorem 5 and following the same steps as in the proof of corollary 1, we can prove a bound estimate for the polar derivative of a polynomial of $p(z)$ having all its zeros in the closed right half-plane. More precisely, the following corollary follows from Theorem 5:

Corollary 2. *Suppose $p(z)$ is a polynomial of degree n, having all its zeros in the closed right half-plane $\mathbb{I} \cup \mathbb{I}^+$. Then, for $z \in \mathbb{I}$, $\alpha \in \mathbb{I}^+$,*

$$|D_\alpha p(z)| \leqslant \sqrt{2}n\frac{|z-\alpha|^n}{|z_0-\alpha|^n}|p(z_0)|,$$

where $z_0 \in \{t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n\}$ with

$$|r(z_0)| = \max\{|r(t_1)|, |r(t_2)|, \ldots, |r(t_n)|, |r(s_1)|, |r(s_2)|, \ldots, |r(s_n)|\},$$

$$r(z) = \frac{p(z)}{(z-\alpha)^n}.$$

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[a] University of Kashmir, Kupwara Campus-190006, India
[b] Department of Mathematics, Central University of Kashmir, Ganderbal-191201, India

M. Y. Mir[a]
E-mail: myousf@cukashmir.ac.in

G. M. Sofi[b]
E-mail: gmsofi@cukashmir.ac.in

S. L. Wali[b]
E-mail: shahlw@yahoo.co.in

W. M. Shah[b]
E-mail: wali@cukashmir.ac.in