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CONVERGENCE ANALYSIS OF CQ METHODS FOR SOLVING SPLIT FEASIBILITY PROBLEMS INVOLVING GAN OPERATORS

Abstract. Optimization algorithms are fundamental to modern machine learning, data analysis, and inverse problems. Among them, CQ-type methods rely on the convergence properties of the Landweber transform. Motivated by recent generalizations of this transform, we study its behavior when applied to Generalized Averaged Nonexpansive (GAN) operators: a class extending averaged nonexpansive mappings with improved local convergence. We show that the Landweber transform preserves both the GAN structure and its regularity properties, allowing us to establish weak and strong convergence of CQ-type algorithms in this broader setting. To demonstrate the applicability of our results, we examine the matrix completion problem: a key task in recommender systems. Numerical experiments support our theory, showing global linear convergence and a local rate of order $o(k^{-1/2})$, confirming the effectiveness of the CQ algorithm for large-scale matrix completion.

Key words: *nonexpansive map, split feasibility, matrix completion*

2020 Mathematical Subject Classification: *47H09, 47H10*

1. Introduction.

Split convex feasibility problems (SCFPs) arise in numerous applications, such as signal and image processing, medical imaging, and machine learning. Given real Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, a nonzero bounded linear operator $B\colon \mathcal{K}_1 \to \mathcal{K}_2$, and nonempty closed convex sets $C \subseteq \mathcal{K}_1$ and $Q \subseteq \mathcal{K}_2$, the split feasibility problem consists of finding

$$p \in C \quad \text{such that} \quad Bp \in Q. \tag{1}$$

Among the many algorithms designed to solve (1), the CQ method of Byrne [6] is particularly notable for its simplicity and broad applicability. Its interpretation through fixed-point iterations has motivated significant

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research, typically relying on contractive or averaged nonexpansive operators to guarantee convergence [15], [16].

However, classical averaged operators may not adequately capture the behavior of certain nonlinear mappings encountered in modern optimization tasks. To address this, Lin and Xu [17] introduced generalized averaged nonexpansive (GAN) operators, a broader class characterized by a positive exponent λ. GAN operators preserve many desirable properties of averaged operators while enabling improved local convergence behavior, making them suitable for a wide range of problems in large-scale optimization and data science.

This paper studies CQ-type methods in the setting where the underlying operators are GAN operators, and examines the effect of the Landweber transform on their structural and convergence properties. We show that the Landweber transform applied to a GAN operator remains GAN and preserves essential regularity properties, such as weak, bounded, and linear regularity, extending existing results previously known only for (firmly/quasi) nonexpansive mappings [22], [7], [9]. By combining the regularity of G_1 with that of $\mathcal{L}\{G_2\}$, we derive corresponding weak, norm, and linear convergence results for CQ-type algorithms, especially under the assumption that B is invertible, thereby avoiding arguments that depend on inner-product structures. Finally, we illustrate the practical relevance of our analysis through an application to the matrix completion problem: an important task in recommender systems and data recovery.

The rest of the paper is organized as follows. Section 2 recalls essential preliminaries from the fixed-point theory, including Fejér monotonicity and regularity concepts, and reviews the Landweber transform and GAN operators. Section 3 establishes that the Landweber transform preserves weak, strong, and linear regularity of GAN operators. Section 4 proves that the GAN property itself is maintained under the Landweber transform. Section 5 applies these results to the matrix completion problem and presents numerical experiments demonstrating the efficiency of the proposed framework.

2. Preliminaries.

This section provides the theoretical foundation for our convergence analysis by recalling key notions from the fixed-point theory in Hilbert spaces. We review Fejér monotonicity and various forms of set and operator regularity, which are central to establishing weak and strong convergence of iterative sequences. We then examine the Landweber transform and its preservation of properties, such as nonexpansiveness and strong quasi-nonexpansiveness. Finally, we discuss Generalized Averaged Nonexpansive (GAN) operators, emphasizing their stability under convex combinations and compositions and their role in ensuring convergence rates of fixed-point iterations.

2.1. Fejér monotone sequences and regular operators.

In this section, we present the key concepts required for the convergence analysis of iterative algorithms in Hilbert spaces. We first recall the notion of *Fejér monotonicity*, a fundamental tool for proving convergence toward a target set without assuming strict contractiveness. We then review several forms of *regularity* for convex sets and operator sequences, following the framework of [10]. These notions provide the essential foundation for studying the stability and efficiency of fixed-point and feasibility algorithms considered in this work.

Definition 1. *Let $\mathcal{H}$ be a real Hilbert space and $F \subseteq \mathcal{H}$ be a nonempty, closed, and convex set, and let $\{z_k\}_{k=0}^{\infty}$ be a sequence in $\mathcal{H}$. We say that $\{z_k\}_{k=0}^{\infty}$ is Fejér monotone with respect to F if for all $z \in F$ and every $k = 0, 1, 2, \ldots$*

$$\|z_{k+1} - z\| \leqslant \|z_k - z\| . \tag{2}$$

We will utilize the following result in the proof of our main theorem:

Theorem 1. *Let the sequence $\{z_k\}_{k=0}^{\infty} \subseteq \mathcal{H}$ be Fejér monotone with respect to F. Then,*

- *(i) $\{z_k\}_{k=0}^{\infty}$ converges weakly to some point $z \in F$ if and only if all its weak cluster points lie in F.*
- *(ii) $\{z_k\}_{k=0}^{\infty}$ converges strongly to some point $z \in F$ if and only if $d(z_k, F) \to 0$.*
- *(iii) If there is some constant $C \in (0,1)$ such that $d(z_{k+1}, F) \leqslant C d(z_k, F)$ holds for every $k \in \mathbb{N}$, then $\|z_k - z\| \leqslant 2d(z_0, F) C^k$ for some $z \in F$.*

The following definition can be found, for example, in [1, Definition 5.1], and [2, Definition 5.7].

Definition 2. *(Regular Sets) Let $S \subseteq K, D_i \subseteq K, i \in I := \{1, \ldots, m\}$, be closed and convex with $D := \bigcap_{i\in I} D_i \neq \varnothing$, and let $\mathcal{D} := \{D_i \mid i \in I\}$. We say that the family $\mathcal{D}$ is*

(i) Regular over S if for any sequence $\{z_k\}_{k=0}^{\infty} \subseteq S$ we have

$$\lim_{k\to\infty} \max_{i\in I} d(z_k, D_i) = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} d(z_k, D) = 0.$$

(ii) Linearly regular over S if there is $\tau_S > 0$, such that for every $z \in S$ we have

$$d(z, D) \leqslant \tau_S \max_{i \in I} d\left(z, D_i\right).$$

Constant τ_S is called a modulus of the linear regularity of $\mathcal{D}$ over S.

If any of the aforementioned regularity conditions apply to every subset $S \subseteq K$, we omit the phrase "over S". If the condition is only valid for bounded subsets $S \subseteq K$, we precede it with the adverb "boundedly".

Example. Let $D_i \subseteq \mathcal{K}, i \in I := \{1, \ldots, m\}$, be closed and convex with $D := \bigcap_{i \in I} D_i \neq \varnothing$, and let $\mathcal{D} := \{D_i \mid i \in I\}$.

1) If $\dim \mathcal{K} < \infty$, then $\mathcal{D}$ is boundedly regular;
2) If all $D_i, i \in I$, are half-spaces, then $\mathcal{D}$ is linearly regular;
3) If $D_1 \cap \operatorname{int}\left(\bigcap_{i=2}^m D_i\right) \neq \varnothing$, then $\mathcal{D}$ is boundedly linearly regular;
4) If $\dim \mathcal{K} < \infty, D_i$ is a half-space, $i = 1, \ldots, p$, and $\bigcap_{i=1}^p D_i \cap \bigcap_{i=p+1}^m \operatorname{ri} D_i \neq \varnothing$, then $\mathcal{D}$ is boundedly linearly regular.

The following definition was introduced in Definition [11, 3.1 & 4.1].

Definition 3. *(Regular Operators) Let $\{G_k\}_{k=0}^{\infty}$ be a sequence of operators $G_k \colon K \to K$ with $F := \bigcap_{k=0}^{\infty}$ Fix $G_k \neq \varnothing$ and let $S \subseteq K$ be nonempty. We say that $\{G_k\}_{k=0}^{\infty}$ is*

(i) Weakly regular over S if for any sequence $\{z_k\}_{k=0}^{\infty} \subseteq S$ and for any point $z_\infty \in K$ we have

$$\left.\begin{array}{l} z_{n_k} \rightharpoonup z_\infty \\ G_k z_k - z_k \to 0 \end{array}\right\} \implies z_\infty \in F.$$

(ii) Regular over S if for any sequence $\{z_k\}_{k=0}^{\infty} \subseteq S$ we have

$$\lim_{k \to \infty} \|G_k z_k - z_k\| = 0 \implies \lim_{k \to \infty} d\left(z_k, F\right) = 0.$$

(iii) Linearly regular over S if there is $\varepsilon_S > 0$, such that for all $z \in S$ we have

$$\|G_k z - z\| \geqslant \varepsilon_S d(z, F).$$

The constant ε_S is called a modulus of linear regularity for G over S.

Theorem 2. *Let $\{G_1^k\}_{k=0}^{\infty}, \ldots, \{G_m^k\}_{k=0}^{\infty}$ be given sequences of σ_i^k-SQNE operators $G_i^k \colon H \to H$ with $Fix G_i^k = F_i$, $i = 1, 2, \ldots, m$, (in particular,* *each sequence can be a constant consisting only of one operator G_i). Define the product operator L^k by*

$$L^k := \prod_{i=1}^{m} G_i^k. \tag{3}$$

Assume that $\sigma := \inf_{i,k} \sigma_i^k > 0$ and $F := \bigcap_{i=1}^m F_i \neq \emptyset$ (L^k is (σ/m)-SQNE and $F = FixL^k$ by Theorem 2.6). Let $B = B(z, r)$ for some $z \in F$ and $r > 0$. Then the following statements hold:

- *(i) If for each $i = 1, \dots, m$, the sequence $\{G_i^k\}_{k=0}^{\infty}$ is weakly regular over B, then $\{L^k\}_{k=0}^{\infty}$ is also weakly regular over B.*
- *(ii) If for each $i = 1, \dots, m$, the sequence $\{G_i^k\}_{k=0}^{\infty}$ is regular over B and the family of sets $\{F_1, \dots, F_m\}$ is regular over B, then $\{L^k\}_{k=0}^{\infty}$ is also regular over B.*
- *(iii) If for each $i = 1, \dots, m$, the sequence $\{G_i^k\}_{k=0}^{\infty}$ is linearly regular over B with modulus ε_i and the family of sets $\{F_1, \dots, F_m\}$ is linearly regular over B with modulus $\kappa > 0$, then $\{L^k\}_{k=0}^{\infty}$ is also linearly regular over B with modulus*

$$\varepsilon_L = \frac{\sigma \varepsilon^2}{2m\kappa^2}, \tag{4}$$

where $\varepsilon := \min_i \varepsilon_i$.

2.2. Landweber and GAN operators. In this subsection, we examine Landweber operators, which naturally arise in inverse and feasibility problems involving compositions of linear and nonlinear mappings. We review key properties preserved under the Landweber transform, including nonexpansiveness, firm nonexpansiveness, and strong quasi-nonexpansiveness. We also discuss GAN (Generalized Averaged Nonexpansive) operators, emphasizing their stability under compositions and convex combinations and their relevance to convergence analysis. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Hilbert spaces, let $B \colon \mathcal{K}_1 \to \mathcal{K}_2$ be a nonzero bounded linear operator, and let $G \colon \mathcal{K}_2 \to \mathcal{K}_2$ be arbitrary.

Definition 4. *The operator $\mathcal{L}\{G\} \colon \mathcal{K}_1 \to \mathcal{K}_1$ defined by*

$$\mathcal{L}\{G\}x := x + \frac{1}{\|B\|^2} B^*(G(Bx) - Bx), \quad x \in \mathcal{K}_1,$$

is called the Landweber operator (corresponding to G). We call the operation $G \mapsto \mathcal{L}\{G\}$ the Landweber transform.

Note that in the literature, the Landweber operator is typically defined for $G = P_Q$, where $Q \subseteq \mathcal{K}_2$ is closed and convex [6].

To introduce our next lemma, we refer to [4], which defines an operator $G\colon \mathcal{K} \to \mathcal{K}$ as β-averaged (β-AV), for $\beta \in (0,1)$, if G is the β-relaxation of some nonexpansive operator T, that is, $G = (1-\beta)\operatorname{Id} + \beta T$. More information on averaged mappings can be found in the early paper by Reich [20]. Bruck and Reich [5] introduced the class of strongly nonexpansive operators. Next, we give the definition of a ρ-strongly quasi-nonexpansive mapping.

Definition 5. *Let $G\colon \mathcal{K} \to \mathcal{K}$ be a mapping defined on a Hilbert space $\mathcal{K}$. G is said to be a ρ-strongly quasi-nonexpansive (ρ-SQNE) map, where $\rho \geqslant 0$, if for all $x \in \mathcal{K}$ and all $w \in Fix\, G$, the following condition holds:*

$$\|Gx - w\|^2 \leqslant \|x - w\|^2 - \rho\|Gx - x\|^2.$$

Before defining the GAN operator [17], we will define a firmly nonexpansive operator.

Definition 6. *Let $\mathcal{K}$ be a Hilbert space and let $G\colon \mathcal{K} \to \mathcal{K}$ be a mapping. The mapping G is said to be* firmly nonexpansive *if*

$$\|Gx - Gy\|^2 \leqslant \|x - y\|^2 - \|(I-G)x - (I-G)y\|^2$$

for all $x, y \in \mathcal{K}$, where I denotes the identity mapping on $\mathcal{K}$.

More information on firmly nonexpansive operators can be found, for example, in Section 11 of the book by Goebel and Reich [13]. Now, we will define the GAN operator introduced in the paper [17].

Definition 7. *An operator $G\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be generalized averaged nonexpansive if there exist $\lambda,\mu \in R+$, such that*

$$\|Gx - Gy\|^\lambda + \mu\|(I-G)x - (I-G)y\|^\lambda \leqslant \|x - y\|^\lambda \; for\ all\ x, y \in \mathbb{R}^n.$$

Now, we present a lemma describing the properties preserved by Landweber operators, taken from [8].

Lemma 1. *Let $\mathcal{L}\{G\}$ be the Landweber operator corresponding to $G\colon \mathcal{K}_2 \to \mathcal{K}_2$, $\beta \in (0,1)$ and $\rho \geqslant 0$.*

(i) If G is (firmly) nonexpansive, then so is $\mathcal{L}\{G\}$.

(ii) If G is $\beta - \mathrm{AV}$, then $\mathcal{L}\{G\}$ is also $\beta - \mathrm{AV}$.

(iii) If G is ρ-SQNE and $\operatorname{im} B \cap FixG \neq \emptyset$, then $\mathcal{L}\{G\}$ is also ρ-SQNE.

Observe that the equality $Fix\mathcal{L}\{G\} = B^{-1}(Fix\, G)$ yields the following equivalence:

$$B^*(G(Bx) - Bx) = 0 \iff Bx \in Fix\, G. \tag{5}$$

Cegielski [10] presented the following properties and definitions of strongly quasi-nonexpansive mappings:

Theorem 3. *Let $G_2 \colon \mathcal{K} \to \mathcal{K}$ be quasi-nonexpansive, $G_1 \colon \mathcal{K} \to \mathcal{K}$ be strictly quasi-nonexpansive, and $FixG_1 \cap FixG_2$. Then $FixG_1G_2 = FixG_2G_1 = FixG_1 \cap FixG_2$. Furthermore, G_1G_2 is quasi-nonexpansive and G_2G_1 is strictly quasi-nonexpansive.*

The following results from [17] state that GAN operators are closed under convex combination and composition:

Proposition 1. *Let $\gamma \in [1, +\infty)$ and $\mu_1, \mu_2 \in \mathbb{R}_+$, $\alpha \in (0, 1)$. If $\mathcal{G}_1 \colon \mathbb{R}^n \to \mathbb{R}^n$ is μ_1-GAN with exponent γ and $\mathcal{G}_2 \colon \mathbb{R}^n \to \mathbb{R}^n$ is μ_2-GAN with exponent γ, then the following statements hold:*

- *(i) $\mathcal{G}_1 \circ \mathcal{G}_2$ is μ-GAN with exponent γ, where $\mu := 2^{1-\gamma} \min\{\mu_1, \mu_2\}$.*
- *(ii) $(1-\alpha)\mathcal{G}_1 + \alpha\mathcal{G}_2$ is μ'-GAN with exponent γ, where $\mu' := \min\{\mu_1, \mu_2\}$.*

Let $Fix(G)$ denote the set of all fixed points of the operator G and $\Gamma = \{G \colon \mathbb{R}^n \to \mathbb{R}^n \mid Fix(G) \neq \varnothing\}$. Throughout this paper, we will assume that $G \in \Gamma$.

Theorem 4. *If $G \in \Gamma$ is a GAN operator with exponent $\gamma \in \mathbb{R}_+$, then for any initial vector $z_0 \in \mathbb{R}^n$, the Picard sequence $\{z_k\}$ of operator G converges to some $z^* \in \mathrm{Fix}(G)$, and*

$$\|z_{k+1} - z_k\| = o(k^{-\frac{1}{\gamma}}). \tag{6}$$

3. Preservation of regularity. In this section, we examine the regularity properties of GAN operators and their behavior under transformation. We show that every μ-GAN operator is strictly quasi-nonexpansive, thereby linking GAN operators to a broader class of fixed-point-preserving mappings. We then analyze the Landweber transform and prove that, under suitable conditions, weak, strong, and linear regularity are preserved when a GAN operator is mapped to $\mathcal{L}\{G\}$. These results ensure that the convergence properties of G extend to its Landweber counterpart, supporting the development of robust iterative algorithms.

Theorem 5. *Every μ-GAN operator is a strictly quasi-nonexpansive operator.*

Proof. From the definition of μ-GAN operator, we have

$$\|Gz - Gy\|^\lambda + \mu\|(I - G)z - (I - G)y\|^\lambda \leqslant \|z - y\|^\lambda, \text{ for all } z, y \in \mathbb{R}^n.$$

Let $z \notin Fix(G)$ and $y \in Fix(G)$;

$$\|Gz - y\|^\lambda + \mu\|(I - G)z\|^\lambda \leqslant \|z - y\|^\lambda.$$

Now, as $z \notin Fix(G)$, therefore we have $\|z - Gz\| \neq 0$, hence we have

$$\|Gz - y\|^\lambda < -\mu\|(I - G)z\|^\lambda + \|z - y\|^\lambda < \|z - y\|^\lambda.$$

Thus G is strictly quasi-nonexpansive. □

Remark. *Using Theorem 1, Theorem 3, and the above Theorem 5, we have: if G_1 and G_2 are μ_1 and μ_2-GAN with exponent λ, then $\mathrm{G}_1 \circ \mathrm{G}_2$ is also μ-GAN with exponent λ, and if $FixG_1 \cap$ Fix $G_2 \neq \emptyset$, then we have $FixG_1 \circ G_2 = FixG_2 \circ G_1 =$ Fix $G_1 \cap$ Fix G_2.*

Theorem 6. *Let $B \colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible matrix and let $G \colon \mathbb{R}^n \to \mathbb{R}^n$ be a μ-GAN operator with exponent λ; and also let $S \subseteq \mathbb{R}^n$ $\quad S \neq \emptyset$. Assume that $imB \cap$ Fix $G \neq \emptyset$. Let $\mathcal{L}\{G\} \colon \mathbb{R}^n \to \mathbb{R}^n$ be the Landweber operator. Then the following holds:*

- *(i) If G is weakly regular over S, then $\mathcal{L}\{G\}$ is weakly regular over $B^{-1}(S)$.*
- *(ii) If G is regular over S, then $\mathcal{L}\{G\}$ is regular over $B^{-1}(S)$.*
- *(iii) If G is linearly regular with modulus $\delta > 0$ over S, then $\mathcal{L}\{G\}$ is linearly regular over $B^{-1}(S)$ with modulus*

$$\delta^* = \frac{\|B^{-1}\| \, \|B^*\| \|B\|}{\delta},$$

that is, for any $z \in B^{-1}(S)$, we have

$$\|\mathcal{L}\{G\}z - z\| \geqslant \frac{\|B^{-1}\| \, \|B^*\| \|B\|}{\delta} d(z, \ Fix\ \mathcal{L}\{G\}).$$

Proof. To show that $\mathcal{L}\{G\}$ is a weakly regular operator over the space S, we need to establish the following: If $z_{n_k} \rightharpoonup z$ and

$\mathcal{L}\{G\}z_k - z_k \to 0$, then z must belong to the fixed point set of $\mathcal{L}\{G\}$, denoted as $Fix\mathcal{L}\{G\} = B^{-1}Fix(G)$. In other words, we need to show that $Bz \in \text{Fix } G$. This can be proven by leveraging the weak regularity of G over S. First, to show that $Bz_{n_k} \rightharpoonup Bz$, given that $z_{n_k} \rightharpoonup z$, we can write:

$$\langle Bz_{n_k} - Bz, z\rangle = \langle z_{n_k} - z, B^* z\rangle \to 0 \text{ as } k \to \infty.$$

Now we need to show that $G(Bz_k) - Bz_k \to 0$ given $\mathcal{L}\{G\}z_k - z_k \to 0$

$$\|\mathcal{L}\{G\}z_k - z_k\| = \frac{1}{\|B\|^2}\|B^*(G-I)Bz_k\|.$$

Thus, $\frac{1}{\|B\|}\|B^*(G-I)Bz_k\| \to 0 \implies \quad \|B^*(G-I)B_{z_k}\| \to 0$. Using this, we obtain that:

$$\begin{aligned} 0 \leqslant \|(G-I)Bz_k\| &= \|\left(B^*\right)^{-1}\left(B^*\right)(G-I)Bz_k\| \\ &\leqslant \|\left(B^*\right)^{-1}\|\|B^*(G-I)Bz_k\| \to 0. \end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\|(G-I)Bz_k\| = 0.$$

(ii) To show that G is regular over S, we need to demonstrate that:

$$\lim_{k\to\infty}\|Gz_k - z_k\| = 0 \implies \lim_{k\to\infty} d\left(z_k, FixG\right) = 0. \tag{7}$$

The objective is to show the regularity of $\mathcal{L}\{G\}$ over $B^{-1}(S)$. For $z_k \in B^{-1}(S)$, we have $Bz_k \in S$. If, for $k \to \infty$,

$$\|\mathcal{L}\{G\}z_k - z_k\| = \frac{1}{\|B\|^2}\|B^*(G-I)Bz_k\| \to 0 \implies B^*(G-I)Bz_k \to 0.$$

Using the regularity of G and the fact that $(B^*)^{-1}$ is a bounded linear operator, we have, for $k \to \infty$:

$$(G-I)Bz_k \to 0 \implies d(Bz_k, FixG) \to 0. \tag{8}$$

Note that $d(Bz_k, FixG) = \|Bz_k - P_{Fix(G)}Bz_k\|$, where $P_{Fix(G)}$ is the projection on the set $Fix(G)$. Using (8) and the invertibility of B, we can conclude that

$$\|z_k - B^{-1}P_{Fix(G)}Bz_k\| \to 0.$$

Note that $P_{Fix(G)}Bz_k \in Fix(G)$ implies $B^{-1}P_{Fix(G)}Bz_k \in B^{-1}Fix(G)$. Now,

$$\begin{aligned} d(z_k, Fix\mathcal{L}\{G\}) &= \|z_k - P_{Fix(\mathcal{L}\{G\})}\| \\ &= \|z_k - P_{B^{-1}Fix(G)}\| \\ &\leqslant \|z_k - B^{-1}P_{Fix(G)}Bz_k\| \to 0. \end{aligned}$$

Thus, we can conclude that

$$d(z_k, Fix\mathcal{L}\{G\}) \to 0.$$

Hence, $\mathcal{L}\{G\}$ is regular over $B^{-1}S$.

(iii) Given that G is linearly regular over S, i.e., there exists some $\delta > 0$, such that $\delta d(z, FixG) \leqslant \|Gz - z\|$, we need to show that $\mathcal{L}\{G\}$ is linearly regular over $B^{-1}(S)$, i.e., there exists $\delta^* > 0$, such that

$$\delta^* d\left(z, Fix\mathcal{L}\{G\}\right) \leqslant \|\mathcal{L}\{G\}z - z\|.$$

As, $P_{FixG}(Bz) \in FixG \implies B^{-1}\left(P_{Fix(G)}Bz\right) \in B^{-1}\left(Fix(G)\right) = Fix(\mathcal{L}\{G\})$. Then,

$$\begin{aligned} d(z, Fix\mathcal{L}\{G\}) &= \left\|z - P_{Fix(\mathcal{L}\{G\})}z\right\| \\ &\leqslant \left\|z - B^{-1}P_{Fix(G)}Bz\right\| \\ &\leqslant \left\|B^{-1}\right\|, d(Bz, Fix(G)) \\ &\leqslant \frac{\|B^{-1}\|}{\delta} \left\|G\left(Bz\right) - Bz\right\|. \end{aligned}$$

Therefore, we obtain that:

$$\delta^* d(z, Fix\mathcal{L}\{G\}) \leqslant \|\mathcal{L}\{G\}z - z\|.$$

Where,

$$\delta^* = \frac{\|B^{-1}\| \, \|B^*\| \|B\|}{\delta}.$$

□

4. Preservation of the GAN property. In this section, we examine whether the GAN property is preserved under certain transformations. Specifically, we focus on the Landweber transform and aim to establish that if an operator G is a GAN operator, then its associated Landweber operator also inherits the GAN structure.

To support this goal, we begin by recalling some auxiliary results from [17] that connect GAN operators with contractive and quasi-nonexpansive mappings. These foundational results will help us build towards the main conclusion that the Landweber operator of a GAN operator remains within the class of GAN operators. Before going to the main results of this section, we first recall some results from [17].

Proposition 2. *If an operator* $G\colon \mathbb{R}^n \to \mathbb{R}^n$ *is* ρ*-contractive for some* $\rho \in (0,1)$*, then it is* $\hat{\rho} - GAN$ *with exponent* λ*, where* $\lambda \in \mathbb{R}_+$ *is an arbitrary fixed number and* $\hat{\rho} = \frac{1-p^\gamma}{(1+p)^\gamma}$.

Proposition 3. *If* $G \in \Lambda$ *is* GAN *with exponent* γ *for some* $\gamma \in (0,1)$*, then it is* $FP - p$*-contractive for some* $\rho \in (0,1)$.

One can easily verify the following Proposition:

Proposition 4. *Let* G *be a* $\mu - GAN$ *with exponent* λ *where* $\mu \geqslant 1$*. Then* $I - G$ *is* $\frac{1}{\mu} - GAN$ *with exponent* λ*.*

Now we will provide results that will help us to establish that the Landweber operator of a GAN operator is also a GAN operator.

Corollary. *Given* G *is a* $\mu - GAN$ *operator with exponent* λ *and* $B\colon \mathbb{R}^n \to \mathbb{R}^n$ *a bounded linear operator with* $\|B\| \leqslant 1$*, we have that* $B^*(I-G)B$ *is also a* GAN *operator.*

Proof. Given G is GAN, then $I - G$ is also GAN from Proposition 4. Since $\|B\| \leqslant 1$ and $\|Bx - By\| \leqslant \|B\|\|x-y\|$, we see that B is a ρ-contractive map and hence it is a GAN operator using Proposition 3. Similarly, B^* is also a GAN operator. Therefore, from Proposition 1, we have $B^*(I-G)B$ is a GAN operator. □

Corollary. $\frac{1}{\|B\|^2}B^*(I-G)B$ *is a* GAN *operator when* $\|B\| \leqslant 1$*.*

Proof. The previous corollary directly implies the result if $\|B\| = 1$. Assume that $\|B\| < 1$; then we have

$$\frac{1}{\|B\|^2}B^*(I-G)B = \frac{B^*}{\|B\|}(I-G)\frac{B}{\|B\|} = S^*(I-G)S,$$

where $S = \frac{B}{\|B\|}$. Therefore, from the previous corollary, we can say that $\frac{1}{\|B\|^2}B^*(I-G)B$ is a GAN operator with exponent λ (same as that of G). □

Remark. *Given that* $\frac{1}{\|B\|^2}B^*(I-G)B$ *is a* GAN *operator, we have* $I - \frac{1}{\|B\|^2}B^*(I-G)B$ *and* $I + \frac{1}{\|B\|^2}B^*(G-I)B$ *are also* GAN *operators. Hence, the Landweber operator of a* GAN *operator is a* GAN *operator.*

5. Split feasibility problem for GAN operators. In this section, we explore the SFP in the context of *generalized averaged nonexpansive* (GAN) operators. Specifically, we aim to find a point $z \in C := \mathrm{Fix}(G_1)$, such that $Bz \in Q := \mathrm{Fix}(G_2)$, where G_1 and G_2 are GAN operators and B is a bounded linear operator. We propose a new type of CQ-like iterative method tailored for GAN operators and establish weak, strong, and linear convergence under suitable regularity assumptions.

Theorem 7. *Let G_1 and G_2 be operators where $G_1\colon \mathbb{R}^n \to \mathbb{R}^n$ and $G_2\colon \mathbb{R}^n \to \mathbb{R}^n$ are $\mu_{G_1}-$ and $\mu_{G_2}-$ GAN operators, respectively. Let the sequence $\{z_k\}_{k=1}^{\infty}$ be defined by the method*

$$z_0 \in \mathbb{R}^n \quad z_{k+1} = G_1\Big(z_k + \frac{1}{\|B\|^2} B^*(G_2 - I)Bz_k\Big).$$

Assume that $\|B\| \leqslant 1$ and the set $F = Fix(G_1) \cap B^{-1}(Fix(G_2))$ is nonempty. Then the following statements hold:

- *(i) If G_1 and G_2 are both weakly regular, then the sequence $\{z_k\}_{k=1}^{\infty}$ converges weakly to some $z_\infty \in F$.*
- *(ii) If G_1 and G_2 are both boundedly regular and the two families of sets $\{\mathrm{im}\, B, FixG_2\}$ and $\{\ FixG_1, B^{-1}(FixG_2\ )\}$ are boundedly regular, then the convergence to z_∞ is in norm.*
- *(iii) If G_1 and G_2 are both boundedly linear regular and the two families of sets $\{\mathrm{im}\, B, FixG_2\}$ and $\{\ FixG_1, B^{-1}(FixG_2\ )\}$ are boundedly linear regular, then the convergence to z_∞ is at least linear, i.e.,*

$$d\left(z_{k+1}, F\right) \leqslant q d\left(z_k, F\right),$$
$$\|z_k - z_\infty\| \leqslant 2d\left(z_0, F\right) q^k.$$

Proof. We can write $z_{k+1} = G_1\mathcal{L}\{G_2\}z_k$, which is a Picard iteration on the operator $G_1\mathcal{L}\{G_2\}$. It follows from Theorem 1 that $G_1\mathcal{L}(G_2)$ is also GAN with exponent λ and $Fix(G_1\mathcal{L}(G_2)) = F$. Therefore,

$$\|G_1\mathcal{L}(G_2)x - G_1\mathcal{L}(G_2)y\|^\lambda \leqslant \|x-y\|^\lambda - \mu\left\|(I - G_1\mathcal{L}(G_2))x - (I - G_1\mathcal{L}(G_2))y\right\|^\lambda.$$

Setting $y = z \in F$ and $x = z_k$, we obtain that

$$\begin{aligned} \|G_1\mathcal{L}(G_2)z_k - z\|^\lambda &\leqslant \|z_k - z\|^\lambda - \mu\|z_{k+1} - z_k\|^\lambda, \\ \|z_{k+1} - z\|^\lambda &\leqslant \|z_k - z\|^\lambda - \mu\|z_{k+1} - z_k\|^\lambda. \end{aligned} \tag{9}$$

$\mathcal{L}(G_2)$ is nonexpansive since GAN implies nonexpansive,

$$\|z_{k+1} - z\| = \|G_1\mathcal{L}(G_2)z_k - z\| \leqslant \|z_k - z\|. \tag{10}$$

Thus $\lim\limits_{k\to\infty} \|z_k - z\| = c$ (say). From (9), we have

$$\mu \|z_{k+1} - z_k\|^\lambda \leqslant \|z_k - z\|^\lambda - \|z_{k+1} - z\|^\lambda.$$

Taking the limit as $k \to \infty$, we get

$$\lim_{k\to\infty} \|G_1\mathcal{L}(G_2)z_k - z_k\| = \lim_{k\to\infty} \|z_{k+1} - z_k\| = 0. \tag{11}$$

Consider

$$\begin{aligned} B_1 &= \{x \in \mathbb{R}^n \mid \|x - P_F z_0\| \leqslant r = d(z_0, F)\}, \\ B_2 &= \{y \in \mathbb{R}^n \mid \|y - BP_F z_0\| \leqslant r\|B\|\}. \end{aligned}$$

Using (10), we can conclude that

$$\|z_k - P_F z_0\| \leqslant \|z_0 - P_F z_0\| = r = d(z_0, F) \implies z_k \in B_1.$$

Now, for $\{Bz_k\}$, we have

$$\|Bz_k - BP_F z_0\| \leqslant \|B\| \|z_k - P_F z_0\| \leqslant \|B\|r.$$

Thus, we have $\{Bz_k\} \in B_2$ and we can easily see that $B_1 \subseteq B^{-1}(B_2)$.

(i): By assumption, both G_1 and G_2 are weakly regular over B_1 and B_2, respectively. From Theorem 6 (i) and the above discussion, the operator $\mathcal{L}(G_2)$ is weakly regular over B_1. Therefore, $G_1\mathcal{L}(G_2)$ is also weakly regular using Theorem 2. For any $z_{n_k} \rightharpoonup z$, and since $\|G_1\mathcal{L}(G_2)z_k - z_k\| \to 0$ (from (11)), using the fact that $G_1\mathcal{L}(G_2)$ is weakly regular, it follows that $z \in F = Fix(G_1) \cap B^{-1}(Fix(G_2))$. Since all weak cluster points of z_k lie in F, we conclude that $\{z_k\}$ converges to some point $z_\infty \in F$, as per Theorem 1 (i).

(ii): We have: $\mathcal{L}(G_2)$ is regular over B_1. By applying Theorem 6 and Theorem 2, we see that $G_1\mathcal{L}(G_2)$ is also regular over B_1, and then, by using the definition of regular operators, Equation (11) and Theorem 1(ii), we obtain that $\|z_k - z_\infty\| \to 0$.

(iii): By assumption, G_2 is linearly regular. Thus, by Theorem 6 and Theorem 2, we have $G_1\mathcal{L}(G_2)$ is linearly regular with modulus τ (say). Thus, we have

$$\begin{aligned} \|z_{k+1} - z_k\| = \|G_1\mathcal{L}(G_2)z_k - z_k\| &\geqslant \tau d(z_k, F), \\ \|z_{k+1} - z_k\|^\lambda &\geqslant \tau^k d(z_k, F), \\ -\mu \|z_{k+1} - z_k\|^\lambda &\leqslant -\tau^k \mu (d(z_k, F))^\lambda. \end{aligned} \tag{12}$$

Observe that $\|z_k - P_F z_k\| = d(z_k, F)$ and $d(z_{k+1}, F) \leqslant \|z_{k+1} - P_F z_k\|$. Thus, by setting $z = P_F z_k$ and using (12), we obtain in (9):

$$\begin{aligned} d(z_{k+1}, F)^\lambda &\leqslant d(z_k, F)^\lambda - \mu\tau^k d(z_k, F)^\lambda, \\ d(z_{k+1}, F) &\leqslant (1 - \mu\tau^k)^{\frac{1}{\lambda}} d(z_k, F), \\ d(z_{k+1}, F) &\leqslant q d(z_k, F). \end{aligned}$$

Where, $q = (1 - \mu\tau^k)^{\frac{1}{\lambda}}$. As $q < 1$, using Theorem 1 (iii) one can claim that $\|z_k - z_\infty\| \leqslant 2d(z_0, F)\, q^k$ for some $z_\infty \in F$. □

Remark. *Using Theorem 4, Theorem 1, and the hypothesis of Theorem 7, we can say that* $\{z_k\}$ *converges with a local convergence rate of* $o(k^{-\frac{1}{\gamma}})$.

6. Matrix Completion. The matrix completion problem aims at recovering a low-rank matrix from a limited set of observed entries and is central to many tasks in machine learning and data analysis. Applications include recommender systems, image and video inpainting, and sensor network localization [12], [21]. Its effectiveness relies on exploiting the inherent low-rank structure present in real data, often driven by latent factors. Significant progress has been made in developing reliable algorithms for this task, with the work of Candès and Recht providing foundational recovery guarantees under suitable conditions [12].

6.1 Low rank matrix recovery. In this subsection, we study low-rank matrix recovery within the split convex feasibility framework. The matrix completion problem is reformulated using the nuclear norm, and the CQ algorithm is applied by iteratively projecting onto two convex sets. The objective is to recover a low-rank matrix A from partial observations M on an index set Ω. Since the rank function is nonconvex, we employ the nuclear norm as its convex surrogate. Given a partially observed matrix M, the feasibility formulation is

$$\text{find } A \in \mathbb{R}^{n\times m} \text{ with } \|A\|_* \leqslant r \quad \text{and} \quad A_{ij} = M_{ij} \text{ for all } (i,j) \in \Omega, \tag{13}$$

where $\|A\|_*$ denotes the nuclear norm of A, i.e., the sum of its singular values. Define $\mathcal{C} = \{A \in \mathbb{R}^{n\times m} \colon \|A\|_* \leqslant r\}$ and $\mathcal{D} = \{A \in \mathbb{R}^{n\times m} \colon A_{ij} = M_{ij} \text{ for all } (i,j) \in \Omega\}$.

Remark. *Problem (13) is equivalent to finding* $A \in \mathcal{C} \cap \mathcal{D}$. *Taking* $B = I$, *this becomes a special case of the SCFP introduced in (1).*

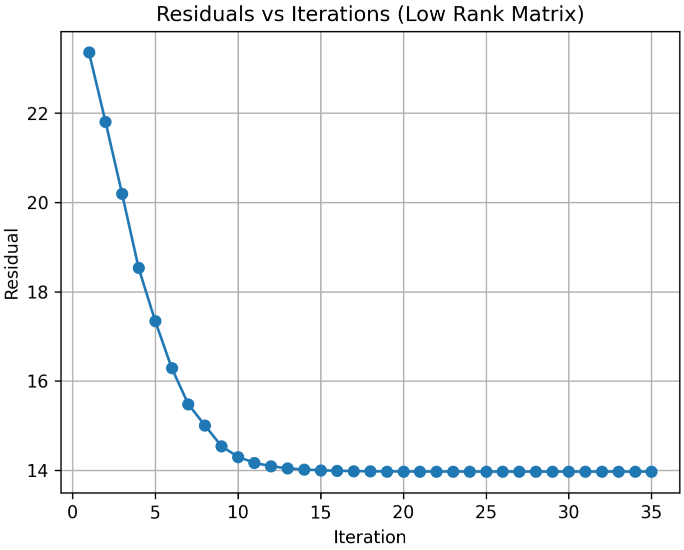

Figure 1: Residual versus iteration for the low-rank matrix case.

To compute the projections onto these sets, note that projecting a matrix A onto $\mathcal{C}$ requires its singular value decomposition $A = U\Sigma V^T$. Soft-thresholding of the singular values yields the projection $P_{\mathcal{C}}(A) = U \max(\Sigma - r, 0)V^T$, where $\max(\Sigma - r, 0)$ denotes replacing each singular value σ_i by $\max(\sigma_i - r, 0)$. The projection onto $\mathcal{D}$ enforces agreement with the observed entries:

$$P_{\mathcal{D}}(A) = \begin{cases} M_{ij}, & (i,j) \in \Omega, \\ A_{ij}, & \text{otherwise.} \end{cases}$$

Remark. *We assume that $\mathcal{C} \cap \mathcal{D} \neq \varnothing$; that is, there exists a matrix completing M with nuclear norm strictly less than r.*

Remark. *Let $G_1 = P_{\mathcal{C}}$ and $G_2 = P_{\mathcal{D}}$. Since projections are firmly nonexpansive and therefore GAN operators with exponent 1, it follows from Remark , Remark , and Example (iii) that Theorem 7 and Remark 4 apply. Consequently, the sequence $\{z_k\}$ generated by the CQ algorithm converges to a solution of (13) with linear global convergence and local convergence of order $o(k^{-1/2})$.*

To demonstrate the algorithm's performance, we conducted a matrix completion experiment in Python using a randomly generated 50×50

matrix of rank 5, with 50% of its entries observed. The CQ algorithm was run for at most 35 iterations with tolerance 10^{-6}, alternating between enforcing the observed entries and applying SVD-based soft-thresholding. The residual decay across iterations is shown in Figure 1, while the original and recovered matrices are displayed in Figure 2. These results confirm the algorithm's ability to recover the underlying low-rank structure.

Original matrix:

$$\begin{bmatrix} 1.40 & -2.83 & \dots & 7.06 & -3.32 \\ -0.76 & -0.74 & \dots & 1.48 & -0.40 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -0.24 & 0.31 & \dots & -0.57 & 0.70 \\ -2.14 & -3.10 & \dots & -2.50 & -0.25 \end{bmatrix}$$

Recovered matrix:

$$\begin{bmatrix} 1.08 & -2.63 & \dots & 6.24 & -3.06 \\ -0.65 & -0.68 & \dots & 1.22 & -0.37 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -0.14 & 0.30 & \dots & -0.48 & 0.62 \\ -1.67 & -2.03 & \dots & -2.16 & -0.03 \end{bmatrix}$$

Figure 2: Original and recovered matrices illustrating the reconstruction performance of the proposed method.

6.2 Positive semi-definite matrix recovery. In this subsection, we address the completion of a partially observed positive semi-definite (PSD) matrix by reformulating the problem as a split convex feasibility problem. The CQ algorithm is employed to enforce both the positive semi-definiteness constraint and the data-consistency constraint. We present the mathematical formulation, the projection steps, and a numerical experiment illustrating the performance of the method.

Given a partially observed matrix M with observed indices Ω, the PSD matrix completion problem can be formulated as finding an $A \in \mathcal{C} \cap \mathcal{D}$. If we take $B = I$, it becomes a special case of the SCFP defined in (1). Here, $\mathcal{C} = \{A \in \mathbb{R}^{n\times n} \mid x^T A x \geqslant 0 \text{ for all } x \in \mathbb{R}^n\}$ is the cone of positive semi-definite matrices, and $\mathcal{D} = \{A \in \mathbb{R}^{n\times n} \mid A_{ij} = M_{ij} \text{ for all } (i,j) \in \Omega\}$ enforces agreement with the observed entries. The projection of a matrix A onto $\mathcal{C}$ is given by the following theorem:

Theorem 8. *[14, Th. 2.1] Let $A \in \mathbb{R}^{n\times n}$. Define $Y = \frac{A+A^T}{2}$ and let $Y = UP$ be a polar decomposition (see [14], Th. 1.1). Then*

$$P_{\mathcal{C}}(A) = \frac{Y+P}{2}.$$

The projection onto the set $\mathcal{D}$ is the same as defined earlier:

$$(P_{\mathcal{D}}(A))_{ij} = \begin{cases} M_{ij}, & (i,j) \in \Omega, \\ A_{ij}, & \text{otherwise.} \end{cases}$$

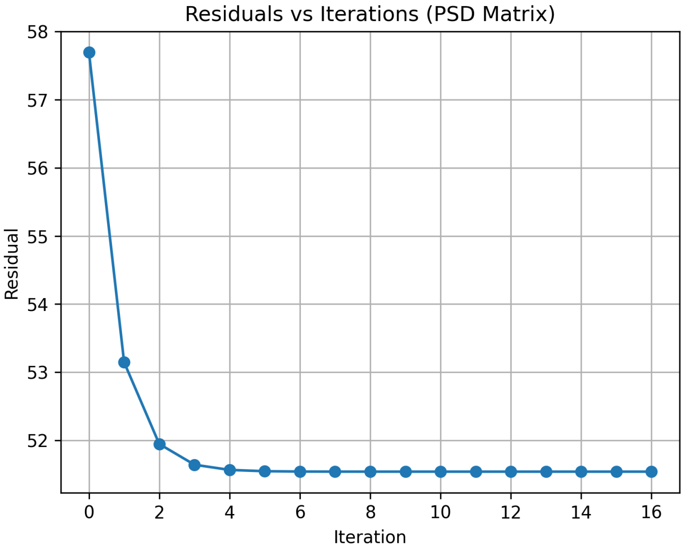

Figure 3: Residual versus iteration for the positive semi-definite matrix case.

Original matrix:

$$\begin{bmatrix} 1.40 & -2.83 & \dots & 7.06 & -3.32 \\ -0.76 & -0.74 & \dots & 1.48 & -0.40 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -0.24 & 0.31 & \dots & -0.57 & 0.70 \\ -2.14 & -3.10 & \dots & -2.50 & -0.25 \end{bmatrix}$$

(a)

Recovered matrix:

$$\begin{bmatrix} 1.08 & -2.63 & \dots & 6.24 & -3.06 \\ -0.65 & -0.68 & \dots & 1.22 & -0.37 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -0.14 & 0.30 & \dots & -0.48 & 0.62 \\ -1.67 & -2.03 & \dots & -2.16 & -0.03 \end{bmatrix}$$

(b)

Figure 4: Original and recovered matrices.

We performed a matrix completion experiment in Python to recover the missing entries of a PSD matrix. A 50 × 50 rank-5 PSD matrix was generated, and 50% of its entries were revealed. The CQ algorithm was executed for up to 35 iterations with tolerance 10^{-5}, alternating between enforcing data consistency and projecting onto the PSD cone via polar decomposition. Figure 4 shows the original and recovered matrices, while the residual decay across iterations is displayed in Figure 3. The experiment demonstrates the algorithm's ability to reconstruct the underlying PSD structure from sparse observations.

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