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FIXED POINT THEOREMS ON PERTURBED METRIC SPACE WITH AN APPLICATION

Abstract. In this paper, following the definition of the perturbed metric space, some fixed point theorems are established for F-perturbed mappings in complete perturbed metric spaces. The result is justified by a counterexample. Finally, an application of this theorem for existence of a solution for the second-order boundary value problem is given.

Key words: *perturbed metric space, F-perturbed mapping, fixed point, boundary value problem*

2020 Mathematical Subject Classification: *46T99, 47H10, 54H25*

1. Introduction. In modern mathematics, metric spaces and normed linear spaces are two widely used concepts in functional analysis. The notion of metric was developed by Frechet, and later Hausdorff presented it axiomatically. Following this, several authors explored different approaches to the concept of a metric, seeking to develop functions that are more general than the standard metric function. In 1993, Gahler [6] introduced the concept of 2-metric, followed by n-metric, and established many fundamental results in functional analysis. Since then, various generalized metrics have been defined by different authors, including D-metric [5], S-metric [15], b-metric [2], G-metric [13], F-metric [9]. In these generalized metrics, authors often modify the triangle inequality found in the definition of a basic metric space. Additionally, several authors have made significant contributions to fixed point results in generalized metric spaces (for references please see [3], [8], [7], [1], [4], [14], [11]).

Measuring the distance between two points is always subject to errors. For instance, imperfections in the instrument calibration can affect measurements. While these errors might be minor, their cumulative effect can be significant. Recognizing this problem, Jleli et al. [10] introduced the concept of a perturbed metric space and established fixed-point theorem within that framework. A review of the literature on perturbed metric spaces uncovers numerous oppertunities for developing fixed point results for various types of contraction mappings within these spaces.

Motivated by these insights, we introduce the concept of F-perturbed mapping in this paper and present a fixed-point theorem for perturbed metric spaces. We also explore application for the existence of a solution to second-order boundary value problem. The effectiveness of this approaches is illustrated through a numerical experiment.

2. Preliminaries. This section provides essential information and terminology that will be important for the rest of the manuscript.

To begin with, we recall the concept of a metric space.

Definition 1. [12] *A metric space is a pair (X, d), where X is a set and d is a metric on X (or a distance function on X), that is, a function defined on $X \times X$, such that for all $x, y, z \in X$ we have*

(M1) d *is a real-valued, finite, and non-negative;*

(M2) $d(x, y) = 0$ *if and only if* $x = y$*;*

(M3) $d(x, y) = d(y, x)$*;*

(M4) $d(x, y) \leqslant d(x, z) + d(z, y)$*.*

Note: Throughout this article, (X, d) is also called exact metric space.

The following definition introduces recently proposed concept of perturbed metric spaces in [10].

Definition 2. [10] *Let $D, P\colon X \times X \to [0, \infty)$ be two given mappings. D is a perturbed metric on X with respect to P if the function*

$$d = D - P\colon X \times X \to \mathbb{R},\ (x, y) \to D(x, y) - P(x, y)$$

is a metric on X. This means that for all $x, y, z \in X$, the following conditions hold:

(P1) $(D - P)(x, y) \geqslant 0$;

(P2) $(D - P)(x, y) = 0 \iff x = y$;

(P3) $(D - P)(x, y) = (D - P)(y, x)$;

(P4) $(D - P)(x, y) \leqslant (D - P)(x, z) + (D - P)(z, y)$.

We call P a perturbed mapping, $d = D - P$ the exact metric, and (X, D, P) a perturbed metric space.

Example 1. [10] Let $D\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^4, \quad \forall\ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2y^4, \quad \forall\ x, y \in \mathbb{R}.$$

In this case, the exact metric is the mapping d defined by

$$d(x, y) = |x - y|, \quad \forall\ x, y \in \mathbb{R}.$$

We present the following basic properties of a perturbed metric space, as established by Jleli and Samet [10].

Proposition 1. [10] *Let $D, P, Q\colon X \times X \to [0, \infty)$ be three given mappings and $\alpha > 0$.*

(i) If two triples (X, D, P) and (X, D, Q) denote two perturbed metric spaces, then the triple $(X, D, \frac{P+Q}{2})$ also forms a perturbed metric space.

(ii) The triple $(X, \alpha D, \alpha P)$ forms a perturbed metric space if the triple (X, D, P) consists of a perturbed metric space.

Jleli and Samat [10] introduced the following topological notations related to perturbed metric space (X, D, P).

Definition 3. [10] *Let (X, D, P) be a perturbed metric space. Let $\{x_n\}$ be a sequence in X, and $T\colon X \to X$ be a mapping.*

(i) We say that $\{x_n\}$ is a perturbed convergent sequence in (X, D, P) if $\{x_n\}$ is convergent in the metric space (X, d), where $d = D - P$ is the exact metric.

(ii) We say that $\{x_n\}$ is a perturbed Cauchy sequence in (X, D, P) if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

(iii) We say that (X, D, P) is a complete perturbed metric space if (X, d) is a complete metric space, or, equivalently, if every perturbed Cauchy sequence in (X, D, P) is a perturbed convergent sequence in (X, D, P).

(iv) *We say that T is a perturbed continuous mapping if T is continuous with respect to the exact metric d.*

We develop a fixed-point theorem in Section 3 using the following type of functions.

Definition 4. [16] *Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be a function satisfying*

$(F1)$ *F is strictly increasing, i.e., for all $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2 \implies F(t_1) < F(t_2)$,*

$(F2)$ *For each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers, $\lim_{n\to\infty} t_n = 0 \iff \lim_{n\to\infty} F(t_n) = -\infty$,*

$(F3)$ *There exists $k \in (0,1)$, such that $\lim_{t\to 0^+} t^k F(t) = 0$.*

Some examples of $F\colon R_+ \to R$ mappings that satisfy the conditions $(F1), (F2)$, and $(F3)$ are as follows.

Example 2. [16]
1) $F(x) = \ln x, \quad x \in (0,\infty)$.
2) $F(x) = \ln x + x, \quad x \in (0,\infty)$.
3) $F(x) = -\frac{1}{\sqrt{x}}, \quad x \in (0,\infty)$.
4) $F(x) = \ln(x^2 + x), \quad x \in (0,\infty)$.

We recall the F-contraction theorem from [16] in exact metric space (X,d) to be used in the main result.

Definition 5. [16] *A mapping $T\colon X \to X$ is said to be an F-contraction if there exists $\tau > 0$, such that*

$$d(Tx,Ty) > 0,\ \forall\, x,y \in X \implies \tau + F(d(Tx,Ty)) \leqslant F(d(x,y)),\ \forall\, x,y \in X.$$

Theorem 1. [16] *Let (X,d) be a complete exact metric space and let $T\colon X \to X$ be a F-contraction. T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

3. Main results. Inspired by the work of D. Wardowski [16] in exact metric spaces, we now extend these results in setting of a perturbed metric space. Now we begin this section by introducing the concept of F-perturbed mapping and a lemma which is important in proving the main results.

Lemma 1. *Let X be a nonempty set and $D(x,y)$ be a perturbed metric on X with respect to the perturbed mapping $P(x,y) \geqslant 0, \quad \forall\, x,y \in X$. Define*

$$D(x,y) = d(x,y) + P(x,y).$$

If $x = y$, then $D(x,y) = P(x,y) \geqslant 0$.

If $x \neq y$, then $d(x,y) > 0$. So that $D(x,y) > 0$. Hence, $D(x,y) \geqslant 0$, $\forall\, x,y \in X$.

Definition 6. *A mapping $T\colon X \to X$ is said to be an F-perturbed mapping if there exists $\tau > 0$, such that*

$$D(Tx,Ty) > 0 \implies \tau + F(D(Tx,Ty)) \leqslant F(D(x,y)), \quad \forall\, x,y \in X. \quad (1)$$

Theorem 2. *Let (X,D,P) be a complete perturbed metric space and $T\colon X \to X$ be a perturbed continuous F-perturbed mapping. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\} \subset X$, $x_{n+1} = Tx_n, n = 1,2,3,\ldots$, and denote $\gamma_n = D(x_{n+1},x_n), n = 1,2,3,\ldots$.

If there exists $n_0 \in \mathbb{N}$, such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is done.

Suppose $x_{n+1} \neq x_n, \quad \forall n \in \mathbb{N}$. Then $\gamma_n > 0, \forall n \in \mathbb{N}$ (using Lemma 1). Using (1), the following holds for every $n \in \mathbb{N}$:

$$\begin{aligned} & F(D(x_{n+1},x_n)) + \tau \leqslant F(D(x_n,x_{n-1})) \\ \implies & F(D(x_{n+1},x_n)) \leqslant F(D(x_n,x_{n-1})) - \tau \\ \implies & F(\gamma_n) \leqslant F(\gamma_{n-1}) - \tau. \end{aligned} \quad (2)$$

Again, from (1), we have

$$F(\gamma_{n-1}) + \tau \leqslant F(\gamma_{n-2}) \implies F(\gamma_{n-1}) \leqslant F(\gamma_{n-2}) - \tau.$$

From (2), we have

$$\begin{aligned} F(\gamma_n) \leqslant F(\gamma_{n-2}) - 2\tau \implies F(\gamma_n) \leqslant F(\gamma_0) - n\tau, \quad n \in \mathbb{N} \implies \\ \lim_{n\to\infty} F(\gamma_n) \leqslant \lim_{n\to\infty}[F(\gamma_0) - n\tau] \leqslant -\infty, \implies \lim_{n\to\infty} F(\gamma_n) = -\infty. \end{aligned} \quad (3)$$

By $(F2)$, we have

$$\lim_{n\to\infty} \gamma_n = 0. \quad (4)$$

By $(F3)$, there exists $k \in (0,1)$, such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{5}$$

From (3), we have

$$\begin{aligned} &F(\gamma_n) \leqslant F(\gamma_0) - n\tau \implies \gamma_n^k F(\gamma_n) \leqslant \gamma_n^k F(\gamma_0) - n\tau\gamma_n^k \\ &\implies \lim_{n\to\infty}[\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0)] \leqslant \lim_{n\to\infty}[-n\tau\gamma_n^k] \implies 0 \leqslant \lim_{n\to\infty}[-n\tau\gamma_n^k] \leqslant 0 \\ &\implies \lim_{n\to\infty} n\gamma_n^k = 0; \end{aligned}$$

i.e., for $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that

$$\begin{aligned} &n\gamma_n^k < \varepsilon, \quad \forall n \geqslant N \\ &\implies D(x_{n+1}, x_n) < (\frac{\varepsilon}{n})^{\frac{1}{k}}, \quad \forall n \geqslant N \\ &\implies d(x_{n+1}, x_n) + P(x_{n+1}, x_n) < (\frac{\varepsilon}{n})^{\frac{1}{k}}, \quad \forall n \geqslant N \\ &\implies d(x_{n+1}, x_n) < (\frac{\varepsilon}{n})^{\frac{1}{k}}, \quad \forall n \geqslant N. \end{aligned}$$

Now we will show that $\{x_n\}$ is a Cauchy sequence:

$$\begin{aligned} d(x_{n+p}, x_n) &\leqslant d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \ldots + d(x_{n+1}, x_n) \\ &< \varepsilon^{\frac{1}{k}} \left[\frac{1}{(n+p-1)^{\frac{1}{k}}} + \frac{1}{(n+p-2)^{\frac{1}{k}}} + \ldots + \frac{1}{n^{\frac{1}{k}}} \right] \\ &= \varepsilon^{\frac{1}{k}} \sum_{i=0}^{p-1} \frac{1}{(n+i)^{\frac{1}{k}}} \\ &\leqslant \varepsilon^{\frac{1}{k}} \cdot \frac{p}{n^{\frac{1}{k}}}, \quad \forall n \geqslant N. \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(x_{n+p}, x_n) = 0, \quad p = 1, 2, 3, \ldots$.

Hence, $\{x_n\}$ is a Cauchy sequence in the exact metric (X, d); that is, $\{x_n\}$ is a perturbed Cauchy sequence in the perturbed metric space (X, D, P). By the completeness of the perturbed metric space (X, D, P), we deduce that there exists $x^* \in X$, such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

We now show that x^* is a fixed point of T. Since T is a perturbed continuous mapping,

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = 0 \quad \text{i.e.,} \quad \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

From the triangle inequality of d, we have

$$\begin{aligned}
& d(x^*, Tx^*) \leqslant d(x^*, Tx_n) + d(Tx_n, Tx^*) = d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\
\Longrightarrow & \lim_{n\to\infty} d(x^*, Tx^*) \leqslant \lim_{n\to\infty} [d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\
\Longrightarrow & d(x^*, Tx^*) = 0 \\
\Longrightarrow & Tx^* = x^*.
\end{aligned}$$

We want to prove that x^* is a unique fixed point.
Suppose there exists $y^* \in X$, such that $Tx^* = x^* \neq y^* = Ty^*$.

$$\begin{aligned}
& \tau + F(D(Tx^*, Ty^*)) \leqslant F(D(x^*, y^*)) \\
\Longrightarrow & \tau + F(D(x^*, y^*)) \leqslant F(D(x^*, y^*)) \\
\Longrightarrow & \tau \leqslant 0, \quad \text{a contradiction.}
\end{aligned}$$

Hence, x^* is a unique fixed point. The proof is completed. □

We now present an example that supports the Theorem 2.

Example 3. Let $D\colon [0,1] \times [0,1] \to [0,\infty)$ be the mapping defined by

$$D(x,y) = |x-y| + (x-y)^4, \quad \forall\ x, y \in [0,1].$$

Then D is a perturbed metric on $[0,1]$ with respect to the perturbed mapping $P\colon [0,1] \times [0,1] \to [0,\infty)$ given by $P(x,y) = (x-y)^4$. In this case, the exact metric is the mapping $d\colon [0,1] \times [0,1] \to [0,\infty)$ defined by

$$d(x,y) = |x-y|, \quad \forall x, y \in [0,1].$$

For $x = 0, y = \frac{1}{2}, z = \frac{1}{3}$, we have

$$D\Big(0, \frac{1}{2}\Big) = 0.5625, D\Big(0, \frac{1}{3}\Big) + D\Big(\frac{1}{3}, \frac{1}{2}\Big) = 0.513,$$

which shows that

$$D\Big(0, \frac{1}{2}\Big) > D\Big(0, \frac{1}{3}\Big) + D\Big(\frac{1}{3}, \frac{1}{2}\Big).$$

Hence, D is not an exact metric on $[0,1]$.

Let us define a mapping $T\colon [0,1] \to [0,1]$ by $Tx = \frac{x}{2}, x \in [0,1]$, which is a continuous perturbed mapping. Take $F(x) = \log x,\ x \in (0,\infty)$, which satisfies the conditions $(F1),(F2),(F3)$ and $\tau = \log 2 > 0$. Then we have

$$D(Tx,Ty) = |Tx - Ty| + (Tx - Ty)^4 = \frac{|x-y|}{2} + \frac{(x-y)^4}{16}.$$

Clearly, we have

$$\begin{aligned}
\tau + F(D(Tx,Ty)) &= \log 2 + \log(D(Tx,Ty)) = \log(2D(Tx,Ty)) \\
&= \log\left[2\left\{\frac{|x-y|}{2} + \frac{(x-y)^4}{16}\right\}\right] \\
&\leqslant \log\left[|x-y| + (x-y)^4\right] \\
&= F(D(x,y)).
\end{aligned}$$

Hence, T is a F-perturbed mapping. So, by the Theorem 2, T has a unique fixed point in X which is $x = 0$.

4. Existence of a solution to the boundary value problem.

We give an application for the second-order boundary-value problem of a unique solution with a binary relation, which is applicable in the Theorem 2.

Let us consider a second-order boundary-value problem as follows:

$$-u''(t) = f(t,u(t)), \quad t \in (0,1) \quad u(0) = u(1) = 0, \tag{6}$$

where the mapping $f\colon (0,1)\times\mathbb{R} \to \mathbb{R}$ is continuous. This type of problem arises in Physics and Engineering, such as in heat conduction, elastic deformation, and electrostatics. One can easily show that the problem (6) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s,u(s)\ ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \begin{cases} s(1-t), & 0 < s \leqslant t \leqslant 1 \\ t(1-s), & 0 < t < s \leqslant 1. \end{cases}$$

Green's function satisfies the boundary conditions $u(0) = u(1) = 0$.

Let $X = C[0,1]$, the class of all real-valued continuous functions defined on $[0,1]$ and define a function

$$D(u_1(t), u_2(t)) = \sup_{t\in[0,1]} |u_1(t) - u_2(t)| + |u_1(0) - u_2(0)|, \quad \forall u_1, u_2 \in X. \tag{7}$$

D is a perturbed metric on $C[0,1]$ with respect to the perturbed mapping $P\colon C[0,1] \times C[0,1] \to [0,\infty)$ defined by the relation

$$P(u_1(t), u_2(t)) = |u_1(0) - u_2(0)|, \quad \forall\, u_1, u_2 \in C[0,1].$$

The exact metric is the function $d\colon C[0,1] \times C[0,1] \to [0,\infty)$ defined by the relation

$$d(u_1(t), u_2(t)) = \sup_{t\in[0,1]} |u_1(t) - u_2(t)|, \quad \forall\, u_1, u_2 \in C[0,1]. \tag{8}$$

It is known that the set $C[0,1]$ endowed with the metric d defined by the relation (8), that is $(C[0,1], d)$, is a complete metric space. Consequently, by (iii) of the Definition 3, it is found that $(C[0,1], D, P)$ is a complete perturbed metric space.

Theorem 3. *Consider the perturbed metric space (X, D, P) defined on (7). Suppose that the boundary value problem (6) satisfies the following condition:*

$$\left|f\big(s, u_1(s)\big) - f\big(s, u_2(s)\big)\right| \leqslant \exp(-\tau) \cdot |u_1(s) - u_2(s)|, \quad \tau > 0. \tag{9}$$

Then the boundary-value problem (6) has a unique solution.

Proof. We define a mapping $T\colon X \to X$ by

$$T\big(u(t)\big) = \int_0^1 G(t,s)\, f\big(s, u(s)\big)\, ds, \quad s \in [0,1].$$

For all $u_1(t), u_2(t) \in X$, we have

$$\begin{aligned}
&\tau + \ln\left(D(Tu_1, Tu_2)\right)\\
&= \tau + \ln\left[\sup_{t\in[0,1]} |Tu_1(t) - Tu_2(t)| + |Tu_1(0) - Tu_2(0)|\right]
\end{aligned}$$

$$= \tau + \ln\Big[\sup_{t\in[0,1]} \Big| \int_0^1 G(t,s) f\left(s, u_1(s)\right) - \int_0^1 G(t,s) f\left(s, u_2(s)\right) \Big|$$

$$+ \Big| \int_0^1 G(0,s) f\left(s, u_1(s)\right) - \int_0^1 G(0,s) f\left(s, u_2(s)\right) \Big| \Big]$$

$$\leqslant \tau + \ln\Big[\sup_{t\in[0,1]} \Big| \int_0^1 G(t,s) ds \Big| \cdot \Big| \int_0^1 \left(f(s, u_1(s)) - f\left(s, u_2(s)\right)\right) ds \Big|$$

$$+ \sup_{t\in[0,1]} \Big| \int_0^1 G(0,s) ds \Big| \cdot \Big| \int_0^1 \left(f(s, u_1(s)) - f(s, u_2(s))\right) ds \Big| \Big]$$

$$< \tau + \ln\Big[\frac{1}{2} \cdot \Big| \int_0^1 \left(f(s, u_1(s)) - f(s, u_2(s))\right) ds \Big|$$

$$+ \frac{1}{2} \cdot \Big| \int_0^1 \left(f(s, u_1(s)) - f(s, u_2(s))\right) ds \Big| \Big]$$

$$< \tau + \ln \Big[\Big| \int_0^1 \left(f(s, u_1(s)) - f(s, u_2(s))\right) ds \Big| \Big]$$

$$\leqslant \tau + \ln\Big[\sup_{s\in[0,1]} \Big| f(s, u_1(s)) - f(s, u_2(s)) \Big| \int_0^1 ds \Big]$$

$$\leqslant \tau + \ln\Big[\exp(-\tau) \cdot \sup_{s\in[0,1]} \Big| u_1(s) - u_2(s) \Big| \Big]$$

$$= \ln\Big[\sup_{t\in[0,1]} \Big| u_1(t) - u_2(t) \Big| + \Big| u_1(0) - u_2(0) \Big| \Big] = \ln\Big[D\big(u_1(t), u_2(t)\big) \Big]$$

$\implies \tau + \ln\big[\big(D(Tu_1, Tu_2)\big)\big] < \ln\big[D\big(u_1(t), u_2(t)\big)\big], \forall u_1(t), u_2(t) \in X$. Thus, the mapping T fulfills the conditions of the Theorem 2 and therefore T has a unique fixed point in X. Consequently, the boundary value problem (6) has a unique solution in X. □

5. Numerical Example. In this section, a numerical example is established to indicate the significance of the given results. Let X be a set of all continuous real-valued functions defined on $[0,1]$, i.e., $X = C[0,1]$, and define $D\colon X \times X \to [0,\infty)$ by

$$D\big(u_1(t), u_2(t)\big) = \sup_{t\in[0,1]} |u_1(t) - u_2(t)| + |u_1(0) - u_2(0)|, \quad \forall u_1(t), u_2(t) \in X.$$

Then D is a perturbed metric on X with respect to the perturbed mapping $P\colon X \times X \to [0,\infty)$ defined by the relation

$$P\big(u_1(t), u_2(t)\big) = |u_1(0) - u_2(0)|, \quad \forall u_1(t), u_2(t) \in X,$$

and the exact metric is the function $d\colon X \times X \to [0,\infty)$ defined by

$$d\big(u_1(t), u_2(t)\big) = \sup_{t\in[0,1]} |u_1(t) - u_2(t)|, \quad \forall u_1(t), u_2(t) \in X.$$

Clearly, (X, D, P) is complete perturbed metric space.

Let T be the operator defined by

$$Tu(t) = \int_0^1 G(t,s) f\big(s, u(s)\big) ds, \quad s \in [0,1], \tag{10}$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 < s \leqslant t, \\ t(1-s), & t \leqslant s < 1. \end{cases}$$

Let $f\big(s, u(s)\big) = \big(\frac{s+0.5}{2}\big) \cdot \sin u(s)$. Then (10) becomes

$$Tu(t) = \int_0^1 G(t,s) \Big(\frac{s+0.5}{2}\Big) \sin u(s) ds, \quad u(t) \in X. \tag{11}$$

Suppose the following condition holds:

$$\big|f\big(s, u_1(s)\big) - f\big(s, u_2(s)\big)\big| \leqslant \exp(-\tau) \cdot \big|u_1(s) - u_2(s)\big|, \quad \tau \in (0.287, 1.386).$$

For all $u_1(t), u_2(t) \in X$ and $\tau \in (0.287, 1.386)$, we have

$$\begin{aligned} &\tau + \ln\left(D(Tu_1(t), Tu_2(t)\right) \\ &= \tau + \ln\Big[\sup_{t\in[0,1]} \Big|Tu_1(t) - Tu_2(t)\Big| + \Big|Tu_1(0) - Tu_2(0)\Big|\Big] \\ &= \tau + \ln\Big[\sup_{t\in[0,1]} \Big|\int_0^1 G(t,s) f(s, u_1(s)) ds - \int_0^1 G(t,s) f(s, (u_2(t)) ds\Big| \end{aligned}$$

$$+\Big|\int_0^1 G(0,t)f(s,u_1(s))ds - \int_0^1 G(0,t)f(s,u_2(s))ds\Big|\Big]$$

$$\leqslant \tau + \ln\Big[\sup_{t\in[0,1]}\Big|\int_0^1 G(t,s)ds\Big| \cdot \Big|\int_0^1 \Big(\big(f(s,u_1(s)) - f\big(s,u_2(s)\big)\Big)ds\Big|$$

$$+ \sup_{t\in[0,1]}\Big|\int_0^1 G(0,s)ds\Big| \cdot \Big|\int_0^1 \Big(\big(f(s,u_1(s)) - f\big(s,u_2(s)\big)\Big)ds\Big|\Big]$$

$$< \tau + \ln\Big[\frac{1}{2} \cdot \Big|\int_0^1 \big(f(s,u_1(s)) - f(s,u_2(s))\big)ds\Big|$$

$$+ \frac{1}{2} \cdot \Big|\int_0^1 \big(f(s,u_1(s)) - f(s,u_2(s))\big)ds\Big|\Big]$$

$$< \tau + \ln\Big[\Big|\int_0^1 \big(f(s,u_1(s)) - f(s,u_2(s))\big)ds\Big|\Big]$$

$$= \tau + \ln\Big[\sup_{s\in[0,1]} |f(s,u_1(s)) - f(s,u_2(s))| \int_0^1 ds\Big]$$

$$\leqslant \tau + \ln\Big[\exp(-\tau) \cdot \sup_{s\in[0,1]}\Big|u_1(s) - u_2(s)\Big|\Big]$$

$$= \ln\Big[\sup_{s\in[0,1]}\Big|u_1(s) - u_2(s)\Big| + \Big|u_1(0) - u_2(0)\Big|\Big]$$

$$= \ln\Big[D(u_1(t), u_2(t))\Big].$$

As a result, the conclusion is that the condition of Theorem 2 are satisfied. Consequently, the Integral Equation (11) has a unique solution. It can be easily checked that $u(t) = t$ is the exact solution of Equation (11). Since $T(0) = 0$, the unique fixed point in X is precisely the zero function.

6. Another generalized fixed-point theorem. In this section, we prove another theorem on the fixed point of an operator that includes the Banach fixed-point theorem of a perturbed metric space [10, Theorem 3.1].

Theorem 4. *Let X be a complete perturbed metric space and let T be* *a mapping of X into itself. Suppose that for each positive integer n,*

$$D(T^n x, T^n y) \leqslant a_n \cdot D(x, y)$$

for all $x, y \in X$, where $a_n > 0$ is independent of x, y. If the series $\sum_{n=1}^{\infty} a_n$ is convergent and T is a perturbed continuous function, then T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary element in X and consider the sequence $\{x_n\}$ of iterations

$$x_n = T^n x_0, \quad n = 1, 2, 3, \ldots.$$

We note that $x_{n+1} = T^{n+1} x_0 = T^n(Tx_0) = T^n x_1$ and $x_{n+1} = T(T^n x_0) = Tx_n$. Now, we have

$$D(x_n, x_{n+1}) = D(T^n x_0, T^{n+1} x_0) = D(T^n x_0, T^n x_1) \leqslant a_n \cdot D(x_0, x_1).$$

Therefore,

$$D(x_n, x_{n+1}) \leqslant a_n \cdot D(x_0, x_1). \tag{12}$$

If $x_0 = x_1$, then a fixed point is obtained. Let, therefore, $x_0 \neq x_1$ and k be a positive integer with $k > D(x_0, x_1)$.
As the series $\sum_{n=1}^{\infty} a_n$ is convergent, $\lim\limits_{n\to\infty} a_n = 0$. From (12), we have

$$\begin{aligned}
& D(x_n, x_{n+1}) \leqslant a_n \cdot D(x_0, x_1) \\
\implies & d(x_n, x_{n+1}) + P(x_n, x_{n+1}) \leqslant a_n \cdot D(x_0, x_1) \\
\implies & \lim_{n\to\infty} d(x_n, x_{n+1}) \leqslant D(x_0, x_1) \cdot \lim_{n\to\infty} a_n \\
\implies & \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.
\end{aligned}$$

Now we will show that $\{x_n\}$ is a Cauchy sequence.

$$\begin{aligned}
& d(x_n, x_{n+p}) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\
\implies & \lim_{n\to\infty} d(x_n, x_{n+p}) \leqslant \lim_{n\to\infty} \big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + \\
& \qquad\qquad d(x_{n+p-1}, x_{n+p})\big] \\
\implies & \lim_{n\to\infty} d(x_n, x_{n+p}) = 0.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and the completeness of X implies existence of $x_0 \in X$, such that

$$\lim_{n\to\infty} x_n = x_0.$$

Since T is a perturbed continuous mapping, then

$$\lim_{n\to\infty} d(Tx_n, Tx_0) = 0 \;\; i.e., \; \lim_{n\to\infty} d(x_{n+1}, x_0) = 0.$$

For a positive integer n, we obtain

$$\begin{aligned} d(x_0, Tx_0) &\leqslant d(x_0, x_{n+1}) + d(x_{n+1}, Tx_0) \\ &= \lim_{n\to\infty} [d(x_0, x_{n+1}) + d(Tx_n, Tx_0)] \leqslant 0. \end{aligned}$$

Therefore, $d(Tx_0, x_0) \implies Tx_0 = x_0$.
We now prove the uniqueness. If y_0 is a fixed point of T, then for any positive integer n:

$$y_0 = T^n y_0 \text{ and } x_0 = T^n x_0.$$

So,

$$D(x_0, y_0) = D(T^n x_0, T^n y_0) \leqslant a_n \cdot D(x_0, y_0) \implies (a_n - 1) \cdot D(x_0, y_0) \geqslant 0.$$

If $x_0 = y_0$, then the uniqueness is done.

From Lemma 1, if $x_0 \neq y_0$, then $D(x_0, y_0) > 0$, so $(a_n - 1) \geqslant 0$ i.e., $a_n \geqslant 1 \; \forall n$.

Therefore, a_n cannot tend to zero and this contradiction shows that $x_0 = y_0$. Thus, T has a unique fixed point in X. □

We now deduce Banach's fixed-point theorem using perturbed metric (see [10]) from fixed-point Theorem 4.

Proof. Since T is a Banach fixed-point theorem, using perturbed mapping [10], we show that there exists $0 < \alpha < 1$, such that

$$D(Tx, Ty) \leqslant \alpha \cdot D(x, y), \quad \forall \; x, y \in X. \tag{13}$$

For $x, y \in X$, we get from (13):

$$\begin{aligned} &D(T^2x, T^2y) \leqslant \alpha \cdot D(Tx, Ty) \leqslant \alpha^2 \cdot D(x, y), \\ &D(T^3x, T^3y) \leqslant \alpha^3 \cdot D(x, y), \end{aligned}$$

and in general $D(T^n x, T^n y) \leqslant \alpha^n \cdot D(x, y)$.
Since the series $\sum_{n=1}^{\infty} \alpha^n$ is convergent, by (4), T has a unique fixed point in X. □

We now present an example, which supports Theorem 4.

Example 4. Let $D\colon [0,1]\times[0,1]\to[0,\infty)$ be the mapping defined by

$$D(x,y) = |x-y| + (x-y)^2, \quad \forall\, x,y \in X.$$

Then D is a perturbed metric on $[0,1]$ with respect to the perturbed mapping $P\colon [0,1]\times[0,1]\to[0,\infty)$ given by $P(x,y)=(x-y)^2$.
Let us define a self-mapping $T\colon [0,1]\to[0,1]$ by

$$Tx = \frac{x}{3}, \quad x\in[0,1].$$

For all $x,y\in[0,1]$, we have

$$D(T^n x, T^n y) = \left|\frac{x}{3^n} - \frac{y}{3^n}\right| + \left(\frac{x}{3^n} - \frac{y}{3^n}\right)^2 = \frac{1}{3^n}\cdot|x-y| + \frac{1}{3^{2n}}\,(x-y)^2\,.$$

We know that $\frac{1}{3^n}, \frac{1}{3^{2n}} < 1$.
Now,

$$D(T^n x, T^n y) = \frac{1}{3^n}\cdot|x-y| + \frac{1}{3^{2n}}\,(x-y)^2 < |x-y| + (x-y)^2\,.$$

Since $a_n > 0$ for all $n\in\mathbb{N}$,

$$D\left(T^n x, T^n y\right) < a_n\cdot\left(|x-y| + (x-y)^2\right) = a_n\cdot D\left(x,y\right), \quad \forall\, x,y\in[0,1].$$

T is also a perturbed continuous mapping. Therefore, all the conditions of Theorem 4 are satisfied. Hence T has a unique fixed point $x=0$.

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