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UNIFORMLY STARLIKE AND UNIFORMLY CONVEX MAPPINGS IN SEVERAL COMPLEX VARIABLES

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In this paper we introduce some subclasses of biholomorphic mappings on the unit ball of $\mathbb{C}^n$. These mappings called uniformly starlike and uniformly convex are introduced by geometric interpretation, like in the case of one variable.

§ 1. Introduction

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the norm $\|z\| = \langle z, z\rangle^{1/2}$. The symbol $'$ means the transpose of vectors and matrices. For $z = (z_1, \ldots, z_n)' \in \mathbb{C}^n$ let $\overline{z} = (\overline{z}_1, \cdots, \overline{z}_n)'$ and for $A = (a_{ij})_{\ 1\le i,j\le n}$ let $\bar{A}$ be the conjugate of matrix A. We denote by 0 the origin of $\mathbb{C}^n$ and by $L(\mathbb{C}^n, \mathbb{C}^n)$ we denote the space of all continuous and linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm. Further, let I denote the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$ and let B_r denote the open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$. The open unit ball is abbreviated by $B = B_1$. In the case of one variable the ball B_r is denoted by U_r and the unit disk U_1 by U.

Also, let $H(G)$ be the set of *holomorphic mappings* from a domain $G \subset \mathbb{C}^n$ into $\mathbb{C}^n$. We say that $f \in H(G)$ is *locally biholomorphic* on G if its Fréchet derivative

$$Df(z) = \left[\frac{\partial f_j}{\partial z_k}(z)\right]_{1\le j,k\le n}$$

as an element of $L(\mathbb{C}^n, \mathbb{C}^n)$ is nonsingular at each point $z \in G$. For a mapping $f \in H(G)$, let $D^2 f(z)(u^2) = D^2 f(z)(u, u)$, for $z \in G$ and

$u \in \mathbb{C}^n$, where $D^2f(z)$ denotes the second order Fréchet derivative of f at z. Also we say that $f \in H(G)$ is *biholomorphic* on G if the inverse f^{-1} exists and is holomorphic on a domain Ω such that $f^{-1}(\Omega) = G$.

The mapping $f \in H(B)$ is said to be *starlike* if $f(0) = 0$, f is biholomorhic on B and $f(B)$ is a starlike domain in $\mathbb{C}^n$ with respect to zero. Also, if $f \in H(B)$, we say that f is *convex* if f is biholomorphic on B and $f(B)$ is a convex domain in $\mathbb{C}^n$.

Matsuno [15] and Suffridge [20] proved that a locally biholomorphic mapping $f \in H(B)$ with $f(0) = 0$ is starlike if and only if

$$\text{Re } \langle [Df(z)]^{-1} f(z), z\rangle > 0, z \in B \setminus \{0\}. \tag{1.1}$$

On the other hand, Kikuchi [8], Gong, Wang and Yu [1] showed that a locally biholomorphic mapping $f \in H(B)$ with $f(0) = 0$ is convex if and only if

$$||v||^2 - \text{Re}\langle Df(z)]^{-1} D^2 f(z)(v, v), z\rangle > 0, \tag{1.2}$$

for all $z \in B \setminus \{0\}$ and $v \in \mathbb{C}^n \setminus \{0\}$ with $\text{Re}\langle z, v\rangle = 0$.

Very recently Goodman [2, 3] introduced and studied the notion of uniform starlikeness and uniform convexity on the unit disk U. Several authors have continued the study of these mappings and deduced important results in this direction (see for details [17, 18], [4, 5, 6], [11], [13, 14], etc.).

Definition 1. *An univalent function f is called uniformly starlike in U if f maps every circular arc γ contained in U with center ζ also in U onto the arc $f(\gamma)$ starlike with respect to $f(\zeta)$.*

In [2] is showed that if

$$f(z) = z + a_2 z^2 + \cdots, \quad z \in U,$$

then f is uniformly starlike in U if and only if

$$\text{Re} \left[\frac{(z - \zeta) f'(z)}{f(z) - f(\zeta)} \right] > 0, \quad (z, \zeta) \in U \times U. \tag{1.3}$$

The class of all functions uniformly starlike in U was denoted by UST.

Definition 2. *An univalent function f is called uniformly convex in U if f maps every circular arc γ, contained in U with center ζ also in U, onto the convex arc $f(\gamma)$.*

In [3] is showed that an univalent function f, normalized by $f(0) = f'(0) - 1 = 0$, is uniformly convex in U if and only if

$$\operatorname{Re}\left[1 + (z - \zeta)\frac{f''(z)}{f'(z)}\right] > 0, \quad (z, \zeta) \in U \times U. \tag{1.4}$$

The class of all univalent functions, uniformly convex in U was denoted by UCV.

As a matter of fact the assumption $\zeta \in U$ in the case of uniform convexity can be dropped. The analytic condition which characterizes such class of functions coincides with (1.4) but with the assumption $\zeta \in \mathbb{C}$ (for details we refer to [4, 5, 6]).

In the present paper we will extend the above definitions to the case of locally biholomorphic mappings on the unit ball of $\mathbb{C}^n$ and we will investigate some properties of such mappings.

§ 2. Main results

We begin this section with the following definitions which are natural extensions in $\mathbb{C}^n$ of Definitions 1.1 and 1.2.

DEFINITION 3. *Let $f : B \to \mathbb{C}^n$ be a biholomorphic mapping on B, normalized by $f(0) = 0$ and $Df(0) = I$. We say that f is uniformly starlike in the unit ball B if for every part Γ of the sphere $\partial B(\zeta, r) = \{z \in \mathbb{C}^n : \|z - \zeta\| = r\}$ contained in B, with center ζ also in B, the hypersurface $f(\Gamma)$ is starlike with respect to $f(\zeta)$.*

We remark that if we denote by $C_r(\zeta) = f(\Gamma)$ then $C_r(\zeta) \subseteq \{w \in \mathbb{C}^n : \psi(w) = 0\}$, where $\psi(w) = \varphi(f^{-1}(w)),\ w \in f(B)$ and $\varphi(z) = \|z - \zeta\|^2 - r^2, z \in B$. Also, the hypersurface $C_r(\zeta)$ is starlike with respect to $f(\zeta)$ if and only if

$$\operatorname{Re}\langle N_w, w - f(\zeta)\rangle > 0, \quad w \in C_r(\zeta), \tag{2.1}$$

where N_w denotes the outward normal vector to $C_r(\zeta)$ at w.

DEFINITION 4. *Let $f : B \to \mathbb{C}^n$ be a biholomorphic mapping on B, normalized by $f(0) = 0$ and $Df(0) = I$.We say that f is uniformly convex in the unit ball B if for every part Γ of the sphere $\partial B(\zeta, r) = \{z \in \mathbb{C}^n : \|z - \zeta\| = r\}$ contained in B, with center $\zeta \in B$, the hypersurface $f(\Gamma)$ is convex.*

Note that the hypersurface $C_r(\zeta) = f(\Gamma)$ is convex if and only if

$$S(u, u) > 0, \quad u \in T_w(C_r(\zeta)), \quad w \in C_r(\zeta), \tag{2.2}$$

where $S(u, u)$ and $T_w(C_r(\zeta))$ mean the second fundamental form of $C_r(\zeta)$ at w, and the real tangent space to C_r at w, respectively.

Taking into account the above definitions, we can prove the following results, which characterize analyticaly the notions of uniform starlikeness and uniform convexity in B.

THEOREM 1. *Let $f : B \to \mathbb{C}^n$ be a locally biholomorphic mapping on B, normalized by $f(0) = 0$ and $Df(0) = I$. Then f is uniformly starlike in B if and only if*

$$\mathrm{Re}\langle [Df(z)]^{-1}(f(z) - f(\zeta)), z - \zeta\rangle > 0, \tag{2.3}$$

for all $z, \zeta \in B, \ z \neq \zeta$.

PROOF. First, assume that f is uniformly starlike on B. Then f is starlike, hence biholomorphic on B. Next, let $\zeta \in B$ and $r > 0$. Also, let $\Gamma = \partial B(\zeta, r) \cap B$ and $C_r(\zeta) = f(\Gamma)$. Then $C_r(\zeta) \subseteq \{w \in f(B) : \psi(w) = 0\}$ is a starlike hypersurface on $\mathbb{C}^n$, with respect to $f(\zeta)$, where $\psi(w) = \varphi \circ f^{-1}(w)$, $w \in f(B)$ and $\varphi(z) = \|z - \zeta\|^2 - r^2, z \in B$. Therefore

$$\mathrm{Re}\langle N_w, w - f(\zeta)\rangle > 0,$$

for all $w \in C_r(\zeta)$.

Now, let $w \in C_r(\zeta)$, then the outward normal vector to $C_r(\zeta)$ at w is $N_w = \frac{\partial}{\partial \overline{w}}\psi(w)$ and a short computation yields that $N_w = ([Df(z)]^{-1})'(\overline{z}-\overline{\zeta})$, where $z = f^{-1}(w)$. Hence, it is obvious to see that the above condition is equivalent to the following

$$\mathrm{Re}\langle [Df(z)]^{-1}(f(z) - f(\zeta)), z - \zeta\rangle > 0,$$

that completes the proof of the first part of our result.

Conversely if the relation (2.3) holds for all z, $\zeta \in B$, with $z \neq \zeta$, then if we let $\zeta = 0$ into (2.3), then we deduce that

$$\mathrm{Re}\langle [Df(z)]^{-1} f(z), z\rangle > 0, z \neq 0,$$

that means f is starlike, hence biholomorphic on B. Further, the relation (2.3) implies that

$$\mathrm{Re}\langle N_w, w - f(\zeta)\rangle > 0,$$

for all $w \in C_r(\zeta)$ and $\zeta \in B$, so, taking into account Definition 2.1, we conclude that f is uniformly starlike on B, as desired conclusion. □

THEOREM 2. *Let $f : B \to \mathbb{C}^n$ be a locally biholomorphic mapping on B, normalized by $f(0) = 0$ and $Df(0) = I$. Then f is uniformly convex in B if and only if*

$$\|v\|^2 - \mathrm{Re}\langle [Df(z)]^{-1} D^2 f(z)(v, v), z - \zeta\rangle > 0, \tag{2.4}$$

for all $z, \zeta \in B,\ z \neq \zeta$ and $v \in \mathbb{C}^n \setminus \{0\}, \mathrm{Re}\ \langle z - \zeta, v\rangle = 0$.

PROOF. First, assume that f is uniformly convex on B, therefore f is biholomorphic on B and each sphere $\partial B(\zeta, r)$ contained in B is mapped by f onto a convex hypersurface in $\mathbb{C}^n$.

Now, let $\zeta \in B$ and $r > 0$. Also, let $\Gamma = \partial B(\zeta, r) \cap B$ and $C_r(\zeta) = f(\Gamma)$. Then it is well known that the second fundamental form $S(u, u)$ of $C_r(\zeta) = f(\partial B(\zeta, r))$ at $w \in C_r(\zeta)$ can be written as follows

$$S(u, u) = \frac{\mathrm{Re}\left[u' \frac{\partial^2 \psi}{\partial w^2}(w) u\right] + \overline{u}' \frac{\partial^2 \psi}{\partial \overline{w} \partial w}(w) u}{\left\| \frac{\partial \psi}{\partial \overline{w}}(w) \right\|} \tag{2.5}$$

for all $u \in T_w(C_r)$, where $\psi(w) = \varphi \circ f^{-1}(w)$ and $\varphi(z) = \|z - \zeta\|^2 - r^2, z \in B$.

After short calculations we deduce the following relations

$$\overline{u}' \frac{\partial^2 \psi}{\partial \overline{w} \partial w}(w) u = \left\| [Df(z)]^{-1} u \right\|^2, \tag{2.6}$$

and

$$u' \frac{\partial^2 \psi}{\partial w^2}(w) u = -(\overline{z} - \overline{\zeta})' [Df(z)]^{-1} D^2 f(z) \Big([Df(z)]^{-1} u, [Df(z)]^{-1} u \Big), \tag{2.7}$$

where $w = f(z)$.

Indeed, using the formulas of matrix derivatives described in [7] we have

$$\begin{aligned} \frac{\partial^2 \psi}{\partial w^2}(w) &= \frac{\partial}{\partial w} \left(\left(\frac{\partial \varphi}{\partial z}(z) \right)' [Df(z)]^{-1} \right) \\ &= \frac{\partial}{\partial z} \left(\left(\frac{\partial \varphi}{\partial z}(z) \right)' [Df(z)]^{-1} \right) \left([Df(z)]^{-1} \times I \right) \end{aligned}$$

$$\begin{aligned} &= \frac{\partial^2 \varphi}{\partial z^2}(z)\left([Df(z)]^{-1} \times [Df(z)]^{-1}\right) \\ &- \left(\frac{\partial \varphi}{\partial z}(z)\right)'[Df(z)]^{-1} D^2 f(z)\left([Df(z)]^{-1} \times [Df(z)]^{-1}\right), \end{aligned}$$

where the sign $\times$ designates the Kronecker product.

Hence

$$\begin{aligned} u' \frac{\partial^2 \psi}{\partial w^2}(w)u &= \left([Df(z)]^{-1}u\right)' \frac{\partial^2 \varphi}{\partial z^2}(z)[Df(z)]^{-1}u \\ &- \left(\frac{\partial \varphi}{\partial z}(z)\right)'[Df(z)]^{-1} D^2 f(z)\left([Df(z)]^{-1}u, [Df(z)]^{-1}u\right). \end{aligned}$$

Since

$$\frac{\partial \varphi}{\partial z}(z) = \overline{z} - \overline{\zeta} \text{ and } \frac{\partial^2 \varphi}{\partial z^2}(z) = 0 \text{ (the null matrix)},$$

then, in view of the above relations, we obtain

$$u' \frac{\partial^2 \psi}{\partial w^2}(w)u = -(\overline{z} - \overline{\zeta})'[Df(z)]^{-1} D^2 f(z)\left([Df(z)]^{-1}u, [Df(z)]^{-1}u\right).$$

In a similar manner we can show that

$$\overline{u}' \frac{\partial^2 \psi}{\partial \overline{w} \partial w}(w)u = \left([Df(z)]^{-1}u\right)' \frac{\partial^2 \varphi}{\partial \overline{z} \partial z}(z)[Df(z)]^{-1}u,$$

and since $\frac{\partial^2 \varphi}{\partial \overline{z} \partial z}(z) = I$, then

$$\overline{u}' \frac{\partial^2 \psi}{\partial \overline{w} \partial w}(w)u = \left\|[Df(z)]^{-1}u\right\|^2.$$

Now if we set $v = [Df(z)]^{-1}u$ and use the fact that $u \in T_w(C_r(\zeta))$ we deduce that

$$\begin{aligned} 0 &= \operatorname{Re}\left\langle \frac{\partial \varphi}{\partial \overline{w}}(w), w - f(\zeta) \right\rangle = \operatorname{Re}\left[u'\left([Df(z)]^{-1}\right)'(\overline{z} - \overline{\zeta})\right] \\ &= \operatorname{Re}\langle z - \zeta, v\rangle, \end{aligned}$$

where $f(z) = w$.

Hence, $\operatorname{Re}\langle z - \zeta, v\rangle = 0$ that means $v \in T_z(\partial B(\zeta, r))$. Finally, combining (2.5), (2.6), (2.7) and the relation (2.2), we conclude that if f is uniformly convex, then the condition (2.4) holds for all $z, \zeta \in B$ with $z \neq \zeta$ and $v \in \mathbb{C}^n \setminus \{0\}$, with $\mathrm{Re}\langle z-\zeta, v\rangle = 0$.

Conversely, if we assume that f satisfies the relation (2.4), then, using the condition (1.2), we conclude that f is convex, hence biholomorphic on the unit ball. Further, using similar kind of arguments as in the first step, we conclude that each hypersurface $C_r(\zeta) = f(\Gamma)$ is convex, where $\Gamma = \partial B(\zeta, r) \cap B$, for all $r > 0$ and $\zeta \in B$, that means f is uniformly convex on B.

This completes the proof. □

REMARK. *Notice that in the case of one variable, we obtain from (2.3) and (2.4) the usual notions of uniform starlikeness and uniform convexity of functions in the unit disk U. For this aim, it suffices to see that if $z, \zeta \in U$, $z \neq \zeta$ and $v \in \mathbb{C} \setminus \{0\}$ such that* $\mathrm{Re}[\overline{v}(z-\zeta)] = 0$, *then*

$$\overline{v}(z-\zeta) + v(\overline{z} - \overline{\zeta}) = 0,$$

hence $|v|^2(z-\zeta) = -v^2(\overline{z}-\overline{\zeta})$. *Therefore, the relation*

$$|v|^2 \left[1 + \mathrm{Re}\, \frac{(z-\zeta)f''(z)}{f'(z)}\right] > 0,$$

i.e.

$$\mathrm{Re}\left[1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right] > 0,$$

for $z, \zeta \in U$, $z \neq \zeta$.

Further on, let $US^*(B)$ and $UK(B)$ denote, respectively, the classes of uniformly starlike and uniformly convex mappings on B, normalized by $f(0) = 0$ and $Df(0) = I$.

We next give some examples of mappings from $US^*(B)$.

EXAMPLE 1. Let $f_1, ..., f_n$ be uniformly starlike functions on the unit disk U and $f(z) = (f_1(z_1), ..., f_n(z_n))', z = (z_1, ..., z_n)'$. Then f is uniformly starlike on B.

Indeed, it is clear that f is normalized and locally biholomorphic on B and also

$$\mathrm{Re}\left\langle [Df(z)]^{-1}(f(z)-f(\zeta)), z-\zeta\right\rangle = \sum_{j=1}^{n} \mathrm{Re}\left\{\frac{f_j(z_j)-f_j(\zeta_j)}{(z_j-\zeta_j)f_j'(z_j)}\right\} |z_j-\zeta_j| \tag{2.8}$$

for $z, \zeta \in B$, with $z \neq \zeta$. Since the functions f_j, $j \in \{1, ..., n\}$ satisfy condition (1.3) then the right hand side of (2.8) is positive and so the left hand side.

Therefore, using results due to Goodman [2], the following mappings belong to $US^*(B)$

$$\text{(i)} \quad f(z) = \left(\frac{z_1}{1 - a_1 z_1}, ..., \frac{z_n}{1 - a_n z_n} \right)', \quad z = (z_1, ..., z_n)' \in B,$$

where

$$a_j \in \mathbb{C}, |a_j| \leq \frac{1}{\sqrt{2}}, \quad \text{for} \quad j \in \{1, ..., n\}.$$

$$\text{(ii)} \quad f(z) = \left(z_1 + a_1 z_1^{l_1}, ..., z_n + a_n z_n^{l_n} \right)', \quad z = (z_1, ..., z_n)' \in B,$$

where

$$a_j \in \mathbb{C}, |a_j| \leq \frac{\sqrt{2}}{2 l_j}, \quad l_j \in \mathbf{N} \setminus \{1\}, \quad j \in \{1, ..., n\}.$$

$$\text{(iii)} \quad f(z) = \left(z_1 - a_1 z_1^2, ..., z_n - a_n z_n^2 \right)', \quad z = (z_1, ..., z_n)' \in B,$$

where

$$a_j \in \mathbb{C}, \quad |a_j| \leq \frac{\sqrt{3}}{4}, \quad j \in \{1, ..., n\}.$$

Example 2. Let $n = 2$ and $f(z) = (z_1 + a z_2^2, z_2)'$ for $z = (z_1, z_2)' \in B$, where $a \in \mathbb{C}$, $|a| \leq \frac{\sqrt{5}-1}{4}$. Then $f \in US^*(B)$.

Clearly, f is normalized and locally biholomorphic on the unit ball of $\mathbb{C}^2$. After simple calculations we deduce that

$$[Df(z)]^{-1}(f(z) - f(\zeta)) =$$
$$= \left(z_1 - \zeta_1 + a({z_2}^2 - {\zeta_2}^2), -2a z_2 (z_1 - \zeta_1 + a({z_2}^2 - {\zeta_2}^2)) \right)'.$$

Therefore, for $|a| \leq \frac{\sqrt{5}-1}{4}$, we obtain

$$\begin{aligned} \operatorname{Re} \left\langle [Df(z)]^{-1}(f(z) - f(\zeta)), z - \zeta \right\rangle \; &\geq \; \|z - \zeta\|^2 \\ -|a| \cdot |z_2 + \zeta_2| \frac{\|z - \zeta\|^2}{2} &- |a| \cdot |z_2| \cdot \|z - \zeta\|^2 \\ -2|a|^2 \cdot |z_2| \cdot |z_2 &- \zeta_2| \cdot |z_2 + \zeta_2| \\ \geq \|z - \zeta\|^2 \left[1 - 2|a| - 4|a|^2 \right] \; &\geq \; 0. \end{aligned}$$

We note that if $f_1, \ldots, f_n$ are uniformly convex functions on the unit disk, then f is not necessarily uniformly convex on B, where $f(z) = (f_1(z_1), ..., f_n(z_n))'$ since this mapping is not always convex (see, for example [1] and [19]).

We shall now give a distortion result for uniformly starlike mappings on B. To this end, we will prove the following lemma.

LEMMA. *Let $P : B \times B \to \mathbb{C}^n$ be a holomorphic mapping, such that*

$$P(z,\zeta) = z - \zeta + \frac{1}{2} D^2P(0,0)(z^2,\zeta^2) + \cdots,$$

for $(z,\zeta) \in B \times B$.

If

$$\text{Re } \langle P(z,\zeta), z-\zeta\rangle > 0,\ z,\ \zeta \in B,\ \ z \neq \zeta,$$

then

$$\left| \frac{1}{2} \left\langle D^2P(0,0)(z^2,\zeta^2), z-\zeta \right\rangle \right| \leq 2, \tag{2.9}$$

for all $z, \zeta \in B,\ \ \|z-\zeta\| = 1$.

PROOF. Let $z, \zeta \in B,\ \ \|z-\zeta\| = 1$ and let $p : U \to \mathbb{C}$, given by

$$p(t) = \begin{cases} \frac{1}{t}\langle P(zt,\zeta t), z-\zeta\rangle, & t \in U \setminus \ \{0\} \\ 1, & t = 0. \end{cases}$$

Then $p \in H(U)$ and Re $p(t) > 0,\ t \in U$, then it is well known that

$$|p'(0)| \leq 2. \tag{2.10}$$

Since

$$p'(t) = \frac{\langle tDP(zt,\zeta t)(z,\zeta) - P(zt,\zeta t), z-\zeta\rangle}{t^2},$$

for $t \in U \setminus \{0\}$, then

$$p'(0) = \lim_{t\to 0} p'(t) = \frac{1}{2} \left\langle D^2P(0,0)(z^2,\zeta^2), z-\zeta \right\rangle.$$

Taking into account the relation as desired. □

Considering this result we can formulate the following distortion theorem.

THEOREM 3. *If $f \in US^*(B)$ then*

$$\left|\frac{1}{2}\left\langle D^2 f(0)(z,z) + D^2 f(0)(\zeta,\zeta) - 2D^2 f(0)(z,\zeta), z-\zeta\right\rangle\right| \leq 2, \quad (2.11)$$

for all $z, \zeta \in B$, $\|z-\zeta\| = 1$. In consequence,

$$\left|\frac{1}{2}\left\langle D^2 f(0)(z,z), z\right\rangle\right| \leq 2,\ \|z\| = 1. \quad (2.12)$$

PROOF. First we will show the inequality (2.11). To this aim, let

$$P(z,\zeta) = [Df(z)]^{-1}(f(z) - f(\zeta))$$

for all $z, \zeta \in B$, $\|z-\zeta\| = 1$. Since $f \in US^*(B)$ then $P \in H(B \times B)$,

$$P(z,\zeta) = z - \zeta + \frac{1}{2}D^2 P(0,0)(z^2,\zeta^2) + \cdots,$$

and

$$\operatorname{Re}\langle P(z,\zeta), z-\zeta\rangle > 0.$$

Hence, considering the result of Lemma 2.5 we conclude that

$$\left|\frac{1}{2}\left\langle D^2 P(0,0)(z^2,\zeta^2), z-\zeta\right\rangle\right| \leq 2. \quad (2.13)$$

On the other hand, since $f(z) - f(\zeta) = Df(z)P(z,\zeta)$, then using the Taylor expansion of f and P and equating the coefficients of the second order in the last equality, we easily deduce that

$$-\frac{1}{2}\left(D^2 f(0)(z,z) - \frac{1}{2}D^2 f(0)(\zeta,\zeta) + D^2 f(0)(z,\zeta)\right) = \frac{1}{2}D^2 P(0,0)(z^2,\zeta^2),$$

for $z, \zeta \in B$.

Thus, considering the relation (2.13) and the above equality, we get inequality (2.11), as desired.

In order to obtain relation (2.12) it suffices to let $\zeta = 0$ into (2.11).

Note that, some similar distortion results for starlike mappings and other biholomorphic mappings have been recently obtained by the second author (cf. [9, 10]).

We shall finish this paper with the following remarks and questions.

Let $US^*_\pi(B)$ be the class of those uniformly starlike mappings on B of the form

$$f(z) = (f_1(z_1), ...f_n(z_n))', \quad z = (z_1, ..., z_n)' \in B,$$

where $f_1, ..., f_n$ are uniformly starlike functions on the unit disk. Clearly, in the case of one variable $US^*_\pi(B) = UST$, but in $\mathbb{C}^n, n > 1, US^*_\pi(B)$ is a proper subclass of $US^*(B)$, since $f(z) = \left(z_1 + \frac{\sqrt{5}-1}{4} z_2^2, z_2\right)'$ is in $US^*(B)$, but not in $US^*_\pi(B)$

In this case, we have the following result.

THEOREM 4. *If $f \in US^*_\pi(B)$, then*

$$\left\| \frac{1}{2} D^2 f(0)(z,z) \right\| \le 1, \quad z \in \mathbb{C}^n, \quad \|z\| = 1.$$

PROOF. This inequality is a simple consequence of the relation (4.2) [2].

Indeed, since f_j is uniformly starlike in the unit disk, for all $j \in \{1, \ldots, n\}$, then

$$\left| \frac{1}{2} f_j''(0) \right| \le 1, \quad j \in \{1, \ldots, n\}.$$

Thus,

$$\left\| \frac{1}{2} D^2 f(0)(z,z) \right\| = \sqrt{\sum_{j=1}^{n} |\frac{1}{2} f_j''(0)| \cdot |z_j|^4} \le \sqrt{\sum_{j=1}^{n} |z_j|^4} \le 1,$$

for all $z \in \mathbf{C}^n$, $\|z\| = 1$. This completes the proof. □

CONJECTURE. If $f \in US^*(B)$, then

$$\left\| \frac{1}{2} D^2 f(0)(z,z) \right\| \le 1, \quad z \in \mathbb{C}^n, \quad \|z\| = 1.$$

OPEN PROBLEMS. Find the best estimates for coefficients of second order for mappings from $UK(B)$ (This problem has been solved by the second author in the case of convex and normalized mappings on B (see [9])).

Find the growth and covering results for uniformly starlike and uniformly convex mappings on the unit ball of $\mathbb{C}^n$.

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