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SIMPSON-TYPE INEQUALITIES VIA (H, M)-LOG-CONVEXITY FOR MULTIPLICATIVE WEIGHTED INTEGRALS

Abstract. The paper presents a new class of modified multiplicative (h, m)-convex functions and formulates the concept of weighted multiplicative generalized integral operators. Two new integral identities are derived, providing a fundamental basis for constructing refined dual Simpson-type inequalities containing fractional weighted integrals. Several illustrative special cases are given, demonstrating that some of the obtained inequalities are reduced to known results described in the literature, which confirms the validity and generality of the proposed framework.

Key words: *multiplicative calculus, modified multiplicative convexity, (h, m)-convex functions, fractional weighted integrals, Simpson-type inequalities, Hölder inequality, power mean inequality, generalized integral operators*

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1. Introduction. In recent years, multiplicative calculus has attracted attention as an alternative approach to traditional methods of analysis, particularly through the introduction of new definitions of derivatives and integrals based on the multiplicative properties of functions. This approach leads to rules and results that differ from those of classical calculus. For example, the concept of multiplicative convexity allows for alternative formulations of certain inequalities. Following the work of Ali et al. [2], many researchers have made significant contributions to the development of integral inequalities within the framework of multiplicative calculus. In particular, Mateen and colleagues [15] have advanced the field considerably by generalizing various integral inequalities and providing their computational analysis.

Several studies have investigated Hermite-Hadamard-type and Milne-type inequalities in the setting of strongly multiplicative convex functions [8], [9], [31]. These works established new inequalities using fractional operators such as the Atangana-Baleanu integral and validated the results through illustrative examples and special function applications. Efforts have also been made to generalize the Ostrowski-type inequalities for multiplicative differentiable convex functions, especially through fractional operators [20]. These contributions introduced new identities and offered practical applications in various analytical contexts. In [29], the authors introduced the concept of multiplicative harmonic s-convex functions and derived Hermite-Hadamard-type inequalities for both the functions and their combinations via multiplicative integrals. These results also encompass classical harmonic convexity as a special case. Further developments include the use of Katugampola and Caputo-hybrid (PCH) operators to derive midpoint-, trapezoid-, and Bullen-type inequalities within multiplicative calculus [28], [10], [16]. Such contributions emphasize the precision of fractional multiplicative techniques in capturing generalized convex behavior and improving estimates for nonlinear functions. The flexibility of generalized convex functions, such as s-convex and harmonic convex functions, has allowed researchers to extend classical results to broader settings. In particular, [33], [28] focused on deriving several families of inequalities including Hermite-Hadamard, midpoint, trapezoid, and Bullen types based on newly proposed identities involving second-order multiplicative differentiability. These studies also included computational analysis to validate the accuracy of their results.

A substantial part of the literature is devoted to Simpson-type and Newton-Cotes-type inequalities, which play a critical role in improving the accuracy of numerical integration. In [19], [30], [21], [33], [1], [11], [18], [3], [17], researchers formulated novel multiplicative identities and employed them to derive fractional and classical Simpson-type inequalities for generalized convex and multiplicatively differentiable functions. Many of these results offer refinements or generalizations of known inequalities, while others demonstrate applications to special means, error bounds, and numerical methods. Taken together, the existing literature highlights the strength of multiplicative calculus in advancing the theory of integral inequalities. Motivated by these developments, the present work aims to contribute to this growing field by establishing new inequalities for generalized convex functions, with a focus on their computational significance and analytical robustness.

Particular attention is due to the work of Bashirov et al. [4], in which a comprehensive mathematical theory of multiplicative calculus was systematically developed.

Definition 1. *[4] The multiplicative derivative of a positive function ψ, denoted by ψ^*, is defined as follows:*

$$\frac{d^*\psi}{d\varkappa} = \psi^*(\varkappa) = \lim_{h\to 0}\left(\frac{\psi(\varkappa+h)}{\psi(\varkappa)}\right)^{\frac{1}{h}}.$$

Remark 1. *[4] Every positive and differentiable function ψ is multiplicatively differentiable. Moreover, the relationship between the classical derivative ψ' and the multiplicative derivative ψ^* is given by:*

$$\psi^*(\varkappa) = e^{(\ln\psi(\varkappa))'} = e^{\frac{\psi'(\varkappa)}{\psi(\varkappa)}}.$$

Definition 2. *[4] The multiplicative integral of a positive function ψ over the interval $[a,b]$ is defined as:*

$$\int_a^b (\psi(\varkappa))^{d\varkappa} = \exp\Big(\int_a^b \ln(\psi(\varkappa))\,d\varkappa\Big).$$

Proposition 1. *[4] Let ψ and ϕ be positive and Riemann-integrable functions. Then both ϕ and ψ are multiplicative integrable, and the following properties hold:*

1) $\displaystyle\int_a^b (\psi(\varkappa)^p)^{d\varkappa} = \Big(\int_a^b \psi(\varkappa)^{d\varkappa}\Big)^p,$ *for all $p\in\mathbb{R}$,*

2) $\displaystyle\int_a^b \psi(\varkappa)^{d\varkappa} = \int_a^c \psi(\varkappa)^{d\varkappa}\cdot\int_c^b \psi(\varkappa)^{d\varkappa},$ *for all $a<c<b$,*

3) $\displaystyle\int_a^a \psi(\varkappa)^{d\varkappa} = 1,$ *and* $\displaystyle\int_b^a \psi(\varkappa)^{d\varkappa} = \Big(\int_a^b \psi(\varkappa)^{d\varkappa}\Big)^{-1},$

4) $\displaystyle\int_a^b (\psi(\varkappa)\phi(\varkappa))^{d\varkappa} = \int_a^b \psi(\varkappa)^{d\varkappa}\cdot\int_a^b \phi(\varkappa)^{d\varkappa},$

5) $$\int_a^b \left(\frac{\psi(\varkappa)}{\phi(\varkappa)}\right)^{d\varkappa} = \frac{\int_a^b \psi(\varkappa)^{d\varkappa}}{\int_a^b \phi(\varkappa)^{d\varkappa}}.$$

Theorem 1. *[4] (Multiplicative integration by parts)*

Let ψ be a positive and multiplicatively differentiable function on the interval $[a,b]$. Also, let $\gamma\colon \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ and $\phi\colon [a,b] \to \mathbb{R}$ be two differentiable functions. Then the following identity holds:

$$\int_a^b \left(\psi^*(\varkappa)^{\phi(\varkappa)}\right)^{d\varkappa} = \frac{\psi(b)^{\phi(b)}}{\psi(a)^{\phi(a)}} \cdot \frac{1}{\int_a^b \left(\psi(\varkappa)^{\phi'(\varkappa)}\right)^{d\varkappa}}.$$

The generalized lemma of integration by parts from [3] (see Lemma 1):

Lemma 1. *Let ψ be a positive and multiplicatively differentiable function on the interval $[a,b]$. Also, let $\gamma\colon \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ and $\delta\colon [a,b] \to \mathbb{R}$ be two differentiable functions. Then the following identity holds:*

$$\int_a^b \left(\psi^*(\gamma(\varkappa))^{\gamma'(\varkappa)\phi(\varkappa)}\right)^{d\varkappa} = \frac{\psi(\gamma(b))^{\phi(b)}}{\psi(\gamma(a))^{\phi(a)}} \times \frac{1}{\int_a^b \left(\psi(\gamma(\varkappa))^{\phi'(\varkappa)}\right)^{d\varkappa}}. \tag{1}$$

2. Preliminaries.

Previously, several works ([6], [7], [23]) presented the following generalized definitions of classes of convex functions ([24] presents quite a few convexity classes known in the literature).

Definition 3. *Let $\psi\colon \mathcal{I} \subseteq [0,\infty) \to [0,\infty)$ and $h\colon [0,1] \to (0,1]$. If inequality*

$$\psi\left(\xi\varkappa + m\varsigma(1-\varkappa)\right) \leqslant \psi(\xi)h^s(\varkappa) + m\psi(\varsigma)(1-h^s(\varkappa))$$

is fulfilled $\forall \varsigma, \xi \in \mathcal{I}$ and $\varkappa \in [0,1]$, where $s \in [0,1]$ and $m \in [0,1]$, then we say that the function ψ is modified (h,m)-convex of the first type on $\mathcal{I}$.

Definition 4. *Let $\psi\colon \mathcal{I} \subseteq [0,\infty) \to [0,\infty)$ and $h\colon [0,1] \to (0,1]$. If inequality*

$$\psi\left(\xi\varkappa + m\varsigma(1-\varkappa)\right) \leqslant \psi(\xi)h^s(\varkappa) + m\psi(\varsigma)(1-h(\varkappa))^s$$

is fulfilled $\forall \varsigma, \xi \in \mathcal{I}$ *and* $\varkappa \in [0,1]$, *where* $s \in [-1,1]$ *and* $m \in [0,1]$, *we say that the function* ψ *is modified* (h,m)*-convex of the second type on* $\mathcal{I}$.

Remark 2. *The class of functions defined in Definition 3 and 4 is denoted by* $N^{s,1}_{h,m}(\mathcal{I})$ *and* $N^{s,2}_{h,m}(\mathcal{I})$.

We introduce a new class of functions, called generalized logarithmic (or multiplicative) convex functions, which encompasses those satisfying the condition of logarithmic convexity or, equivalently, multiplicative convexity. This approach allows us to work within an additive structure, while preserving the essence of multiplicative convexity via logarithmic transformation.

Definition 5. *Let* $\psi\colon \mathcal{I} = [0,\infty) \to [0,\infty)$ *and* $h\colon [0,1] \to (0,1]$. *If inequality*

$$\psi\left(\upsilon_1 t + m(1-t)\upsilon_2\right) \leqslant \left[\psi(\upsilon_1)\right]^{h^s(t)} \left[m\psi(\upsilon_2)\right]^{(1-h^s(t))}$$

is fulfilled $\forall a, b \in \mathcal{I}$ *and* $t \in [0,1]$, *where* $s \in [0,1]$ *and* $m \in [0,1]$, *then we say that the function* ψ *is modified* (h,m)*-logarithmically (respectively, multiplicatively) convex of the first type on* $\mathcal{I}$.

Definition 6. *Let* $\psi\colon \mathcal{I} = [0,\infty) \to [0,\infty)$ *and* $h\colon [0,1] \to (0,1]$. *If inequality*

$$\psi\left(\upsilon_1 t + m(1-t)\upsilon_2\right) \leqslant \left[\psi(\upsilon_1)\right]^{h^s(t)} \left[m\psi(\upsilon_2)\right]^{(1-h(t))^s}$$

is fulfilled $\forall \upsilon_2, \upsilon_1 \in \mathcal{I}$ *and* $t \in [0,1]$, *where* $s \in [0,1]$ *and* $m \in [0,1]$ *then we say that the function* ψ *is modified* (h,m)*-logarithmically (respectively, multiplicatively) convex of the second type on* $\mathcal{I}$.

Remark 3. *The class of functions defined in Definition 5 and 6 is denoted by* $M^{s,1}_{h,m}(\mathcal{I})$ *and* $M^{s,2}_{h,m}(\mathcal{I})$.

Note that Definition 6 generalizes several well-known classes of logarithmically convex functions as special cases, depending on the choices of the parameters h, m, and s:

1) Classical Logarithmic Convexity [5], [12], [13], [25], [27].
   If $h(t) = t$, $s = 1$, and $m = 1$ are satisfied, the inequality becomes:

   $$\psi(t\upsilon_1 + (1-t)\upsilon_2) \leqslant [\psi(\upsilon_1)]^t \cdot [\psi(\upsilon_2)]^{1-t},$$

   which corresponds to the standard log-convex (multiplicative convex) functions.

2) m-Logarithmic Convexity [26].
If $h(t) = t$, $s = 1$, and $m \in (0, 1)$, then:

$$\psi(t\upsilon_1 + m(1-t)\upsilon_2) \leqslant [\psi(\upsilon_1)]^t \cdot [m\psi(\upsilon_2)]^{1-t},$$

which defines the class of m-logarithmically convex functions.

3) s-Logarithmic Convexity [32].
If $h(t) = t$, $m = 1$, and $s \in (0, 1)$, then the inequality becomes:

$$\psi(t\upsilon_1 + (1-t)\upsilon_2) \leqslant [\psi(\upsilon_1)]^{t^s} \cdot [\psi(\upsilon_2)]^{(1-t)^s},$$

which corresponds to the class of s-logarithmically convex functions.

4) (h, s)-Logarithmic Convexity.
If $m = 1$, and $h\colon [0, 1] \to (0, 1]$, $s \in (0, 1]$, then:

$$\psi(t\upsilon_1 + (1-t)\upsilon_2) \leqslant [\psi(\upsilon_1)]^{h^s(t)} \cdot [\psi(\upsilon_2)]^{(1-h(t))^s},$$

which defines the class of (h, s)-logarithmically convex functions.

Definition 7. *Let $\psi \in L_1[\upsilon_1, \upsilon_2]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$ are defined by (right and left respectively):*

$$I^{\alpha}_{\upsilon_1^+}\psi(r) = \frac{1}{\Gamma(\alpha)} \int_{\upsilon_1}^{r} (r - \varkappa)^{\alpha-1}\psi(\varkappa)\, d\varkappa, \quad r > \upsilon_1,$$

and

$$I^{\alpha}_{\upsilon_2^-}\psi(r) = \frac{1}{\Gamma(\alpha)} \int_{r}^{\upsilon_2} (\varkappa - r)^{\alpha-1}\psi(\varkappa)\, d\varkappa, \quad r < \upsilon_2.$$

Here $\Gamma(.)$s the Euler gamma function.

Definition 8. *Let $\psi \in L_1[\upsilon_1, \upsilon_2]$. Then multiplicative Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$ are defined by (right and left respectively):*

$${}_*I^{\alpha}_{\upsilon_1^+}\psi(r) = \exp\left[{}^{\alpha}I_{\upsilon_1^+} \ln\psi(r)\right], \quad r > \upsilon_1,$$

and

$${}_*I^{\alpha}_{\upsilon_2^-}\psi(r) = \exp\left[{}^{\alpha}I_{\upsilon_2^-} \ln\psi(r)\right], \quad r < \upsilon_2.$$

Definition 9. *[22], [14] Let $\upsilon_1, \upsilon_2 \in \mathcal{I}$, $\psi \in L[\upsilon_1, \upsilon_2]$, and let $\varpi\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$ be a continuous and differentiable function with $\varpi' > 0$ and $\varpi' \in L[\upsilon_1, \upsilon_2]$. The weighted fractional integrals, in both right-hand and left-hand forms, are defined by:*

$$J^{\varpi}_{\upsilon_1^+}\psi(r) = \int_{\upsilon_1}^{r} \varpi'\left(\frac{r-\varkappa}{r-\upsilon_1}\right)\psi(\varkappa)\,d\varkappa, \quad r > \upsilon_1,$$

and

$$J^{\varpi}_{\upsilon_2^-}\psi(r) = \int_{r}^{\upsilon_2} \varpi'\left(\frac{x-r}{\upsilon_2-r}\right)\psi(\varkappa)\,d\varkappa, \quad r < \upsilon_2.$$

Remark 4. *Some special cases of Definition 9 are given in [22] (see Remark 3).*

Definition 10. *The multiplicative weighted fractional integrals of a function ψ, defined in both right-hand and left-hand forms, are given by:*

$$^*J^{\varpi}_{\upsilon_1^+}\psi(r) = \exp\left\{\int_{\upsilon_1}^{r} \varpi'\left(\frac{r-\varkappa}{r-\upsilon_1}\right)\ln(\psi(\varkappa))\,d\varkappa\right\}, \quad r > \upsilon_1,$$

and

$$^*J^{\varpi}_{\upsilon_2^-}\psi(r) = \exp\left\{\int_{r}^{\upsilon_2} \varpi'\left(\frac{\varkappa-r}{\upsilon_2-r}\right)\ln(\psi(\varkappa))\,d\varkappa\right\}, \quad r < \upsilon_2.$$

Lemma 2. *Let $b \geqslant a \geqslant 0$ and some $p > 1$; then*

$$\left(pa^{p-1}\right)^{\frac{1}{p}}(b-a)^{\frac{1}{p}} \leqslant \left(b^p - a^p\right)^{\frac{1}{p}} \leqslant \left(pb^{p-1}\right)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}. \tag{2}$$

Proof. Using Lagrange's theorem, we can write:

$$b^p - a^p = pc^{p-1}(b-a), \text{ where } c \in [a,b].$$

The following double inequality is obvious:

$$pa^{p-1}(b-a) \leqslant b^p - a^p \leqslant pb^{p-1}(b-a).$$

Raising this double inequality to the power $\frac{1}{p}$, we get (2). □

3. Main Results. Throughout this paper, we use the following notation:

$$\begin{cases} \mathbf{v}_k := \upsilon_1 + k\delta, \text{ where } \delta = \frac{\upsilon_2-\upsilon_1}{2n}, \ k = 0,1,2,\dots,2n, \\ \mathcal{W}_k(p,\varpi_k) := \int_0^1 |\varpi_k(\varkappa)|^p \, d\varkappa, \ k = 1,2,\dots,2n, \\ \mathcal{A}_1(\phi) := \int_0^1 |\phi(\varkappa)| \, (1-h(\varkappa))^s \, d\varkappa, \\ \mathcal{A}_2(\phi) := \int_0^1 |\phi(\varkappa)| \; h^s(\varkappa) d\varkappa. \end{cases} \tag{3}$$

Lemma 3. *Assume that $\psi > 0$ is a multiplicatively differentiable function on $[\upsilon_1, \upsilon_2] \subset \mathcal{I} = [0,\infty)$ and that $\psi^* > 0$ is multiplicatively integrable on $[\upsilon_1, \upsilon_2]$. Let ϖ_k be weight functions. Then the following identity holds:*

$$\prod_{k=1}^{2n} \mathcal{L}_k = \prod_{k=1}^{2n} \left[\int_0^1 \left(\psi^*((1-\varkappa)\mathbf{v}_{k-1} + \varkappa \mathbf{v}_k)^{\varpi_k(\varkappa)} \right)^{d\varkappa} \right]^{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}}, \tag{4}$$

where $n \in \mathbb{N}$ and

$$\mathcal{L}_k = \frac{\psi^{\frac{\varpi_k(1)}{2n}}(\mathbf{v}_k)}{\psi^{\frac{\varpi_k(0)}{2n}}(\mathbf{v}_{k-1})} \left[{}^*J^{\varpi_k}_{\mathbf{v}_k^-} \psi(\mathbf{v}_{k-1}) \right]^{-\frac{1}{2n(\mathbf{v}_k - \mathbf{v}_{k-1})}}, \tag{5}$$

and $\mathbf{v}_k$ are defined in (3) for all k with $\mathbf{v}_k \in [\upsilon_1, \upsilon_2]$.

Proof. Let

$$\mathbb{I}_k = \left[\int_0^1 \left(\psi^* \left((1-\varkappa)\mathbf{v}_{k-1} + \varkappa \mathbf{v}_k \right)^{\varpi_k(\varkappa)} \right)^{d\varkappa} \right]^{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}}. \tag{6}$$

Employing the property 1 of Proposition 1, the multiplicative integration by parts (1) and Definition 2, for $\mathbb{I}_k$, where $k = 1,2,\dots,2n$, we get

$$\mathbb{I}_k = \left[\int_0^1 \left(\psi^* \left((1-\varkappa)\mathbf{v}_{k-1} + \varkappa \mathbf{v}_k \right)^{\varpi_k(\varkappa)} \right)^{d\varkappa} \right]^{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}} \tag{7}$$

$$= \int_0^1 \left(\psi^* \left((1-\varkappa)\mathbf{v}_{k-1} + \varkappa \mathbf{v}_k \right)^{(\mathbf{v}_k - \mathbf{v}_{k-1})\frac{\varpi_k(\varkappa)}{2n}} \right)^{d\varkappa}$$

$$= \frac{\psi^{\frac{\varpi_k(1)}{2n}}(\mathbf{v}_k)}{\psi^{\frac{\varpi_k(0)}{2n}}(\mathbf{v}_{k-1})} \exp\left\{-\frac{1}{2n}\int_0^1 \varpi_k'(\varkappa)\ln\psi\left((1-\varkappa)\mathbf{v}_{k-1}+\varkappa\mathbf{v}_k\right)d\varkappa\right\}$$

$$= \frac{\psi^{\frac{\varpi_k(1)}{2n}}(\mathbf{v}_k)}{\psi^{\frac{\varpi_k(0)}{2n}}(\mathbf{v}_{k-1})} \exp\left\{-\frac{1}{2n(\mathbf{v}_k-\mathbf{v}_{k-1})}\int_{\mathbf{v}_{k-1}}^{\mathbf{v}_k} \varpi_k'\left(\frac{\sigma-\mathbf{v}_{k-1}}{\mathbf{v}_k-\mathbf{v}_{k-1}}\right)\ln\psi(\sigma)\,d\sigma\right\}$$

$$= \frac{\psi^{\frac{\varpi_k(1)}{2n}}(\mathbf{v}_k)}{\psi^{\frac{\varpi_k(0)}{2n}}(\mathbf{v}_{k-1})} \left[{}^{*}J^{\varpi_{\mathbf{k}}}_{\mathbf{v}_k^-}\psi(\mathbf{v}_{k-1})\right]^{-\frac{1}{2n(\mathbf{v}_k-\mathbf{v}_{k-1})}}.$$

By multiplying all $\mathbb{I}_k$, the asserted identity is established. This completes the proof. □

Remark 5. *For certain choices of the weight functions, we recover results already established in the literature. For instance, for* $\alpha > 0$, $n = 4$, $\varpi_1(\varkappa) = \varkappa^\alpha$, $\varpi_2(\varkappa) = -\frac{2}{15} - (1-\varkappa)^\alpha$, $\varpi_3(\varkappa) = \varkappa^\alpha + \frac{2}{15}$, *and* $\varpi_4(\varkappa) = -(1-\varkappa)^\alpha$, *with* $h(t) = t$ *and* $s = m = 1$, *we obtain Lemma* 3.1 *of [19]. Moreover, if* $\varpi_1(\varkappa) = t$, $\varpi_2(\varkappa) = t - \frac{5}{3}$, $\varpi_3(\varkappa) = t + \frac{2}{3}$, *and* $\varpi_4(\varkappa) = t - 1$, *we recover Lemma* 3.1 *of [17].*

Theorem 2. *Let* ψ, ψ^*, ϖ_k *be as in Lemma 3. If* $\psi^* \in M^{s,2}_{h,m}(\mathcal{I})$, *then it is true that:*

$$\left|\prod_{k=1}^{2n}\mathcal{L}_k\right| \leqslant \prod_{k=1}^{2n}\left[\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1(\varpi_k)}\left(\psi^*(\mathbf{v}_k)\right)^{\mathcal{A}_2(\varpi_k)}\right]^{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}}, \quad (8)$$

where $\mathcal{L}_k$ *is defined above in Lemma 3 and* $\mathcal{A}_1$, $\mathcal{A}_2$, *and* $\mathbf{v}_k$ *are defined in (3) for all* k, *with* $\mathbf{v}_k \in [\upsilon_1, \upsilon_2]$. *Moreover* $\mathbf{v}_{k-1} \leqslant \frac{\mathbf{v}_{k-1}}{m} < \mathbf{v}_k < \infty$, *where* $m \in (0, 1]$.

Proof. By using the properties of multiplicative integrals and the fact that $\psi^* \in M^{s,2}_{h,m}(\mathcal{I})$, for the $\mathbb{I}_k$, where $k = 1, 2, \ldots, 2n$ from (6), we have

$$|\mathbb{I}_k| \leqslant \exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\int_0^1 |\varpi_k(\varkappa)|\,|\ln\psi^*((1-\varkappa)\mathbf{v}_{k-1}+\varkappa\mathbf{v}_k)|\,d\varkappa\right\}$$

$$\leqslant \exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\int_0^1 |\varpi_k(\varkappa)|\left|\ln\left[\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{(1-h(\varkappa))^s}\left(\psi^*(\mathbf{v}_k)\right)^{h^s(\chi)}\right]\right|d\varkappa\right\}$$

$$\leqslant \exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\left(\left|\ln\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)\right|\int_0^1 |\varpi_k(\varkappa)|\,(1-h(\varkappa))^s\,d\varkappa\right.\right.$$

$$+\left|\ln\psi^{*}\left(\mathbf{v}_k\right)\right|\int_0^1\left|\varpi_k(\varkappa)\right|h^s(\varkappa)\,d\varkappa\Big)\Big\}$$

$$=\exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\left[\mathcal{A}_1(\varpi_k)\left|\ln\left(m\psi^{*}\left(\mathbf{v}_{k-1}\right)\right)\right|+\mathcal{A}_2(\varpi_k)\left|\ln\psi^{*}\left(\mathbf{v}_k\right)\right|\right]\right\}$$

$$=\left[\left(m\psi^{*}\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1(\varpi_k)}\left(\psi^{*}\left(\mathbf{v}_k\right)\right)^{\mathcal{A}_2(\varpi_k)}\right]^{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}}.$$

Taking into account (4), we obtain (8). Thus, the proof is complete. □

Corollary 1. *When the conditions of Theorem 2 are satisfied, if we take for* $n=2$, $\varpi_1(\varkappa)=\varkappa^{\alpha}$, $\varpi_2(\varkappa)=-\frac{2}{15}-(1-\varkappa)^{\alpha}$, $\varpi_3(\varkappa)=\varkappa^{\alpha}+\frac{2}{15}$, *and* $\varpi_4(\varkappa)=-(1-\varkappa)^{\alpha}$ *with* $\alpha>0$ *and* (ψ^{*}), *corresponding to the classical* s*-log-convex case* ($m=1$ *and* $h(\varkappa)=\varkappa$), *we get*

$$\psi^{\frac{8}{15}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{-\frac{1}{15}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{8}{15}}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\times\mathcal{F}\left(\psi,\upsilon_1,\upsilon_2\right)$$
$$\leqslant\left[\left(\psi^{*}\left(\upsilon_1\right)\psi^{*}\left(\upsilon_2\right)\right)^{B(\alpha+1,s+1)}\left(\psi^{*}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{2B(\alpha+1,s+1)+\frac{4}{15(s+1)}}\right.$$
$$\left.\times\left(\psi^{*}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{*}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{2(\alpha+16s+16)}{15(s+1)(\alpha+s+1)}}\right]^{\frac{\upsilon_2-\upsilon_1}{16}},\quad(9)$$

where

$$\mathcal{F}\left(\psi,\upsilon_1,\upsilon_2\right)=\left[\left({}_{*}I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^-}\ln\psi(\upsilon_1)\right)\left({}_{*}I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^+}\ln\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)\right.$$
$$\left.\times\left({}_{*}I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^-}\ln\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)\left({}_{*}I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^+}\ln\psi\left(\upsilon_2\right)\right)\right]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}},$$

$B(.,.)$ *and* $\Gamma(.)$ *are the Euler beta and gamma functions.*

Proof. *Indeed,* $\forall k$, $\mathbf{v}_k-\mathbf{v}_{k-1}=\frac{\upsilon_2-\upsilon_1}{4}$ *from (5) for* $\mathcal{L}_1$ *we obtain*

$$\mathcal{L}_1=\frac{\psi^{\frac{\varpi_1(1)}{4}}\left(\mathbf{v}_1\right)}{\psi^{\frac{\varpi_1(0)}{4}}\left(\mathbf{v}_0\right)}\left[{}^{*}J^{\varpi_{\mathbf{k}}}_{\mathbf{v}_1^-}\psi\left(\mathbf{v}_0\right)\right]^{-\frac{1}{4(\mathbf{v}_1-\mathbf{v}_0)}}$$
$$=\frac{\psi^{\frac{\varpi_1(1)}{4}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)}{\psi^{\frac{\varpi_1(0)}{4}}\left(\upsilon_1\right)}\left[{}^{*}J^{\varpi_{\mathbf{1}}}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^-}\psi(\upsilon_1)\right]^{-\frac{1}{\upsilon_2-\upsilon_1}}$$
$$=\frac{\psi^{\frac{1}{4}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)}{\psi^{\frac{\varpi_1(0)}{4}}\left(\upsilon_1\right)}\left[\exp\left(\int_{\upsilon_1}^{\frac{3\upsilon_1+\upsilon_2}{4}}\alpha\left(\frac{x-\upsilon_1}{\frac{3\upsilon_1+\upsilon_2}{4}-\upsilon_1}\right)^{\alpha-1}\ln\psi(x)dx\right)\right]^{-\frac{1}{\upsilon_2-\upsilon_1}}$$

$$= \psi^{\frac{1}{4}}\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)\Big[\exp\Big(\frac{1}{\Gamma(\alpha)}\int_{\upsilon_1}^{\frac{3\upsilon_1+\upsilon_2}{4}}(x-\upsilon_1)^{\alpha-1}\ln\psi(x)dx\Big)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}}$$

$$= \psi^{\frac{1}{4}}\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)\Big[{}_*I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^-}\ln\psi(\upsilon_1)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}}.$$

Similarly, for other $\mathcal{L}$ we can write

$$\mathcal{L}_2 = \frac{\psi^{\frac{\varpi_2(1)}{4}}(\mathbf{v}_2)}{\psi^{\frac{\varpi_2(0)}{4}}(\mathbf{v}_1)}\Big[{}^*J^{\varpi_{\mathbf{k}}}_{\mathbf{v}_2^-}\psi(\mathbf{v}_1)\Big]^{\frac{-1}{4(\mathbf{v}_2-\mathbf{v}_1)}}$$

$$= \frac{\psi^{-\frac{1}{30}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)}{\psi^{-\frac{17}{60}}\left(\frac{3\upsilon_1+\upsilon_2}{2}\right)}\Big[{}_*I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^+}\ln\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}},$$

$$\mathcal{L}_3 = \frac{\psi^{\frac{\varpi_3(1)}{4}}(\mathbf{v}_3)}{\psi^{\frac{\varpi_3(0)}{4}}(\mathbf{v}_2)}\Big[{}^*J^{\varpi_{\mathbf{k}}}_{\mathbf{v}_3^-}\psi(\mathbf{v}_2)\Big]^{\frac{-1}{4(\mathbf{v}_3-\mathbf{v}_2)}}$$

$$= \frac{\psi^{\frac{17}{60}}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)}{\psi^{\frac{1}{30}}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)}\Big[{}_*I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^-}\ln\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}},$$

$$\mathcal{L}_4 = \frac{\psi^{\frac{\varpi_4(1)}{4}}(\mathbf{v}_4)}{\psi^{\frac{\varpi_4(0)}{4}}(\mathbf{v}_3)}\Big[{}^*J^{\varpi_{\mathbf{k}}}_{\mathbf{v}_4^-}\psi(\mathbf{v}_3)\Big]^{\frac{-1}{4(\mathbf{v}_4-\mathbf{v}_3)}}$$

$$= \psi^{\frac{1}{4}}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\Big[{}_*I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^+}\ln\psi(\upsilon_2)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}}.$$

Thus, we get

$$\prod_{k=1}^{4}\mathcal{L}_k = \psi^{\frac{1}{4}}\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)\Big[{}_*I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^-}\ln\psi(\upsilon_1)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}}$$

$$\times\frac{\psi^{\frac{17}{60}}\left(\frac{3\upsilon_1+\upsilon_2}{2}\right)}{\psi^{\frac{1}{30}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)}\Big[t_*I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^+}\ln\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}}$$

$$\times\frac{\psi^{\frac{17}{60}}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)}{\psi^{\frac{1}{30}}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)}\Big[{}_*I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^-}\ln\psi\Big(\frac{\upsilon_1+\upsilon_2}{2}\Big)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}}$$

$$\times\psi^{\frac{1}{4}}\Big(\frac{\upsilon_1+3\upsilon_2}{4}\Big)\Big[{}_*I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^+}\ln\psi(\upsilon_2)\Big]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}},$$

or

$$\prod_{k=1}^{4}\mathcal{L}_k = \psi^{\frac{8}{15}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{-\frac{1}{15}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{8}{15}}\left(\frac{\upsilon_1+4\upsilon_2}{4}\right)$$
$$\times\left[\left({}_{*}I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^-}\ln\psi(\upsilon_1)\right)\left({}_{*}I^{\alpha}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^+}\ln\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)\right.$$
$$\left.\times\left({}_{*}I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^-}\ln\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)\left({}_{*}I^{\alpha}_{\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)^+}\ln\psi\left(\upsilon_2\right)\right)\right]^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha}}}. \quad (10)$$

For the right-hand side of (8)*, we have*

$$\prod_{k=1}^{4}\left[\left(m\psi^{*}\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1(\varpi_k)}\left(\psi^{*}\left(\mathbf{v}_k\right)\right)^{\mathcal{A}_2(\varpi_k)}\right]^{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{4}}$$
$$=\left[\left(\psi^{*}\left(\upsilon_1\right)\right)^{\mathcal{A}_1(\varpi_1)}\left(\psi^{*}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\right)^{\mathcal{A}_2(\varpi_1)}\right.$$
$$\times\left(\psi^{*}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\right)^{\mathcal{A}_1(\varpi_2)}\left(\psi^{*}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\mathcal{A}_2(\varpi_2)}$$
$$\times\left(\psi^{*}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\mathcal{A}_1(\varpi_3)}\left(\psi^{*}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\mathcal{A}_2(\varpi_3)}$$
$$\left.\times\left(\psi^{*}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\mathcal{A}_1(\varpi_4)}\left(\psi^{*}\left(\upsilon_2\right)\right)^{\mathcal{A}_2(\varpi_4)}\right]^{\frac{\upsilon_2-\upsilon_1}{16}}. \quad (11)$$

We calculate $\mathcal{A}_1$ and $\mathcal{A}_2$ integrals:

$$\mathcal{A}_1(\varpi_1) = B(\alpha+1,s+1), \qquad \mathcal{A}_2(\varpi_1) = \tfrac{1}{\alpha+s+1},$$
$$\mathcal{A}_1(\varpi_2) = \tfrac{2}{15(s+1)} + \tfrac{1}{\alpha+s+1}, \qquad \mathcal{A}_2(\varpi_2) = B(\alpha+1,s+1) + \tfrac{2}{15(s+1)},$$
$$\mathcal{A}_1(\varpi_3) = B(\alpha+1,s+1) + \tfrac{2}{15(s+1)}, \qquad \mathcal{A}_2(\varpi_3) = \tfrac{2}{15(s+1)} + \tfrac{1}{\alpha+s+1},$$
$$\mathcal{A}_1(\varpi_4) = \tfrac{1}{\alpha+s+1}, \qquad \mathcal{A}_2(\varpi_4) = B(\alpha+1,s+1).$$

Thus, the right-hand side of (8) *will get the form:*

$$\leqslant\left[\left(\psi^{*}\left(\upsilon_1\right)\psi^{*}\left(\upsilon_2\right)\right)^{B(\alpha+1,s+1)}\left(\psi^{*}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{*}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{1}{\alpha+s+1}}\right.$$
$$\times\left(\psi^{*}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{*}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{2}{15(s+1)}+\frac{1}{(\alpha+s+1)}}$$
$$\left.\times\left(\psi^{*}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{2B(\alpha+1,s+1)+\frac{4}{15(s+1)}}\right]^{\frac{\upsilon_2-\upsilon_1}{16}}$$

$$= \left[\left(\psi^*\left(\upsilon_1\right)\psi^*\left(\upsilon_2\right)\right)^{B(\alpha+1,s+1)} \left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{2B(\alpha+1,s+1)+\frac{4}{15(s+1)}} \right.$$
$$\left. \times \left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{2(\alpha+16s+16)}{15(s+1)(\alpha+s+1)}} \right]^{\frac{\upsilon_2-\upsilon_1}{16}}. \tag{12}$$

It is obvious that (9) *follows from inequalities* (10) *and* (12)*. The proof is complete.* □

Remark 6. *This is an analogue of the inequality obtained in Theorem 3.2 from [19].*

Corollary 2. *Under the conditions of Theorem 2, if we take for* $n = 2$, $\varpi_1(\varkappa) = \varkappa$, $\varpi_2(\varkappa) = \varkappa - \frac{5}{3}$, $\varpi_3(\varkappa) = \varkappa + \frac{2}{3}$, $\varpi_4(\varkappa) = \varkappa - 1$, *and* (ψ^*) *corresponding to the* s-log-convex *case* ($m = 1$ *and* $h(\varkappa) = \varkappa$)*, we get*

$$\psi^{\frac{2}{3}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{-\frac{1}{3}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{2}{3}}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\left(\int_{\upsilon_1}^{\upsilon_2}(\psi(x))^{dx}\right)^{-\frac{1}{\upsilon_2-\upsilon_1}}$$
$$\leqslant \left[\left(\psi^*\left(\upsilon_1\right)\psi^*\left(\upsilon_2\right)\right)\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{4s+14}{3}} \right.$$
$$\left. \times \left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{8s+10}{3}} \right]^{\frac{1}{(s+1)(s+2)}\frac{\upsilon_2-\upsilon_1}{16}}. \tag{13}$$

Proof. *Indeed, from* (5) *for* $\mathcal{L}_1$ *we obtain*

$$\mathcal{L}_1 = \frac{\psi^{\frac{\varpi_1(1)}{4}}(\mathbf{v}_1)}{\psi^{\frac{\varpi_1(0)}{4}}(\mathbf{v}_0)}\left({}^*J^{\boldsymbol{\varpi}_1}_{(\mathbf{v}_1)^-}\psi(\mathbf{v}_0)\right)^{-\frac{1}{4(\mathbf{v}_1-\mathbf{v}_0)}}$$
$$= \frac{\psi^{\frac{\varpi_1(1)}{4}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)}{\psi^{\frac{\varpi_1(0)}{4}}(\upsilon_1)}\left({}^*J^{\boldsymbol{\varpi}_1}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^-}\psi(\upsilon_1)\right)^{-\frac{1}{\upsilon_2-\upsilon_1}}$$
$$= \frac{\psi^{\frac{1}{4}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)}{\psi^{0}(\upsilon_1)}\left({}^*J^{\boldsymbol{\varpi}_1}_{\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)^-}\psi(\upsilon_1)\right)^{-\frac{1}{\upsilon_2-\upsilon_1}}$$
$$= \psi^{\frac{1}{4}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\left(\int_{\upsilon_1}^{\frac{3\upsilon_1+\upsilon_2}{4}}(\psi(x))^{dx}\right)^{-\frac{1}{4(\upsilon_2-\upsilon_1)}}.$$

Similarly, for other $\mathcal{L}$ we can write

$$\mathcal{L}_2 = \frac{\psi^{\frac{\varpi_2(1)}{4}}(\mathbf{v}_2)}{\psi^{\frac{\varpi_2(0)}{4}}(\mathbf{v}_1)}\left[{}^*J^{\varpi_\mathbf{2}}_{\mathbf{v}_2^-}\psi(\mathbf{v}_1)\right]^{\frac{-1}{4(\mathbf{v}_2-\mathbf{v}_1)}}$$

$$= \frac{\psi^{-\frac{1}{6}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)}{\psi^{-\frac{5}{12}}\left(\frac{3\upsilon_1+\upsilon_2}{2}\right)}\left(\int_{\frac{3\upsilon_1+\upsilon_2}{4}}^{\frac{\upsilon_1+\upsilon_2}{2}}(\psi(x))^{dx}\right)^{-\frac{1}{\upsilon_2-\upsilon_1}},$$

$$\mathcal{L}_3 = \frac{\psi^{\frac{\varpi_3(1)}{4}}(\mathbf{v}_3)}{\psi^{\frac{\varpi_3(0)}{4}}(\mathbf{v}_2)}\left[{}^*J^{\varpi_\mathbf{k}}_{\mathbf{v}_3^-}\psi(\mathbf{v}_2)\right]^{-\frac{1}{4(\mathbf{v}_3-\mathbf{v}_2)}}$$

$$= \frac{\psi^{\frac{5}{12}}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)}{\psi^{\frac{1}{6}}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)}\left(\int_{\frac{\upsilon_1+\upsilon_2}{2}}^{\frac{\upsilon_1+3\upsilon_2}{4}}(\psi(x))^{dx}\right)^{-\frac{1}{\upsilon_2-\upsilon_1}},$$

$$\mathcal{L}_4 = \frac{\psi^{\frac{\varpi_4(1)}{4}}(\mathbf{v}_4)}{\psi^{\frac{\varpi_4(0)}{4}}(\mathbf{v}_3)}\left[{}^*J^{\varpi_\mathbf{k}}_{\mathbf{v}_4^-}\psi(\mathbf{v}_3)\right]^{-\frac{1}{4(\mathbf{v}_4-\mathbf{v}_3)}}$$

$$= \psi^{\frac{1}{4}}\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\left(\int_{\frac{\upsilon_1+3\upsilon_2}{2}}^{\upsilon_2}(\psi x))^{dx}\right)^{-\frac{1}{\upsilon_2-\upsilon_1}}.$$

Thus, we get

$$\prod_{k=1}^{4}\mathcal{L}_k = \psi^{\frac{2}{3}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{-\frac{1}{3}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{2}{3}}\left(\frac{\upsilon_1+4\upsilon_2}{4}\right)$$

$$\times\left[\left(\int_{\upsilon_1}^{\frac{3\upsilon_1+\upsilon_2}{4}}(\psi(x))^{dx}\right)\left(\int_{\frac{3\upsilon_1+\upsilon_2}{4}}^{\frac{\upsilon_1+\upsilon_2}{2}}(\psi(x))^{dx}\right)\right.$$

$$\left.\times\left(\int_{\frac{\upsilon_1+\upsilon_2}{2}}^{\frac{\upsilon_1+3\upsilon_2}{4}}(\psi(x))^{dx}\right)\left(\int_{\frac{\upsilon_1+3\upsilon_2}{2}}^{\upsilon_2}(\psi(x))^{dx}\right)\right]^{-\frac{1}{\upsilon_2-\upsilon_1}}$$

$$= \psi^{\frac{2}{3}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{-\frac{1}{3}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{2}{3}}\left(\frac{\upsilon_1+4\upsilon_2}{4}\right)\left(\int_{\upsilon_1}^{\upsilon_2}(\psi(x))^{dx}\right)^{-\frac{1}{\upsilon_2-\upsilon_1}}. \tag{14}$$

For the powers appearing in (11), *we have*

$$\begin{aligned}
\mathcal{A}_1(\varpi_1) &= \tfrac{1}{4(s+1)(s+2)}, & \mathcal{A}_2(\varpi_1) &= \tfrac{1}{4(s+2)},\\
\mathcal{A}_1(\varpi_2) &= \tfrac{5s+7}{12(s+1)(s+2)}, & \mathcal{A}_2(\varpi_2) &= \tfrac{2s+7}{12(s+1)(s+2)},\\
\mathcal{A}_1(\varpi_3) &= \tfrac{2s+7}{12(s+1)(s+2)}, & \mathcal{A}_2(\varpi_3) &= \tfrac{5s+7}{12(s+1)(s+2)},\\
\mathcal{A}_1(\varpi_4) &= \tfrac{1}{4(s+2)}, & \mathcal{A}_2(\varpi_4) &= \tfrac{1}{4(s+1)(s+2)}.
\end{aligned} \tag{15}$$

Thus, the right-hand side of (8) *will get the form:*

$$\begin{aligned}
&\leqslant \Bigg[\left(\psi^*\left(\upsilon_1\right)\right)^{\frac{1}{4(s+1)(s+2)}} \left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\right)^{\frac{1}{4(s+2)}}\\
&\quad \times \left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\right)^{\frac{5s+7}{12(s+1)(s+2)}} \left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{2s+7}{12(s+1)(s+2)}}\\
&\quad \times \left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{2s+7}{12(s+1)(s+2)}} \left(\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{5s+7}{12(s+1)(s+2)}}\\
&\quad \times \left(\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{1}{4(s+2)}} \left(\psi^*\left(\upsilon_2\right)\right)^{\frac{1}{4(s+1)(s+2)}} \Bigg]^{\frac{\upsilon_2-\upsilon_1}{16}},
\end{aligned}$$

or, finally, this form:

$$\begin{aligned}
\left|\prod_{k=1}^{4}\mathcal{L}_k\right| &\leqslant \Bigg[\left(\psi^*\left(\upsilon_1\right)\psi^*\left(\upsilon_2\right)\right)\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{4s+14}{3}}\\
&\quad \times \left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\frac{8s+10}{3}} \Bigg]^{\frac{1}{(s+1)(s+2)}\frac{\upsilon_2-\upsilon_1}{16}}.
\end{aligned} \tag{16}$$

It is obvious that (13) *follows from inequalities* (14) *and* (16). *The proof is complete.* □

Remark 7. *For $s = 1$ in* (13), *we obtain the result of Theorem 3.2 [17].*

Theorem 3. *Let ψ, ψ^*, ϖ_k be as in Lemma 3. If $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, ψ is increasing and $(\ln\psi^*)^q \in N^{s,2}_{h,m}(\mathcal{I})$, then it is true that*

$$\left|\prod_{k=1}^{2n}\mathcal{L}_k\right| \leqslant \prod_{k=1}^{2n}\Bigg[\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1^{\frac{1}{q}}(1)} \left(\psi^*\left(\mathbf{v}_k\right)\right)^{\mathcal{A}_2^{\frac{1}{q}}(1)} \Bigg]^{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{\frac{1}{p}}(p,\varpi_k)}. \tag{17}$$

Here $\mathcal{L}_k$ is defined above in Lemma 3 and $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{W}_k$, and $\mathbf{v}_k$, are defined in (3) for all k, with $\mathbf{v}_k \in [\upsilon_1, \upsilon_2]$. Moreover, $\mathbf{v}_{k-1} \leqslant \frac{\mathbf{v}_{k-1}}{m} < \mathbf{v}_k < \infty$, where $m \in (0,1]$.

Proof. Based on Lemma 3, the Hölder's inequality, and the modified (h,m) convexity of $(\ln \psi^*)^q$, we can write the following for the $\mathbb{I}_k$ from (6):

$$
\begin{aligned}
|\mathbb{I}_k| &\leqslant \exp\left\{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}\int_0^1 |\varpi_k(\varkappa)| \ln \psi^*\left((1-\varkappa)\mathbf{v}_{k-1} + \varkappa \mathbf{v}_k\right) d\varkappa\right\} \\
&\leqslant \exp\left\{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}\left(\int_0^1 |\varpi_k(\varkappa)|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1 (\ln \psi^*((1-\varkappa)\mathbf{v}_{k-1} + \varkappa \mathbf{v}_k))^q d\varkappa\right)^{\frac{1}{q}}\right\} \\
&\leqslant \exp\left\{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{\frac{1}{p}}(p, \varpi_k)\right. \\
&\quad \left.\times \left[\ln^q\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)\int_0^1 (1-h(\varkappa))^s d\varkappa + \ln^q \psi^*(\mathbf{v}_k)\int_0^1 h^s(\varkappa) d\varkappa\right]^{\frac{1}{q}}\right\} \\
&= \exp\left\{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{\frac{1}{p}}(p, \varpi_k)\right. \\
&\quad \left.\times \left[\mathcal{A}_1(1)\ln^q\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right) + \mathcal{A}_2(1)\ln^q \psi^*(\mathbf{v}_k)\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Employing the fact that $u^q + v^q \leqslant (u+v)^q$ for $u \geqslant 0$, $v \geqslant 0$ with $q \geqslant 1$, we get

$$
\begin{aligned}
|\mathbb{I}_k| &\leqslant \exp\left\{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{\frac{1}{p}}(p, \varpi_k)\right. \\
&\qquad \left.\times \left[\mathcal{A}_1^{\frac{1}{q}}(1)\ln\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right) + \mathcal{A}_2^{\frac{1}{q}}(1)\ln \psi^*(\mathbf{v}_k)\right]\right\} \\
&= \exp\left\{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{\frac{1}{p}}(p, \varpi_k)\right. \\
&\qquad \left.\times \left[\ln\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1^{\frac{1}{q}}(1)} + \ln\left(\psi^*(\mathbf{v}_k)\right)^{\mathcal{A}_2^{\frac{1}{q}}(1)}\right]\right\} \\
&= \left[\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1^{\frac{1}{q}}(1)}\left(\psi^*(\mathbf{v}_k)\right)^{\mathcal{A}_2^{\frac{1}{q}}(1)}\right]^{\frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{\frac{1}{p}}(p, \varpi_k)}. \qquad (18)
\end{aligned}
$$

Taking into account (18), we obtain (17): as we wanted. □

Remark 8. *We recover an analogue Theorem 3.6 of [19] by setting ψ, $n = 2$, $\varpi_1(\varkappa)$, $\varpi_2(\varkappa)$, $\varpi_3(\varkappa)$, $\varpi_4(\varkappa)$ as in Corollary 1, and supposing that $(\ln \psi^*)$ is s-convex.*

Corollary 3. *Under the conditions of Theorem 3, if we take for $n = 4$, $\varpi_1(\varkappa) = \varkappa$, $\varpi_2(\varkappa) = \varkappa - \frac{5}{3}$, $\varpi_3(\varkappa) = \varkappa + \frac{2}{3}$, and $\varpi_4(\varkappa) = \varkappa - 1$ and (ψ^*), corresponding to the s-log-convex case ($m = 1$ and $h(\varkappa) = \varkappa$), we obtain*

$$\psi^{\frac{2}{3}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{-\frac{1}{3}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{2}{3}}\left(\frac{\upsilon_1+4\upsilon_2}{4}\right)\Bigg(\int\limits_{\upsilon_1}^{\upsilon_2}(\psi(x))^{dx}\Bigg)^{-\frac{1}{\upsilon_2-\upsilon_1}} \tag{19}$$

$$\leqslant \Bigg[\psi^*(\upsilon_1)\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{10}{3}(p+1)^{\frac{1}{p}}}\psi^*(\upsilon_2)$$

$$\times\left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{1+\frac{5}{3}(p+1)^{\frac{1}{p}}}\Bigg]^{\frac{\upsilon_2-\upsilon_1}{16}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}}.$$

Proof. *Indeed, for the powers on the right-hand side of (17), we have*

$$\mathcal{W}_1^{\frac{1}{p}}(p,\varpi_1) = \Bigg(\int\limits_0^1 \varkappa^p\, d\varkappa\Bigg)^{\frac{1}{p}} = \left(\frac{1}{p+1}\right)^{\frac{1}{p}},$$

$$\mathcal{W}_2^{\frac{1}{p}}\left(p,\varpi_2\right) = \Bigg(\int\limits_0^1 \left(\frac{5}{3}-\varkappa\right)^p d\varkappa\Bigg)^{\frac{1}{p}} = \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{3}\right)^{\frac{1}{p}+1}\left(5^{p+1}-2^{p+1}\right)^{\frac{1}{p}},$$

$$\mathcal{W}_3^{\frac{1}{p}}(p,\varpi_3) = \Bigg(\int\limits_0^1 \left(\varkappa+\frac{2}{3}\right)^p d\varkappa\Bigg)^{\frac{1}{p}} = \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{3}\right)^{\frac{1}{p}+1}\left(5^{p+1}-2^{p+1}\right)^{\frac{1}{p}}$$

$$\mathcal{W}_4^{\frac{1}{p}}(p,\varpi_4) = \Bigg(\int\limits_0^1 (1-\varkappa)^p\, d\varkappa\Bigg)^{\frac{1}{p}} = \left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

$$\mathcal{A}_1^{\frac{1}{q}}(1) = \mathcal{A}_2^{\frac{1}{q}}(1) = \Bigg(\int\limits_0^1 (1-\varkappa)^s\, d\varkappa\Bigg)^{\frac{1}{q}} = \Bigg(\int\limits_0^1 \varkappa^s d\varkappa\Bigg)^{\frac{1}{q}} = \left(\frac{1}{s+1}\right)^{\frac{1}{q}}.$$

Then, for the right-hand side (17), we get

$$\leqslant \Bigg[\psi^*(\upsilon_1)\left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\right)^{\left(\frac{1}{3}\right)^{1+\frac{1}{p}}\left(5^{p+1}-2^{p+1}\right)^{\frac{1}{p}}}$$

$$\times\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\left(\frac{1}{3}\right)^{1+\frac{1}{p}}\left(5^{p+1}-2^{p+1}\right)^{\frac{1}{p}}}\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)$$
$$\times\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)^{\left(\frac{1}{3}\right)^{1+\frac{1}{p}}\left(5^{p+1}-2^{p+1}\right)^{\frac{1}{p}}}\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\right.$$
$$\left.\times\left(\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\left(\frac{1}{3}\right)^{1+\frac{1}{p}}\left(5^{p+1}-2^{p+1}\right)^{\frac{1}{p}}}\psi^*\left(\upsilon_2\right)\right]^{\frac{\upsilon_2-\upsilon_1}{16}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}};$$

taking into account inequality $(b^p-a^p)^{\frac{1}{p}}\leqslant\left(pb^{p-1}\right)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}$ *in* (2), *we have:*

$$\left(5^{p+1}-2^{p+1}\right)^{\frac{1}{p}}\leqslant\left((p+1)\,5^p\right)^{\frac{1}{p}}3^{\frac{1}{p}}=5\,(p+1)^{\frac{1}{p}}\,3^{\frac{1}{p}};$$

$$\leqslant\left[\psi^*\left(\upsilon_1\right)\left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\right)^{1+\frac{5}{3}(p+1)^{\frac{1}{p}}}\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{10}{3}(p+1)^{\frac{1}{p}}}\right.$$
$$\left.\times\left(\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{1+\frac{5}{3}(p+1)^{\frac{1}{p}}}\psi^*\left(\upsilon_2\right)\right]^{\frac{\upsilon_2-\upsilon_1}{16}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}}.$$

Take account (14) *to get* (19). *The proof is finished.* □

Remark 9. *In* (19), *it is sufficient to take* $s=1$ *to obtain an estimate for the classic* log-convex *function.*

Theorem 4. *Let* ψ, ψ^*, *and* ϖ_k *be as in Lemma* 3. *If* $q\geqslant 1$, ψ *is increasing and* $(\ln\psi^*)^q\in N^{s,2}_{h,m}(\mathcal{I})$, *then it is true that*

$$\left|\prod_{k=1}^{2n}\mathcal{L}_k\right|\leqslant\prod_{k=1}^{2n}\left[\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1^{\frac{1}{q}}(\varpi_k)}\left(\psi^*\left(\mathbf{v}_k\right)\right)^{\mathcal{A}_2^{\frac{1}{q}}(\varpi_k)}\right]^{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{1-\frac{1}{q}}(1,\varpi_k)}. \tag{20}$$

Here $\mathcal{L}_k$ *is defined above in Lemma* 3 and $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{W}_k$, *and* $\mathbf{v}_k$, *are defined in* (3) *for all* k, *with* $\mathbf{v}_k\in[\upsilon_1,\upsilon_2]$. *Moreover,* $\mathbf{v}_{k-1}\leqslant\frac{\mathbf{v}_{k-1}}{m}<\mathbf{v}_k<\infty$, *where* $m\in(0,1]$.

Proof. Let us adopt a method similar to the one applied in the Theorem 3 but using the Power-mean integral inequality; consider $\mathbb{I}_k$ from (6) and get the following expression:

$$|\mathbb{I}_k|=\exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\int_0^1|\varpi_k(\varkappa)|\,\ln\psi^*\left((1-\varkappa)\mathbf{v}_{k-1}+\varkappa\mathbf{v}_k\right)\,d\varkappa\right\}$$

$$\leqslant \exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\Big(\int\limits_0^1 |\varpi_k(\varkappa)|\,d\varkappa\Big)^{1-\frac{1}{q}}\right.$$

$$\left.\times\Big(\int\limits_0^1 |\varpi_k(\varkappa)|\Big[\ln\psi^*\left((1-\varkappa)\mathbf{v}_{k-1}+\varkappa\mathbf{v}_k\right)\Big]^q d\varkappa\Big)^{\frac{1}{q}}\right\}$$

$$=\exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{1-\frac{1}{q}}(1,\varpi_k)\left[\ln^q\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)\right.\right.$$

$$\left.\left.\times\int\limits_0^1|\varpi_k(\varkappa)|\,(1-h(\varkappa))^s d\varkappa+\ln^q\left(\psi^*\left(\mathbf{v}_k\right)\right)\int\limits_0^1|\varpi_k(\varkappa)|\,h^s(\varkappa)d\varkappa\right]^{\frac{1}{q}}\right\}$$

$$=\exp\left\{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{1-\frac{1}{q}}(1,\varpi_k)\right.$$

$$\left.\times\left[\mathcal{A}_1(\varpi_k)\ln^q\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)+\mathcal{A}_2(\varpi_k)\ln^q\left(\psi^*\left(\mathbf{v}_k\right)\right)\right]^{\frac{1}{q}}\right\}.$$

And finally, by analogy with (18), we have

$$\mathbb{I}_k\leqslant\left[\left(m\psi^*\left(\frac{\mathbf{v}_{k-1}}{m}\right)\right)^{\mathcal{A}_1^{\frac{1}{q}}(\varpi_k)}\left(\psi^*\left(\mathbf{v}_k\right)\right)^{\mathcal{A}_2^{\frac{1}{q}}(\varpi_k)}\right]^{\frac{\mathbf{v}_k-\mathbf{v}_{k-1}}{2n}\mathcal{W}_k^{1-\frac{1}{q}}(1,\varpi_k)}.$$

Hence, the proof is finished. □

Corollary 4. *Under conditions of Theorem 3, if we take for $n=2$, $\varpi_1(\varkappa)=\varkappa$, $\varpi_2(\varkappa)=\varkappa-\frac{5}{3}$, $\varpi_3(\varkappa)=\varkappa+\frac{2}{3}$, and $\varpi_4(\varkappa)=\varkappa-1$ and (ψ^*) corresponding to the $s-$log–convex case ($m=1$ and $h(\varkappa)=\varkappa$), we obtain*

$$\left|\psi^{\frac{2}{3}}\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)\psi^{-\frac{1}{3}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{2}{3}}\Big(\frac{\upsilon_1+4\upsilon_2}{4}\Big)\Big(\int\limits_{\upsilon_1}^{\upsilon_2}(\psi(\varkappa))^{d\varkappa}\Big)^{-\frac{1}{\upsilon_2-\upsilon_1}}\right|$$

$$\leqslant\left[\left(\psi^*\left(\upsilon_1\right)\psi^*\left(\upsilon_2\right)\right)\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{14}{3}\left(\frac{2}{7}s+1\right)^{\frac{1}{q}}}\right.$$

$$\left.\times\left(\psi^*\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)\psi^*\Big(\frac{\upsilon_1+3\upsilon_2}{4}\Big)\right)^{(s+1)^{\frac{1}{q}}+\frac{7}{3}\left(\frac{5}{7}s+1\right)^{\frac{1}{q}}}\right]^{\frac{\upsilon_2-\upsilon_1}{32}\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}}. \quad (21)$$

Proof. *The left-hand side of (21) was obtained in (14). In the right-hand* *side of (20), the exponents of the expressions are given by*

$$\mathcal{W}_1(1,\varpi_1)=\int_0^1 \varkappa d\varkappa=\tfrac{1}{2},\qquad \mathcal{W}_2(1,\varpi_2)=\int_0^1\left(\tfrac{5}{3}-\varkappa\right)d\varkappa=\tfrac{5}{3}-\tfrac{1}{2}=\tfrac{7}{6},$$

$$\mathcal{W}_3(1,\varpi_3)=\int_0^1\left(\varkappa+\tfrac{2}{3}\right)d\varkappa=\tfrac{7}{6},\quad \mathcal{W}_4(1,\varpi_4)=\int_0^1(-\varkappa+1)\,d\varkappa=\tfrac{1}{2}.$$

Moreover, the quantities $\mathcal{A}_1(\varpi_i)$ *and* $\mathcal{A}_2(\varpi_i)$, *for* $i=1,2,3,4$, *are given in (15). Then for the right-hand side of (20) we get*

$$\begin{aligned}\leqslant\Bigg[&\left(\psi^*(\upsilon_1)\,\psi^*(\upsilon_2)\right)^{\left(\frac{1}{4(s+1)(s+2)}\right)^{\frac{1}{q}}}\\&\times\left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\left(\frac{1}{4(s+2)}\right)^{\frac{1}{q}}}\Bigg]^{\frac{\upsilon_2-\upsilon_1}{2^{5-\frac{1}{q}}}}\\&\times\Bigg[\left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{\left(\frac{5s+7}{12(s+1)(s+2)}\right)^{\frac{1}{q}}}\\&\times\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{2\left(\frac{2s+7}{12(s+1)(s+2)}\right)^{\frac{1}{q}}}\Bigg]^{\frac{\upsilon_2-\upsilon_1}{16}\left(\frac{7}{6}\right)^{1-\frac{1}{q}}}.\end{aligned}$$

Or, after some transformations,

$$\begin{aligned}\leqslant\Bigg[&\left(\psi^*(\upsilon_1)\,\psi^*(\upsilon_2)\right)\left(\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right)^{\frac{14}{3}\left(\frac{2}{7}s+1\right)^{\frac{1}{q}}}\\&\times\left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^{(s+1)^{\frac{1}{q}}+\frac{7}{3}\left(\frac{5}{7}s+1\right)^{\frac{1}{q}}}\Bigg]^{\frac{\upsilon_2-\upsilon_1}{32}\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}}.\end{aligned}$$

Thus, the proof is complete. □

Remark 10. *For* $s=1$ *and* $q=1$, *from (21) we obtain an estimate for the classical* log-convex *function:*

$$\left|\psi^{\frac{2}{3}}\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^{-\frac{1}{3}}\left(\frac{\upsilon_1+\upsilon_2}{4}\right)\psi^{\frac{2}{3}}\left(\frac{\upsilon_1+4\upsilon_2}{4}\right)\left(\int_{\upsilon_1}^{\upsilon_2}(\psi(\varkappa))^{d\varkappa}\right)^{-\frac{1}{\upsilon_2-\upsilon_1}}\right.$$

$$\leqslant\left[\psi^*(\upsilon_1)\left(\psi^*\left(\frac{3\upsilon_1+\upsilon_2}{4}\right)\psi^*\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\psi^*\left(\frac{\upsilon_1+3\upsilon_2}{4}\right)\right)^6\psi^*(\upsilon_2)\right]^{\frac{\upsilon_2-\upsilon_1}{32}}.$$

4. Conclusions. In this paper, we have introduced a new framework for Simpson-type inequalities within the setting of multiplicative calculus by employing modified (h,m)-logarithmically convex functions and weighted multiplicative fractional integral operators. By deriving novel integral identities, we have established several refined inequalities that extend and generalize classical Simpson-type results. The proposed approach unifies and broadens existing inequalities but also highlights the analytical strength of multiplicative methods.

Furthermore, we have shown that some of the obtained results reduce to previously known inequalities for particular choices of parameters and weight functions, thereby confirming the validity and generality of the presented framework. These findings emphasize the flexibility of the multiplicative setting in capturing various generalized convexity structures and improving integral estimates.

We believe that the methods and results developed here can stimulate further research on multiplicative convexity and its applications. Future directions may include exploring extensions Newton-Cotes formulas, applications to special means, and connections with computational methods.

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