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MULTIVALENT α — CONVEX HARMONIC MAPPINGS

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In this paper we give coefficient conditions for complex-valued harmonic functions that are multivalent, sense-preserving and α - convex. We determine the extreme points, distortion and covering theorems for these mappings.

§ 1. Introduction

A continuous function $f = u + iv$ is said to be a complex-valued harmonic function in a domain $\Omega \subset \mathbb{C}$ if both u and v are real harmonic in Ω. In any simply connected domain D such a mapping f can be written in the form

$$f(z) = h(z) + \overline{g(z)}, \tag{1}$$

where $h(z)$ and $g(z)$ are analytic in D (see [1]). The Jacobian of f is then given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 .$$

A harmonic function f of the form (1) will be called sense-preserving at a point $z_0 \in D$ if $h'(z) \not\equiv 0$ and the second dilatation function $\omega(z) = \frac{g'(z)}{h'(z)}$ is analytic at z_0 (possibly with a removable singularity), and $|\omega(z_0)| \leq 1$. Note that if $J_f(z) > 0$ for each $z \in D$ then f is sense-preserving in D.

Suppose that $f(z_0) = 0$ at some $z_0 \in D$ where f is sense-preserving, then we may express the analytic functions h and g as

$$h(z) = a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n, \quad g(z) = b_0 + \sum_{n=1}^{\infty} b_n (z - z_0)^n.$$

We see at once that $b_0 = -\overline{a_0}$, because $f(z_0) = 0$. Since $h'(z) \neq 0$ in D, it follows that some a_n must be nonzero. We denote the first such a coefficient by a_m. Then we have $b_n = 0$ for $1 \leq n < m$ and $|b_m| < |a_m|$, because the second dilatation $\omega(z)$ is analytic at the point z_0 and $|\omega(z_0)| < 1$. In this case we say that $f(z)$ has a zero of order m at $z_0 \in D$.

Let D be the open unit disk $\Delta = \{z : |z| < 1\}$. We can certainly assume that $a_n = b_n = 0$ for $0 \leq n < m$ and $a_m = 1$. Denote by $S_H(m)$ the set of all m-valent harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} \overline{b_{n+m-1} z^{n+m-1}}, \tag{2}$$

that are sense-preserving in Δ. Since $|b_m| < |a_m|$, we see that $|b_m| < 1$. By the argument principle for harmonic functions [2] and the above arguments, if $J_f(z) > 0$ in $\Delta \setminus \{0\}$ then f of the form (2) belongs to the class $S_H(m)$.Note that $S_H(1)$ is the familiar class S_H of harmonic univalent and sense-preserving functions (see [1]).

We say that $f \in S_H(m)$ is harmonic convex of order α (e.g. see [4]), $0 \leq \alpha < 1$ in Δ if

$$\frac{\partial}{\partial\theta}\left\{\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right\} \geq \alpha, \tag{3}$$

for each z, $|z| = r < 1$.

Let us denote by $K_H(m, \alpha)$ the subclass of $S_H(m)$ consisting of functions f that are convex of order α. In particular, we will denote by $K_H(m)$ the class $K_H(m, 0)$.

We further denote by $TK_H(m, \alpha)$ the subclass of $K_H(m, \alpha)$ consisting of functions $f = h + \overline{g}$ so that h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}|\, z^{n+m-1}, \quad g(z) = -\sum_{n=1}^{\infty} |b_{n+m-1}|\, z^{n+m-1}. \tag{4}$$

§ 2. Coefficient conditions

In this section we proved a sufficient coefficient condition for the class $K_H(m, \alpha)$.It is also shown that those condition is necessary when $f \in TK_H(m, \alpha)$.Those results are a generalization of the theorems for the classes $K_H(1, \alpha)$ and $TK_H(1, \alpha)$ of convex univalent harmonic mappings of order α and convex univalent harmonic mappings of order α with negative coefficients, respectively (see [3]).

THEOREM 1. *Let $f(z) = h(z) + \overline{g(z)}$ be of the form (2). If*

$$\sum_{n=1}^{\infty} \frac{n+m-1}{m} \left(\frac{n+m-\alpha-1}{m-\alpha} |a_{n+m-1}| + \frac{n+m+\alpha-1}{m-\alpha} |b_{n+m-1}| \right) \leq 2, \tag{5}$$

where $a_m = 1$ and $m \geq 1$, then $f(z) \in K_H(m, \alpha)$.

PROOF. We first prove that the coefficient condition (5) is sufficient for the function

$$f(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} \overline{b_{n+m-1} z^{n+m-1}}$$

to be sense-preserving in Δ.

Let us first observe that for each pair of numbers $m, n \in \mathbb{N}$ and $0 \leq \alpha < 1$ we have

$$1 \leq \frac{n+m-\alpha-1}{m-\alpha} \leq \frac{n+m+\alpha-1}{m-\alpha}. \tag{6}$$

Set $h(z) = z^m + \sum\limits_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}$ and $g(z) = \sum\limits_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}$. Obviously, the second dilatation function $\omega(z) = \frac{h'(z)}{g'(z)}$ has the removable singularity at $z_0 = 0$.

From (5) and (6) we conclude that for $0 < |z| < 1$ we have

$$\begin{aligned}
|h'(z)| &\geq m|z|^{m-1} - \sum_{n=2}^{\infty} (n+m-1)|a_{n+m-1}||z|^{n+m-2} = \\
&= m|z|^{m-1} \Big[1 - \sum_{n=2}^{\infty} \frac{n+m-1}{m} |a_{n+m-1}||z|^{n-1} \Big] > \\
&> m|z|^{m-1} \Big[1 - \sum_{n=2}^{\infty} \frac{n+m-1}{m} |a_{n+m-1}| \Big] \geq \\
&\geq m|z|^{m-1} \Big[1 - \sum_{n=2}^{\infty} \frac{(n+m-1)(n+m-\alpha-1)}{m(m-\alpha)} |a_{n+m-1}| \Big] \geq \\
&\geq m|z|^{m-1} \Big[\sum_{n=1}^{\infty} \frac{(m+n-1)(n+m+\alpha-1)}{m(m-\alpha)} |b_{n+m-1}| \Big] \geq
\end{aligned}$$

$$\geq m|z|^{m-1}\Big[\sum_{n=1}^{\infty}\frac{n+m-1}{m}|b_{n+m-1}|\Big] \geq$$

$$\geq \sum_{n=1}^{\infty}(n+m-1)|b_{n+m-1}|\,|z|^{n+m-2} \geq$$

$$\geq \left|\sum_{n=1}^{\infty}(n+m-1)b_{n+m-1}z^{n+m-2}\right| = |g'(z)|\,.$$

Therefore the harmonic function $f = h + \overline{g}$ of the form (2) is sense-preserving in Δ.

We next show that $f \in K_H(m,\alpha)$. By the definition of the class $K_H(m,\alpha)$, it remains to prove that

$$\frac{\partial}{\partial\theta}\left\{\arg\Big(\frac{\partial}{\partial\theta}f(re^{i\theta})\Big)\right\} = \mathrm{Re}\Big[\frac{zh'(z)+z^2h''(z)+\overline{zg'(z)+z^2g''(z)}}{zh'(z)-\overline{zg'(z)}}\Big] \geq \alpha,$$

or equivalently if

$$\mathrm{Re}\Big[\frac{zh'(z)+z^2h''(z)+\overline{zg'(z)+z^2g''(z)}}{zh'(z)-\overline{zg'(z)}} - \alpha\Big] \geq 0,$$

for each $z = re^{i\theta}$, $|z| < 1$. For $z \in \Delta$ we have

$$\mathrm{Re}\Big[\frac{zh'(z)+z^2h''(z)+\overline{zg'(z)}+\overline{z^2g''(z)}}{zh'(z)-\overline{zg'(z)}} - \alpha\Big] =$$

$$= \mathrm{Re}\Big[\Big((m-\alpha)z^m + \sum_{n=2}^{\infty}\frac{n+m-1}{m}(n+m-\alpha-1)a_{n+m-1}z^{n+m-1} +$$

$$+\sum_{n=1}^{\infty}\frac{n+m-1}{m}(n+m+\alpha-1)\overline{b_{n+m-1}z^{n+m-1}}\Big)\Big/$$

$$\Big/\Big(z^m + \sum_{n=2}^{\infty}\frac{n+m-1}{m}a_{n+m-1}z^{n+m-1} - \sum_{n=1}^{\infty}\frac{n+m-1}{m}\overline{b_{n+m-1}z^{n+m-1}}\Big)\Big].$$

Let $z = re^{i\theta}$, $0 < r < 1$, then by the above

$$\frac{\partial}{\partial\theta}\left\{\arg\Big(\frac{\partial}{\partial\theta}f(re^{i\theta})\Big)\right\} - \alpha = (m-\alpha)\,\mathrm{Re}\Big[\frac{1+p(re^{i\theta})}{1-p(re^{i\theta})}\Big],$$

where

$$p(re^{i\theta}) = (\sum_{n=2}^{\infty} \frac{n+m-1}{m}(n-1)a_{n+m-1}r^{n-1}e^{i(n-1)\theta} +$$
$$+ \sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+2m-1)\overline{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta})\Big/$$
$$\Big/(2(m-\alpha) + \sum_{n=2}^{\infty} \frac{n+m-1}{m}(n+2m-2\alpha-1)a_{n+m-1}r^{n-1}e^{i(n-1)\theta} +$$
$$+ \sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+2\alpha-1)\overline{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}),$$

as it is easy to check.

The proof will be complete if we can show that $\left|p(re^{i\theta})\right| < 1$. We have

$$\left|p(re^{i\theta})\right| \leq (\sum_{n=2}^{\infty} \frac{n+m-1}{m}(n-1)\left|a_{n+m-1}\right| r^{n-1} +$$
$$+ \sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+2m-1)\left|b_{n+m-1}\right| r^{n-1})\Big/$$
$$\Big/(4(m-\alpha) - \sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+2m-2\alpha-1)\left|a_{n+m-1}\right| r^{n-1} -$$
$$- \sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+2\alpha-1)\left|b_{n+m-1}\right| r^{n-1}) <$$
$$< (\sum_{n=2}^{\infty}(n-1)\frac{n+m-1}{m}\left|a_{n+m-1}\right| + \sum_{n=1}^{\infty}\frac{n+m-1}{m}(n+2m-1)\left|b_{n+m-1}\right|)\Big/$$
$$\Big/(4(m-\alpha) - \sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+2m-2\alpha-1)\left|a_{n+m-1}\right| -$$
$$- \sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+2\alpha-1)\left|b_{n+m-1}\right|) = \frac{R(m)}{Q(m,\alpha)} \leq 1,$$

which is due to the fact that

$$Q(m,\alpha)-R(m)=$$
$$=2(m-\alpha)\Big[2-\sum_{n=1}^{\infty}\frac{n+m-1}{m}\Big(\frac{n+m-\alpha-1}{m-\alpha}\,|a_{n+m-1}|+$$
$$+\frac{n+m+\alpha-1}{m-\alpha}\,|b_{n+m-1}|\Big)\Big]\geq 0,$$

by (5).

From this we conclude that $\mathrm{Re}\left[\frac{1+p(re^{i\theta})}{1-p(re^{i\theta})}\right]\geq 0$, which is the desired conclusion. □

The restrictions in the above Theorem placed on the moduli of the coefficients enable us to conclude for arbitrary rotations of the coefficients of f that the resulting functions would still be in the class $K_H(m,\alpha)$. Now we show that such coefficient bounds can't be improved.

THEOREM 2. *Let $f(z)=h(z)+\overline{g(z)}$ be of the form (4). Then $f\in TK_H(m,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{n+m-1}{m}\left(\frac{n+m-\alpha-1}{m-\alpha}\,|a_{n+m-1}|+\frac{n+m+\alpha-1}{m-\alpha}|b_{n+m-1}|\right)\leq 2, \tag{7}$$

where $a_m=1$ and $m\geq 1$.

PROOF. In view of Theorem 1, we need only show that $f\notin TK_H(m,\alpha)$ if the coefficient condition (7) does not hold.We examine the required condition (3) for $f=h+\overline{g}\in TK_H(m,\alpha)$. By the above this is equivalent to

$$\mathrm{Re}\Big[\frac{zh'(z)+z^2h''(z)+\overline{zg'(z)+z^2g''(z)}}{zh'(z)-\overline{zg'(z)}}-\alpha\Big]=$$
$$=\mathrm{Re}\Big[\Big((m-\alpha)z^m-\sum_{n=2}^{\infty}\frac{(n+m-1)(n+m-\alpha-1)}{m}|a_{n+m-1}|z^{n+m-1}-$$
$$-\sum_{n=1}^{\infty}\frac{(n+m-1)(n+m+\alpha-1)}{m}|b_{n+m-1}|\overline{z^{n+m-1}}\Big)\Big/\Big(z^m-$$
$$-\sum_{n=2}^{\infty}\frac{n+m-1}{m}|a_{n+m-1}|z^{n+m-1}+\sum_{n=1}^{\infty}|b_{n+m-1}|\frac{n+m-1}{m}\overline{z^{n+m-1}}\Big)\Big]\geq 0.$$

The above condition must hold for all $z \in \Delta$.Upon choosing the values of z on positive real axis and such that $0 \leq z = r < 1$ we must have

$$((m-\alpha) - le\sum_{n=2}^{\infty} \frac{n+m-1}{m}(n+m-\alpha-1)\,|a_{n+m-1}|\,r^{n-1} -$$

$$-\sum_{n=1}^{\infty} \frac{n+m-1}{m}(n+m+\alpha-1)\,|b_{n+m-1}|\,r^{n-1})\Big/$$

$$\Big/(1- \leq \sum_{n=2}^{\infty} \frac{n+m-1}{m}\,|a_{n+m-1}|\,r^{n-1} + \sum_{n=1}^{\infty} \frac{n+m-1}{m}\,|b_{n+m-1}|\,r^{n-1}) \geq 0.$$

If the condition (7) does not hold then the numerator in (2) is negative for r sufficiently close to 1.Thus there exists $r_0 \in (0,1)$ for which the quotient in (2) is negative, and we arrive at a contradiction. □

Next theorem shows that class $TK_H(m, \alpha)$ is closed under forming convex combinations.

THEOREM 3. *If* $f_i(z) \in TK_H(m,\alpha)$ *for* $i = 1, 2, \ldots,$ *and if* $\sum_{i=1}^{\infty} \lambda_i = 1,\ 0 \leq \lambda_i \leq 1$, *then* $g(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z)$ *is a member of the class* $TK_H(m,\alpha)$.

PROOF. Since

$$f_i(z) = z^m - \sum_{n=2}^{\infty} |a^i_{n+m-1}| z^{n+m-1} - \sum_{n=1}^{\infty} |b^i_{n+m-1}| \overline{z^{n+m-1}} \in TK_H(m,\alpha).$$

Theorem 2 shows that for each $i \in \mathbb{N}$ we have

$$\sum_{n=2}^{\infty} \frac{n+m-1}{m}\Big[\frac{n+m-\alpha-1}{m-\alpha}|a^i_{n+m-1}| + \frac{n+m+\alpha-1}{m-\alpha}|b^i_{n+m-1}|\Big] \leq 2. \tag{8}$$

For $\sum_{n=1}^{\infty} \lambda_i = 1,\ 0 \leq \lambda_i \leq 1$, the convex combination $g(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z)$ is of the form

$$g(z) = z^m - \sum_{n=2}^{\infty}(\sum_{i=1}^{\infty} \lambda_i |a^i_{n+m-1}|) z^{n+m-1} - \sum_{n=1}^{\infty}(\sum_{i=1}^{\infty} \lambda_i |b^i_{n+m-1}|)\overline{z^{n+m-1}}.$$

We check at once that

$$\sum_{n=2}^{\infty}\frac{n+m-1}{m}\Big[\frac{n+m-\alpha-1}{m-\alpha}(\sum_{i=1}^{\infty}\lambda_i|a^i_{n+m-1}|)+$$

$$+\frac{n+m+\alpha-1}{m-\alpha}(\sum_{i=1}^{\infty}\lambda_i|b^i_{n+m-1}|)\Big]\le 1,$$

which is clear from (8).Theorem 2 implies that $g(z)\in TK_H(m,\alpha)$. □

§ 3. Distortion bounds and extreme points

We now give the distortion bounds for functions in $TK_H(m,\alpha)$, which yield a covering result for this class.

THEOREM 4. *If $f\in TK_H(m,\alpha)$, then*

$$(i)\quad |f(z)|\le(1+|b_m|)r^m+\frac{m(m-\alpha)-m(m+\alpha)\,|b_m|}{(m+1)(m-\alpha+1)}r^{m+1},$$

and

$$(ii)\quad |f(z)|\ge(1-|b_m|)r^m-\frac{m(m-\alpha)-m(m+\alpha)\,|b_m|}{(m+1)(m-\alpha+1)}r^{m+1},$$

where $|z|=r<1$.

Equalities are rendered by the function

$$f(z)=z^m-|b_m|\,\overline{z^m}+\frac{m(m-\alpha)-m(m+\alpha)\,|b_m|}{(m+1)(m-\alpha+1)}\overline{z^{m+1}}$$

and its rotations.

PROOF.

We shall justify the (i) right hand inequality only. For $|z|=r$, we have

$$|f(z)|=\left|z^m-\sum_{n=2}^{\infty}|a_{n+m-1}|\,z^{n+m-1}+\sum_{n=1}^{\infty}|b_{n+m-1}|\,\overline{z^{n+m-1}}\right|\le$$

$$\le r^m+\sum_{n=2}^{\infty}|a_{n+m-1}|\,r^{n+m-1}+\sum_{n=1}^{\infty}|b_{n+m-1}|\,r^{n+m-1}=$$

$$= (1+|b_{n+m-1}|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|)r^{n+m-1} \leq$$

$$\leq (1+|b_{n+m-1}|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|)r^{m+1}.$$

Theorem 1 now shows that

$$m(m-\alpha)+m(m+\alpha)\,|b_m|+(m+1)(m-\alpha+1)\sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|) \leq$$

$$\leq \sum_{n=1}^{\infty}(n+m-1)\big[\,|a_{n+m-1}|\,(n+m-\alpha-1)+|b_{n+m-1}|\,(n+m+\alpha-1)\big] \leq$$

$$\leq 2m(m-\alpha),$$

and hence

$$\sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|) \leq \frac{m(m-\alpha)-m(m+\alpha)\,|b_m|}{(m+1)(m-\alpha+1)},$$

which establishes the formula. □

REMARK. *Bounds given in Theorem 4 also are valid for $f \in K_H(m,\alpha)$ if the coefficient condition (5) is satisfied.*

Letting $r \to 1^-$ in the left hand inequality in Theorem 4 we obtain a covering result for the class $TK_H(m,\alpha)$.

COROLARY. *If $f \in TK_H(m,\alpha)$, then*

$$\left\{w : |w| < \frac{2m-\alpha+1-|b_m|\,(2m+1)(1-\alpha)}{(m+1)(m-\alpha+1)}\right\} \subset f(\Delta).$$

In particular, if $f \in TK_H(m,0) = TK_H$ then $\left\{w : |w| < \frac{2m+1}{(m+1)^2}(1-|b_m|)\right\} \subset f(\Delta)$.

For any compact family, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull.Since $TK_H(m,\alpha)$ is a convex family, we will use the necessary and sufficient condition of Theorem 2 to determine the extreme points.

THEOREM 5. *Let*

$$h_m(z) = z^m,\ h(z)_{n+m-1}(z) = z^m - \frac{m(m-\alpha)}{(n+m-1)(n+m-\alpha-1)} z^{n+m-1},$$

$n = 2, 3, \ldots$ *and*

$$g_{n+m-1}(z) = z^m - \frac{m(m-\alpha)}{(n+m-1)(n+m+\alpha-1)} \overline{z^{n+m-1}},\ n = 1, 2, \ldots .$$

Then $f = h + \overline{g} \in TK_H(m, \alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \Big[\lambda_{n+m-1} h_{n+m-1}(z) + \mu_{n+m-1} g_{n+m-1}(z) \Big],$$

where $\lambda_{n+m-1} \geq 0,\ \mu_{n+m-1} \geq 0$ *and* $\sum_{n=1}^{\infty} (\lambda_{n+m-1} + \mu_{n+m-1}) = 1.$

In particular, the extreme points of $TK_H(m, \alpha)$ are $\{h_{n+m-1}(z)\}$ and $\{g_{n+m-1}(z)\}$.

PROOF. Suppose that

$$f(z) = \sum_{n=1}^{\infty} \Big[\lambda_{n+m-1} h_{n+m-1}(z) + \mu_{n+m-1} g_{n+m-1}(z) \Big] =$$

$$= z^m - \sum_{n=2}^{\infty} \frac{m(m-\alpha)}{(n+m-1)(n+m-\alpha-1)} \lambda_{n+m-1} z^{n+m-1} -$$

$$- \sum_{n=1}^{\infty} \frac{m(m-\alpha)}{(n+m-1)(m+n+\alpha-1)} \mu_{n+m-1} \overline{z^{n+m-1}}.$$

Then

$$\sum_{n=1}^{\infty} \frac{n+m-1}{m} \Big[\frac{n+m-\alpha-1}{m-\alpha} \Big(\frac{m(m-\alpha)}{n+m-\alpha-1} \lambda_{n+m-1} \Big) +$$

$$+ \frac{n+m+\alpha-1}{m-\alpha} \Big(\frac{m(m-\alpha)}{(n+m-1)(n+m+\alpha-1)} \mu_{n+m-1} \Big) \Big] \leq 2,$$

and by Theorem 2 $f \in TK_H(m, \alpha)$. Conversely, if

$$f = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}|\, z^{n+m-1} - \sum_{n=1}^{\infty} |b_{n+m-1}|\, z^{n+m-1} \in TK_H(m, \alpha),$$

then by Theorem 2 we have

$$|a_{n+m-1}| \leq \frac{m(m-\alpha)}{(n+m-1)(n+m-\alpha-1)} \quad \text{and} \quad |b_{n+m-1}| \leq \frac{m(m-\alpha)}{(n+m-1)(n+m+\alpha-1)}.$$

Consider

$$\lambda_{n+m-1} = \frac{(n+m-1)(n+m-\alpha-1)}{m(m-\alpha)} |a_{n+m-1}|, \ n = 2, 3, ...,$$

$$\mu_{n+m-1} = \frac{(n+m-1)(n+m+\alpha-1)}{m(m-\alpha)} |b_{n+m-1}|, \ n = 1, 2, ...,$$

$$\lambda_m = 1 - \sum_{n=2}^{\infty} (\lambda_{n+m-1} + \mu_{n+m-1}).$$

Then we obtain $f(z) = \sum_{n=1}^{\infty} \left[\lambda_{n+m-1} h_{n+m-1}(z) + \mu_{n+m-1} g_{n+m-1}(z)\right]$, as required. □

REMARK. *If the co-analytic part of $f = h + \overline{g} \in S^*_H(m, \alpha)$ is is zero, i.e. the function g(z) is identically zero, then we have analogous properties of m-valent analytic convex functions of order α in the unit disk.*

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