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OPTIMAL CONTROL PROBLEM OF SOME DIFFERENTIAL INCLUSION AND APPROXIMATIO N

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In this paper we present the optimal control problem governed by a variational inclusion with the monotone operator and a quadratic costfunctional. We apply standart Galerkin method to the approximation of the problem. After giving some results on the existance of optimal control we shall prove the existance of weak condensation points of a set of solution of approximate problems. Each of these points is a solution of the initial optimization problem. Finally we shall give a simple example using the obtaned results.

§ 1. Introduction

The problems connected with inclusions were considered by many authors. The recent results were published by [2, 5, 9] and others. The recent results concerning optimal control for systems governed by the inclusions were published by [1, 7, 11].

We consider the optimal control problem governed by a second order differential inclusion with a linear continuous operator and a nonlinear multivalued maximal monotone operator. We apply standard Galerkin technique (see [4, 6]) and provide a convergence analysis.

The main result of our paper is the theorem proving the convergence of optimal values for approximated control problems to those of the original problem.

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Let V, H be two real Hilbert spaces such that $V \subset H$ and the inclusion mapping of V into H is continuous and compact. V^* denotes the dual space of V and H is identified with its own dual H^* (see [2, 13]).

We shall consider the following nonlinear second order differential inclusion

$$y''(t) + Ay(t) + \partial\varphi(y'(t)) \ni Bu(t), \quad \text{for } t \in (0,T) \tag{1}$$

with the initial conditions

$$y(0) = y_0 \quad \text{and} \quad y'(0) = y_1 \tag{2}$$

where y' denotes the generalised derivative on the interval $(0,T)$ of the function $y : [0,T] \to V$ and $0 < T < \infty$ ([8, 13]).

We assume that:

(i) The operator $A : V \to V^*$ is a linear continuous symmetric and coercive operator i.e. there existes a positive constant α such that $\langle Av, v\rangle \geq \alpha ||v||_V^2 \ \ \forall\, v \in V$ where $\langle \cdot,\cdot \rangle$ denotes the duality relation between the adequate spaces (see [2]).

(ii) $\partial\varphi$ is the subdifferential of a lower-semicontinuous proper convex function $\varphi : V \to \mathbb{R} \cup \{\infty\}$ and $0 \in \partial\varphi(0)$ (see [2, 13]).

(iii) The operator $B : U \to V^*$ is a linear continuous operator and U is a Hilbert space.

(iv) The function $f : [0,T] \ni t \to f(t) = Bu(t)$ is of the class $W^{1,2}(0,T;H)$ (see [2]), $y_0 \in V$, $\varphi(y_1) < \infty$ and $\{Ay_0 + \partial\varphi(y_1)\} \cap H \neq \emptyset$.

The differential inclusion (1) is equivalent to the next variational inequality (see [2])

$$\langle y''(t) + Ay(t) - Bu(t), z - y'(t)\rangle \geq \varphi(y'(t)) - \varphi(z) \qquad \text{a.e. } t \in (0,T) \text{ and } \forall\, z \in V \tag{3}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality relation between the adequate spaces [8].

Theorem 1. *Let the assumptions (i) – (iv) be satisfied. Then there exists a unique solution y of the problem (1) – (2) such that: $y \in C([0,T];V)$, $y' \in C([0,T];H) \cap L^\infty(0,T;V)$, $y'' \in L^\infty(0,T;H)$ (see [2]).*

Let F denote the operator $F : L^2(0,T;U) \to L^2(0,T;V) \times L^2(0,T;H)$ such that $F(u) = (y, y')$, where y is the solution of (1) – (2).

LEMMA 1. *If the assumptions (i) – (iv) are satisfied then the operator F is the Lipschitz map. Moreover, the operator F is weakly continuous map.*

PROOF. In the first part of the proof we shall present Lipschitz continuity of the operator F. Let $F(u) = (y, y')$ and $F(\overline{u}) = (\overline{y}, \overline{y}')$ for $u, \overline{u} \in U$. Using the monotonicity of the subdifferential $\partial\varphi$ in (1) we obtain

$$\langle Bu(t) - y''(t) - Ay(t) - B\overline{u}(t) + \overline{y}''(t) + A\overline{y}(t), y'(t) - \overline{y}'(t)\rangle \geq 0$$

and using linearity of the operators A and B we have

$$\begin{aligned}\langle y''(t) - \overline{y}''(t), y'(t) - \overline{y}'(t)\rangle + \langle A(y(t) - \overline{y}(t)), y(t) - \overline{y}'(t)\rangle \leq \\ \leq \langle B(u(t) - \overline{u}(t)), y'(t) - \overline{y}'(t)\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|y'(t) - \overline{y}'(t)\|^2\right) + \langle A(y(t) - \overline{y}(t)), y'(t) - \overline{y}'(t)\rangle \leq \\ \leq \langle B(u(t) - \overline{u}(t), y'(t) - \overline{y}'(t)\rangle.\end{aligned} \tag{4}$$

By integration (4) over an arbitrary interval $[0, t] \subset [0, T]$ with the assumptions (i) – (iv), and with $2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2$ for $\varepsilon > 0$ and $a, b \in \mathbb{R}$ (applying Schwartz's inequality) we have

$$\begin{aligned}\|y'(t) - \overline{y}'(t)\|_H^2 + \alpha\|y(t) - \overline{y}(t)\|_V^2 \leq \\ \leq c_1\left(\int_0^t \|u(s) - \overline{u}(s)\|_U^2 ds + \int_0^t \|y'(s) - \overline{y}'(s)\|_H^2 ds\right)\end{aligned} \tag{5}$$

for certain $c_1 > 0$ and a.a. $t \in [0, T]$. From (5) by Gronwall's inequality (see [8]) we obtain

$$\|y'(t) - \overline{y}'(t)\|_H^2 + \|y(t) - \overline{y}(t)\|_V^2 \leq c_2 \int_0^T \|u(s) - \overline{u}(s)\|_U^2 ds \tag{6}$$

for certain $c_2 > 0$. Inequality (6) implies that the operator F is Lipschitz map. Thus we have proved the first part of the Lemma. Let a sequence (u_n) satisfy the following condition

$$u_n \rightharpoonup \overline{u} \quad \text{weaklyin } \mathrm{L}^2(0, \mathrm{T}; \mathrm{U}) \tag{7}$$

and y_n satisfy differential inclusion (1) with $u = u_n$ i.e

$$y_n''(t) + Ay_n(t) + \partial\varphi(y_n'(t)) \ni Bu_n(t) \tag{8}$$

and initial conditions (2) i.e.

$$y_n(0) = y_0, \quad y_n'(0) = y_1. \tag{9}$$

From the Theorem 1 we know that the problem (8) – (9) has exactly one solution y_n for $n \in \mathbb{N}$. From the assumption of Lemma and from the first part of the proof we obtain

$$||y_n'(t)||_H^2 + ||y(t)||_V^2 \le c||u_n||_{L^2(0,T;U)}^2 \tag{10}$$

for certain $c > 0$ and a.a. $t \in (0,T)$. Further from (8) we infer that the subsequence (y_n'') is bounded in $L^2(0,T;H)$ too. From the assumption (7) and (10) follows that there exists a subsequence, which we also denote (y_n), converging weakly to an element $\overline{y}$ in $L^2(0,T;V)$ and respectively its subsequence (y_n') converges weakly to $\overline{y}'$ in $L^2(0,T;H)$ and also subsequence (y_n'') converges weakly to $\overline{y}''$ in

$L^2(0,T;H)$. This implies that the operator F is weakly continuous map by the demiclosedness of $\partial\varphi$ and by the unique solution of the problem (1) – (2). □

§ 2. Optimal Control Problem

Let there be given a space of controls $L^2(0,T;U)$ and elements $y_d, y_d' \in L^2(0,T;H)$.

The optimal control problem (P) can be stated as follows [3, 10]: find a control $u^0 \in L^2(0,T;U)$ which minimizes the integral functional

$$\begin{gathered} J(y,u) = \lambda_1 \int_0^T ||y(t) - y_d||_H^2 dt + \\ +\lambda_2 \int_0^T ||y'(t) - y_d'||_H^2 dt + \int_0^T ||u(t)||_U^2 dt, \end{gathered} \tag{11}$$

where $y = y(u)$ is a solution of (1) – (2) for $u \in L^2(0,T;U)$ and $\lambda_1, \lambda_2 \ge 0$ and $\lambda_1^2 + \lambda_2^2 > 0$.

We put $\Phi(u) = J(y(u),u)$. Using the definition of an optimal control u^0 we obtain that $\Phi(u^0) = \inf_{u \in L^2(0,T;U)} \Phi(u)$.

THEOREM 2. *Let the assumptions (i) – (iv) be satisfied. Then the optimal control problem (P) has at least one optimal solution $u^0 \in L^2(0,T;U)$.*

The proof of this Theorem is standard by applying a minimizing sequence for the functional J because the functional (11) is weakly lower-semicontinuous in $L^2(0,T;H) \times L^2(0,T;U)$.

§ 3. Approximation of the Control Problem

Let denote the approximate family of all finite-dimensional subspaces of the original space V i.e. $W \in$ implies $W \subset V$, $\dim W < \infty$ and $\overline{\bigcup_{W\in} W} = V$. The family approximates the space H too. The approximation of space $L^2(0,T;V)$ is here understood as the family of spaces $\{L^2(0,T;W)\}$ (see [13]).

As an approximation of the control space U we assume a family of all finite-dimensional subspaces of the original space U i.e. $Y \subset U$, $\dim Y < \infty$ and $\overline{\bigcup_{Y\in} Y} = U$.

We shall study the following approximated optimal problem (P_{YW}): find a control $u^0_{YW} \in L^2(0,T;Y)$ which minimizes the cost functional

$$\Phi(u_Y) = J(y_W, u_Y) = \lambda_1 \int_0^T \|y_W(t) - y_{dW}\|^2_H dt + \tag{12}$$

$$+\lambda_2 \int_0^T \|y'_W(t) - y'_{dW}\|^2_H dt + \int_0^T \|u_Y(t)\|^2_U dt$$

where $y_W = y_W(u_Y)$ is the solution of the inclusion

$$y''_W(t) + Ay_W(t) + \partial\varphi(y'_W(t)) \ni Bu_Y(t), \quad \text{for } t \in (0,T) \tag{13}$$

with the initial conditions

$$y_W(0) = y_{0W} \quad \text{and} \quad y'_W(0) = y_{1W} \tag{14}$$

where y_{0W} and y_{1W} are the orthogonal projections of y_0 and y_1 onto W, y_{dW} and y'_{dW} are the orthogonal projections of y_d and y'_d onto $L^2(0,T;W)$ with the norm from the space $L^2(0,T;H)$.

Theorem 3. *Under the assumption from Theorem 2 the optimal control problem (P_{YW}) has at least one optimal solution u^0_{YW}.*

The proof of this theorem can be made in the same way as the proof of Theorem 2 because the inclusion (13) with the initial conditions (14) has the unique solution $y_W = y_W(u_Y)$.

From Lemma 1 we have the following corollary

LEMMA 2. *Let (u_Y) be a sequence of elements in $L^2(0,T;Y)$ and (y_W) be a sequence of solutions of (13) – (14). If the assumptions of Lemma 1 are satisfied then the following conditions hold:*

(a) *If $u_Y \rightharpoonup \overline{u}$ weakly in $L^2(0,T;U)$ for $\dim W \to \infty$ then $y_W \rightharpoonup \overline{y}$ weakly in $L^2(0,T;V)$ and $y_W \to \overline{y}$ strongly in $L^2(0,T;H)$ and $y'_W \rightharpoonup \overline{y}'$ weakly in $L^2(0,T;H)$ for $\dim Y \to \infty$ and $\dim W \to \infty$.*

(b) *If $u_Y \to \overline{u}$ strongly in $L^2(0,T;U)$ for $\dim Y \to \infty$ then $y_W \to \overline{y}$ strongly in $L^2(0,T;V)$ and $y'_W \to \overline{y}$ strongly in $L^2(0,T;H)$ for $\dim Y \to \infty$ and $\dim W \to \infty$.*

The proof of parts (a) and (b) follows immediately from Lemma 1.

Let us now consider the problem of convergence of the approximation.

THEOREM 4. *Let the assumptions of (i) – (iv) be satisfied. Then there exist weak condensation points of a set of solutions of the optimal problems (P_{YW}) in $L^2(0,T;H) \times L^2(0,T;U)$ and each of these points is the solution of the optimal problem (P).*

PROOF. The sequence (u^0_{YW}) is a minimizing sequence for functional (12). According to the approximation of the space U for u^0 (solution of problem (P)) there exists a sequence $(\overline{u}_Y)$ such that $\overline{u}_Y \to u^0$ strongly in $L^2(0,T;U)$ for $\dim Y \to \infty$ and (from Lemma 2) $\overline{y}_W = y_W(\overline{u}_Y) \to y^0 = y(u^0)$, $\overline{y}'_W \to y^{0'}$ strongly in $L^2(0,T;H)$ for $\dim Y \to \infty$ and $\dim W \to \infty$ where $\overline{y}_W$ is a solution of the problem (13) – (14) for $u_Y = \overline{u}_Y$. Since

$$\inf_{u \in L^2(0,T;U)} \Phi(u) = J(y^0,u^0) \leq J(y^0_W, u^0_{YW}) \leq J(\overline{y}_W, \overline{u}_Y)$$

where $y^0_W = y_W(u^0_{YW})$ is the solution of the problem (13) – (14) for $u_Y = u^0_{YW}$. Then because the functional J is continuous on $L^2(0,T;H) \times L^2(0,T;U)$ we have

$$\lim J(y^0_W, u^0_{YW}) = J(y^0, u^0)$$

for $\dim Y \to \infty$ and $\dim W \to \infty$. The functional J is coercive, therefore the sequence (u^0_{YW}) is bounded in $L^2(0,T;U)$. It follows that there exists a subsequence which we also denote (u^0_{YW}) such that $u^0_{YW} \rightharpoonup \tilde{u}$ weakly in $L^2(0,T;U)$ for $\dim Y \to \infty$ and $\dim W \to \infty$. Then Lemma 2 implies that $y^0_W = y_W(u^0_{YW}) \rightharpoonup \tilde{y}$, $y'_W(u^0_{YW} \rightharpoonup \tilde{y}'$ weakly in $L^2(0,T;H)$ for $\dim Y \to \infty$ and $\dim W \to \infty$ where $\tilde{y}$ is a solution of the problem (1) – (2) for

$u = \tilde{u}$. The functional J is weakly lower-semicontinuous on $L^2(0,T;H) \times L^2(0,T;U)$. Then we have

$$\inf_{u \in L^2(0,T;U)} \Phi(u) = \lim \Phi(u^0_{YW}) = \liminf J(y^0_W, u^0_{YW}) \geq J(\tilde{y}, \tilde{u})$$

for $\dim Y \to \infty$ and $\dim W \to \infty$. This implies that $\tilde{u}$ is one of the solutions of the optimal control problem (P). □

Example. We denote $V = H^1_0(\Omega)$ and $H = L^2(\Omega)$ where $\Omega \subset^n$ is an open bounded set with a sufficiently regular boundary Γ (see [8]) and $Q = \Omega \times (0,T)$.

We shall consider the following control problem:

$$\text{minimize } J(y,u) = \int_0^T \|y(t)\|^2_{L^2(\Omega)} dt + \int_0^T \|u(t)\|^2_{L^2(\Omega)} dt$$

subject to

$$\begin{aligned} &\frac{\partial^2 y(t,x)}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 y(t,x)}{\partial x_i^2} + \partial\chi\left(\frac{\partial y(t,x)}{\partial t}\right) \ni u(t,x) \quad \text{a.e. } Q, \\ &y(0,x) = y_0(x), \quad \frac{\partial y(0,x)}{\partial t} = y_1(x) \quad \text{a.e. } \Omega, \\ &y(t,x) = 0 \quad \text{a.e. } \Gamma \times (0,T) \end{aligned} \tag{15}$$

where

$$\chi\left(\frac{\partial y(t,x)}{\partial t}\right) = \begin{cases} 0 & \text{for } \frac{\partial y(t,x)}{\partial t} \in C, \\ +\infty & \text{for } \frac{\partial y(t,x)}{\partial t} \in H^1_0(\Omega) \setminus C. \end{cases}$$

The set C is any nonempty convex closed subset of $H^1_0(\Omega)$, $y_0 \in H^1_0(\Omega) \cap H^2(\Omega)$ and $y_1 \in C$. We assume that $u \in L^2(Q)$. From Theorem 1 there exists a unique solution of the equation (15) $y \in C([0,T]; H^1_0(\Omega))$ and $\frac{\partial y}{\partial t} \in L^\infty(0,T;H^1_0(\Omega)) \cap C([0,T];L^2(\Omega))$. Using the notations from Section 3 we transform the problem (15) to the system of differential inclusions [7]:

$$\frac{\partial^2 y_W(t,x)}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 y_W(t,x)}{\partial x^2} + \chi\left(\frac{\partial y_W(t,x)}{\partial t}\right) \ni u_Y(t,x)$$

with the initial conditions

$$y_W(0) = y_{0W} \quad \text{and} \quad y'_W(0) = y_{1W}$$

where y_{0W} and y_{1W} are the orthogonal projections of $y_0(x)$ and $y_1(x)$ onto W.

The above system of differential inclusions is equivalent to the next systems described by functional differential inequalities

$$\left\langle \frac{\partial^2 y_W(t,x)}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 y_W(t,x)}{\partial x^2} - u_Y(t,x), z_W(x) - \frac{\partial y_W(t,x)}{\partial t} \right\rangle$$
$$\geq \chi\left(\frac{\partial y_W(t,x)}{\partial t}\right) - \chi(z_W(x)) \quad \forall\, z_W \in W$$

with the initial conditions

$$y_W(0) = y_{0W} \quad \text{and} \quad y'_W(0) = y_{1W}.$$

Now, we can study the following optimal approximated problem: find a control $u^0_{YW} \in Y$ which minimizes a cost functional

$$J(y_W, u_Y) = ||y_W||^2_{L^2(Q)} + ||u_Y||^2_{L^2(Q)}.$$

The above example may be compared with the study of the vibrating string with an obstacle (see [12]).

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