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SHARP FEJÉR TRAPEZOIDAL TYPE INEQUALITIES

Abstract. This paper presents a new sharp Fejér trapezoidal type inequality, utilizing the concepts of absolute continuity, Chebyshev functionals, Grüss type inequalities, and some properties of the integral function $\mathcal{M}$ defined on the interval $[a, b]$. Unlike classic versions, the absolute value of the derivative of the considered function is not assumed to be convex, a setting that broadens the applicability of the results. Additionally, several special cases, such as sharp inequalities related to Riemann-Liouville fractional integrals, sharp Hermite-Hadamard trapezoidal type inequalities, and trapezoidal formulas for the approximation of definite integrals are discussed, offering several examples and applications to trigonometric functions and Euler's beta and gamma functions.

Key words: *Fejér inequality, trapezoidal type inequality, fractional integrals, trigonometric functions, Gamma and Beta functions*

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1. Introduction and preliminaries. In 1906, L. Fejér [7] presented a weighted version of the celebrated Hermite-Hadamard inequality [14], which gives upper and lower bounds for the weighted Riemann integral of a function based on special criteria. In fact, for a convex function $\mathcal{H}\colon [a, b] \to \mathbb{R}$ the following inequality holds:

$$\mathcal{H}\Big(\frac{a+b}{2}\Big) \int_a^b \mathcal{W}(x)dx \leqslant \int_a^b \mathcal{H}(x)\mathcal{W}(x)dx \leqslant \frac{\mathcal{H}(a)+\mathcal{H}(b)}{2} \int_a^b \mathcal{W}(x)dx, \quad (1)$$

where the function $\mathcal{W}\colon [a, b] \to \mathbb{R}^+ \cup \{0\} = [0, +\infty)$ is integrable on $[a, b]$ and symmetric with respect to $x = \frac{a+b}{2}$; that is, $\mathcal{W}(x) = \mathcal{W}(a + b - x)$ for

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all $x \in [a,b]$. If we consider $\mathcal{W} \equiv 1$ in (1), then we recapture the Hermite-Hadamard inequality:

$$\mathcal{H}\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \mathcal{H}(x)\,dx \leqslant \frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}. \tag{2}$$

Over the past century, significant efforts have been devoted to generalizing, refining, and extending inequality (1) and its consequences by using various concepts related to the generalized convex functions or their modified domains and ranges. For advanced results, generalization, and applications of (1), we refer the readers to the references [11], [12], [19], [20], [26], [28] and the sources cited therein, where there exist some well-known concepts related to the Fejér and Hermite-Hadamard inequality.
A key question arising from (1) concerns the comparability of its constituent parts. To estimate the difference between the middle and right-hand parts of (1), which is generally called the *trapezoidal type inequality*, D-Y. Hwang [10] obtained the following result:

Theorem 1. *Let $\mathcal{H} : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I°, with $a, b \in I$ and $a < b$. Consider $\mathcal{W} : [a,b] \to \mathbb{R}^+ \cup \{0\}$ to be symmetric with respect to $\frac{a+b}{2}$. If the mapping $|\mathcal{H}'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}\int_a^b \mathcal{W}(x)dx - \int_a^b \mathcal{H}(x)\mathcal{W}(x)dx\right| \\ &\leqslant \frac{(b-a)}{4}\Big[\big|\mathcal{H}'(a)\big| + \big|\mathcal{H}'(b)\big|\Big]\int_0^1 \int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} \mathcal{W}(x)dxdt.\end{aligned} \tag{3}$$

Also in [22] (see also [25]) we can find another presentation of Fejér trapezoidal type inequality with the same assumptions as above:

$$\begin{aligned}&\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}\int_a^b \mathcal{W}(x)dx - \int_a^b \mathcal{H}(x)\mathcal{W}(x)dx\right| \\ &\leqslant \Big[\big|\mathcal{H}'(a)\big| + \big|\mathcal{H}'(b)\big|\Big]\int_{\frac{a+b}{2}}^b (b-x)\mathcal{W}(x)dx\end{aligned} \tag{4}$$

$$= \left[\left|\mathcal{H}'(a)\right| + \left|\mathcal{H}'(b)\right|\right] \int_a^{\frac{a+b}{2}} (x-a)\mathcal{W}(x)dx.$$

Both inequalities (3) and (4) are generalizations of Theorem 2.2 in [5] (see (17)). For more results related to the Fejér trapezoidal type inequality see [21], [33], [10]. Note that in almost all the results related to the Fejér trapezoidal type inequality, the derivative absolute value of the considered function is assumed to be convex, which imposes some restrictions on the function usage. Additionally, most trapezoidal type inequalities are not sharp, which means that the inequality could be more optimized by changing the criteria. On the other hand, in some references, such as [13], [15], [34], the concepts of the Chebyshev functional and Grüss type inequalities have played essential roles in obtaining valuable sharp estimation-type results in connection with the Simpson and Ostrowski inequalities, which can also be helpful in obtaining some new results, especially those related to trapezoidal type inequalities. Here we recall these concepts briefly:

Definition 1. *[4] For any pair of integrable functions $\mathcal{H}, \mathcal{W}: [a,b] \to \mathbb{R}$, the Chebyshev functional is defined as*

$$T(\mathcal{H},\mathcal{W}) = \frac{1}{b-a}\int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}s - \frac{1}{(b-a)^2}\int_a^b \mathcal{H}(x)\mathrm{d}x \int_a^b \mathcal{W}(x)\mathrm{d}x.$$

The Average (mean) of $\mathcal{H}$ is defined as

$$\mathcal{A}(\mathcal{H}) = \frac{1}{b-a}\int_a^b \mathcal{H}(x)\mathrm{d}x.$$

For the case $\mathcal{H} = \mathcal{W}$, we use the characterization

$$T(\mathcal{H}) = \mathcal{A}(\mathcal{H}^2) - \mathcal{A}^2(\mathcal{H}).$$

The following result known as the Grüss inequality [8], is an answer to finding an upper bound for the Chebyshev functional as well:

Theorem 2. *For two integrable functions $\mathcal{H}, \mathcal{W}$ defined on $[a,b]$ with $\theta \leqslant \mathcal{H}(x) \leqslant \Theta$ and $\psi \leqslant \mathcal{W}(x) \leqslant \Psi$ for all $x \in [a,b]$, the following*

inequality holds:

$$\left|T(\mathcal{H},\mathcal{W})\right| \leqslant \frac{1}{4}(\Theta-\theta)(\Psi-\psi).$$

Also the concept of absolute continuity for real functions is defined as follows [17]:

Definition 2. *A real function $\mathcal{H}$ is absolutely continuous on $[a,b]$ if for any $\epsilon>0$, there exists a $\delta>0$, such that any open disjoint choice $\{(s_i,t_i)\}_1^n$ of $[a,b]$ with $\sum\limits_{i=1}^{n}(t_i-s_i)<\delta$, implies that $\sum\limits_{i=1}^{n}|\mathcal{H}(t_i)-\mathcal{H}(s_i)|<\epsilon$.*

Motivated by these observations, we consider the class of absolutely continuous functions to derive sharp Fejér trapezoidal-type inequalities. This approach allows for a broader class of functions to be used, leading to new and more refined results. For the main results, after introducing the integral function $\mathcal{M}$ defined on interval $[a,b]$, we provide some functional properties for it and then utilize some of these properties to present a new and sharp form of the Fejér trapezoidal type inequality. In the corollaries and applications section, we extract some results for special cases, such as sharp inequalities related to Riemann-Louville fractional integrals, sharp Hermite-Hadamard trapezoidal type inequalities, and trapezoidal formulas along with some examples and applications.

2. Main results. The main results are obtained by first introducing the integral function $\mathcal{M}$ on the interval $[a,b]$, then establishing several of its functional properties. These properties are subsequently used to derive a new and sharp Fejér trapezoidal-type inequality. Consider the integral function $\mathcal{M}(x)\colon [a,b]\to\mathbb{R}$ defined as:

$$\mathcal{M}(x)=\int\limits_{\frac{a+b}{2}}^{x}\mathcal{W}(t)\mathrm{d}t, \qquad x\in[a,b],$$

where $\mathcal{W}$ is assumed to be continuous on $[a,b]$. Note that an equivalent presentation of $\mathcal{M}$ is

$$\mathcal{M}(x)=\mathcal{G}(x)-\mathcal{G}\Big(\frac{a+b}{2}\Big), \qquad x\in[a,b],$$

where

$$\mathcal{G}(x)=\int\limits_{a}^{x}\mathcal{W}(t)\mathrm{d}t.$$

The integral function $\mathcal{M}$ is the generalized version of the function $f\colon [a,b] \to \mathbb{R}$ representing *the deviation of* x from the midpoint of the interval $[a,b]$, i.e., $f(x) = x - \frac{a+b}{2}$. This function is used in mathematical analysis frequently, especially in statistics; this shows that the definition of $\mathcal{M}$ is valid and is applicable to achieve new results as well.

Here, we discuss some properties of the function $\mathcal{M}$ that will be used in the main result and its applications. We use Hölder inequality [23], which simply is presented as

$$\int_a^b |f(x)g(x)|\,\mathrm{d}x \leqslant \Big(\int_a^b |f(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_a^b |g(x)|^q dx\Big)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1,$$

where $f \in L^p[a,b]$ *and* $g \in L^q[a,b]$.

Theorem 3. *Consider* $\mathcal{M}$ *and* $\mathcal{W}$ *defined as above.*
(i) *If* $\frac{1}{p}+\frac{1}{q}=1$, *where* $p,q \geqslant 1$, *then*

$$\|\mathcal{M}\|_1 \leqslant \|\mathcal{M}\|_p (b-a)^{\frac{1}{q}}. \tag{5}$$

Also, we have

$$\|\mathcal{M}\|_1 \leqslant \|\mathcal{W}\|_p \frac{q(b-a)^{\frac{1}{q}+1}}{2^{\frac{1}{q}}(q+1)}. \tag{6}$$

(ii) *If we consider* $\|\mathcal{W}\|_\infty = \sup\limits_{x\in[a,b]} |\mathcal{W}(x)|$, *then*

$$\|\mathcal{M}\|_1 \leqslant \frac{(b-a)^2\|\mathcal{W}\|_\infty}{4}, \tag{7}$$

and

$$\|\mathcal{M}\|_2 \leqslant \sqrt{\|\mathcal{M}\|_p\|\mathcal{M}\|_q} \leqslant \frac{(b-a)^{\frac{3}{2}}\|\mathcal{W}\|_\infty}{2\sqrt[2p]{p+1}\sqrt[2q]{q+1}}, \tag{8}$$

where $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.
(iii) $\mathcal{W}$ *and* $\mathcal{M}$ *cannot both have a root on* $[a,b]$.
(iv) *The following characterization holds, whenever* $\mathcal{H}\colon [a,b] \to \mathbb{R}$ *is an absolutely continuous function defined on* $[a,b]$.

$$\int_a^b \mathcal{M}(x)\mathcal{H}'(x)\mathrm{d}x = \frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}\int_a^b \mathcal{W}(x)\mathrm{d}x - \int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x. \tag{9}$$

Further, if $\mathcal{W}$ is symmetric with respect to $\frac{a+b}{2}$, then

(v) *The function $\dfrac{d\mathcal{M}}{dx}(x)$ is symmetric with respect to $\frac{a+b}{2}$.*

(vi) *The integral function $\mathcal{M}$ is of zero mean value:*

$$\int_a^b \mathcal{M}(x)\mathrm{d}x = 0.$$

(vii) *Equality (9) has the following presentation:*

$$\int_a^b \mathcal{M}(x)\mathcal{H}'(x)\mathrm{d}x = (\mathcal{H}(a)+\mathcal{H}(b))\mathcal{M}(b) - \int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x. \tag{10}$$

Proof. (i) Hölder inequality can be applied as follows to obtain (5):

$$\int_a^b |\mathcal{M}(x)|\mathrm{d}x \leqslant \Big(\int_a^b |\mathcal{M}(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_a^b \mathrm{d}x\Big)^{\frac{1}{q}}.$$

To prove (6), we consider

$$\Big|\int_{\frac{a+b}{2}}^x \mathcal{W}(t)\mathrm{d}t\Big| \leqslant \Big(\int_{\frac{a+b}{2}}^x |\mathcal{W}(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} \Big|x-\frac{a+b}{2}\Big|^{\frac{1}{q}} \leqslant \|\mathcal{W}\|_p \Big|x-\frac{a+b}{2}\Big|^{\frac{1}{q}}. \tag{11}$$

By integration with respect to the variable x over $[a,b]$ from both sides of (11), we obtain the expected result.

(ii) To see (7), put $p=+\infty$ and $q=1$ in (6). To prove (8), by considering Hölder inequality we conclude

$$\begin{aligned}
\|\mathcal{M}\|_2^2 &\leqslant \Big(\int_a^b |\mathcal{M}(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_a^b |\mathcal{M}(x)|^q \mathrm{d}x\Big)^{\frac{1}{q}} \leqslant \|\mathcal{M}\|_p \|\mathcal{M}\|_q \\
&\leqslant \|\mathcal{W}\|_\infty^2 \Big(\int_a^b \Big|x-\frac{a+b}{2}\Big|^p \mathrm{d}t\Big)^{\frac{1}{p}} \Big(\int_a^b \Big|x-\frac{a+b}{2}\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&= \|\mathcal{W}\|_\infty^2 \Big(\frac{(b-a)^{p+1}}{2^p(p+1)}\Big)^{\frac{1}{p}} \Big(\frac{(b-a)^{q+1}}{2^q(q+1)}\Big)^{\frac{1}{q}} = \frac{\|\mathcal{W}\|_\infty^2 (b-a)^3}{4\sqrt[p]{p+1}\sqrt[q]{q+1}}.
\end{aligned}$$

(iii) If $\mathcal{W}$ has no root on $[a,b]$, it is either positive or negative all over $[a,b]$. Suppose that it is negative on $[a,b]$, which implies that $\mathcal{M}(a)>0$ and that $\mathcal{M}(b)<0$. By the definition of the cumulative function in the fundamental theorem of integral calculus [23] and the fact that $\mathcal{W}$ is integrable, we conclude that $\mathcal{M}(x)$ is continuous on [a, b]. The intermediate value theorem (Bolzano's Theorem [2]) ensures that there is at least one root for $\mathcal{M}$ on $[a,b]$.

(iv) Integration by parts rule for $\mathcal{M}$ and $\mathcal{H}'$ in $\int_a^b \mathcal{M}(x)\mathcal{H}'(x)$ and fundamental theorems of integral calculus

$$\mathcal{M}'(x)=\mathcal{G}'(x)=\mathcal{W}(x), \qquad x\in[a,b],$$

imply the expected result.

(v) Since $\mathcal{W}$ is symmetric with respect to $\frac{a+b}{2}$, using the Leibniz integral rules [16] we conclude

$$\frac{d\mathcal{M}}{dx}(a+b-x)=-\frac{d}{dx}\Big(\int_{\frac{a+b}{2}}^{a+b-x}\mathcal{W}(t)\mathrm{d}t\Big)=\mathcal{W}(a+b-x)$$

$$=\mathcal{W}(x)=\frac{d}{dx}\Big(\int_{\frac{a+b}{2}}^{x}\mathcal{W}(t)\mathrm{d}t\Big)=\frac{d\mathcal{M}}{dx}(x).$$

(vi) Consider the fact $\mathcal{M}(a+b-x)=-\mathcal{M}(x)$, for all $x\in[a,b]$ to get

$$\int_a^{\frac{a+b}{2}}\mathcal{M}(x)\mathrm{d}x=-\int_a^{\frac{a+b}{2}}\mathcal{M}(a+b-x)\mathrm{d}x=-\int_a^{\frac{a+b}{2}}\int_{\frac{a+b}{2}}^{a+b-x}\mathcal{W}(t)\mathrm{d}t\mathrm{d}x$$

$$=-\int_{\frac{a+b}{2}}^{b}\int_{\frac{a+b}{2}}^{x}\mathcal{W}(t)\mathrm{d}t\mathrm{d}x=-\int_{\frac{a+b}{2}}^{b}\mathcal{M}(x)\mathrm{d}x.$$

(vii) This identity holds because

$$\mathcal{M}(b)=\frac{1}{2}\int_a^b\mathcal{W}(t)\mathrm{d}t,$$

which is a consequence of the symmetry of $\mathcal{W}$ on $[a,b]$ with respect to $\frac{a+b}{2}$. □

Remark. *All the results in Theorem 3 remain valid if we consider*

$$\mathcal{M}(x) = \int_x^{\frac{a+b}{2}} \mathcal{W}(t)dt,$$

unless instead of (10) we have

$$\int_a^b \mathcal{M}(x)\mathcal{H}'(x)\mathrm{d}x = (\mathcal{H}(a)+\mathcal{H}(b))\mathcal{M}(a) - \int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x.$$

Here we establish a new sharp Fejér trapezoidal type inequality related to absolutely continuous functions and Chebyshev functional.

Theorem 4. *Suppose that $\mathcal{W}\colon [a,b]\to\mathbb{R}$ is symmetric with respect to $\frac{a+b}{2}$. Let $\mathcal{H}\colon [a,b]\to\mathbb{R}$ be an absolutely continuous function, such that $\mathcal{H}'\in L^2[(a,b)]$. Then*

$$\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}\int_a^b \mathcal{W}(x)\mathrm{d}x - \int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x\right| \leqslant \frac{(b-a)^2\|\mathcal{W}\|_\infty T^{\frac{1}{2}}(\mathcal{H}')}{2\sqrt{3}}. \quad (12)$$

Inequality (12) is sharp, which means that $\frac{1}{2\sqrt{3}}$ is the best possible constant.

Proof. Consider

$$\begin{aligned}
&\left|\int_a^b \mathcal{M}(x)\mathcal{H}'(x)\,\mathrm{d}x\right| \\
&= \left|\int_a^b \mathcal{M}(x)\mathcal{H}'(x)\mathrm{d}x - \frac{\mathcal{H}(\mathrm{b})-\mathcal{H}(\mathrm{a})}{b-a}\int_a^b \mathcal{M}(x)\mathrm{d}x\right| \quad \big(\text{by Theorem 3(vi)}\big) \\
&= \left|\int_a^b \mathcal{M}(x)\mathcal{H}'(x)\,\mathrm{d}x - \mathcal{A}(\mathcal{H}')\int_a^b \mathcal{M}(x)\,\mathrm{d}x\right| \\
&= \left|\int_a^b \mathcal{M}(x)\left[\mathcal{H}'(x) - \mathcal{A}(\mathcal{H}')\right]\mathrm{d}x\right|
\end{aligned}$$

$$\leqslant \Big\{ \int_a^b \mathcal{M}^2 \Big\}^{\frac{1}{2}} \Big\{ \int_a^b \Big(\mathcal{H}'(x) - \mathcal{A}(\mathcal{H}') \Big)^2 \mathrm{d}x \Big\}^{\frac{1}{2}},$$

where we have used Cauchy-Schwarz inequality [30] in the last part. On the other hand, a direct computation of the last term reveals that

$$\begin{aligned}
&\int_a^b \Big(\mathcal{H}'(x) - \mathcal{A}(\mathcal{H}') \Big)^2 dx \\
&= \int_a^b (\mathcal{H}'(x))^2 dx - 2\mathcal{A}(\mathcal{H}') \int_a^b \mathcal{H}'(x) dx + \int_a^b \mathcal{A}^2(\mathcal{H}') dx \\
&= \int_a^b (\mathcal{H}'(x))^2 dx - 2(b-a)\mathcal{A}^2\left(\mathcal{H}'\right) + (b-a)\mathcal{A}^2\left(\mathcal{H}'\right) \\
&= (b-a)(\mathcal{A}((\mathcal{H}')^2) - \mathcal{A}^2(\mathcal{H}')) = (b-a)T(\mathcal{H}'),
\end{aligned}$$

and

$$\int_a^b \mathcal{M}^2(x) \mathrm{d}x \leqslant \frac{\|\mathcal{W}\|_\infty^2 (b-a)^3}{12}. \qquad \text{(by Theorem 3(ii))} \tag{13}$$

Therefore, by Theorem 3(iv), which states

$$\int_a^b \mathcal{M}\left(x\right) \mathcal{H}^{'}\left(x\right) \mathrm{d}x = \frac{\mathcal{H}\left(a\right) + \mathcal{H}\left(b\right)}{2} \int_a^b \mathcal{W}\left(x\right) \mathrm{d}x - \int_a^b \mathcal{H}\left(x\right) \mathcal{W}\left(x\right) \mathrm{d}x,$$

we deduce (12).

For the sharpness of (12), define the function $\mathcal{H}\colon [a,b] \to \mathbb{R}^+$ as

$$\mathcal{H}(x) = \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2, \quad x \in [a,b],$$

and for the convenience consider $\mathcal{W} \equiv 1$. Suppose that there exists a positive real number $\mathcal{C}$, such that

$$L = \left| \frac{\mathcal{H}(a) + \mathcal{H}(b)}{2} \int_a^b \mathcal{W}(x) \mathrm{d}x - \int_a^b \mathcal{H}(x) \mathcal{W}(x) \mathrm{d}x \right|$$

$$\leqslant \mathcal{C}(b-a)^2\|\mathcal{W}\|_\infty T^{\frac{1}{2}}(\mathcal{H}') = \mathcal{C}R.$$

Straight calculations result in:

$$L = \int_a^b \mathcal{M}(x)\mathcal{H}'(x)\mathrm{d}x = \frac{(b-a)^3}{12}, \qquad \text{(by Theorem 3(iv))}$$
$$R = (b-a)^2 T^{\frac{1}{2}}(\mathcal{H}') = \frac{(b-a)^3}{\sqrt{12}}.$$

This implies that $L \leqslant \mathcal{C}R \Longrightarrow \mathcal{C} \geqslant \frac{1}{\sqrt{12}}$, which means that all possible values of $\mathcal{C}$ satisfying (12) are eventually equal or greater than the constant $\frac{1}{\sqrt{12}}$. □

For a generalized version of Theorem 4, we need the following result:

Lemma 1. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\sqrt[p]{p+1}.\sqrt[q]{q+1} \leqslant 3.$$

Equality holds whenever $p = q = 2$.

Proof. Consider the concave function $f(x) = \log(x+1)$, $x > 0$, to obtain

$$\log(3) = f\Big(\frac{1}{p}p + \frac{1}{q}q\Big) \geqslant \frac{1}{p}f(p) + \frac{1}{q}f(q) = \log\big(\sqrt[p]{p+1}.\sqrt[q]{q+1}\big),$$

which implies the desired result. Note that the conditions $p, q > 2$ and $p, q < 2$ are impossible and only the choices $1 < p < 2$, $q > 2$ and $1 < q < 2$, $p > 2$ are available. □

The following result is a generalized version of Theorem 4 by the use of Lemma 1.

Theorem 5. *With all the assumptions in Theorem 4, the following inequality holds:*

$$\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}\int_a^b \mathcal{W}(x)\mathrm{d}x - \int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x\right| \leqslant \frac{(b-a)^2\|\mathcal{W}\|_\infty T^{\frac{1}{2}}(\mathcal{H}')}{2\ \sqrt[2p]{p+1}.\ \sqrt[2q]{q+1}},$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. This is a consequence of Theorem 4 and Lemma 1. □

3. Corollaries and applications. In this section, we discuss some special cases of the main results where several special choices exist for the symmetric function $\mathcal{W}$. Some sharp inequalities related to Riemann-Louville fractional integrals are mentioned, and several sharp Fejér and Hermite-Hadamard trapezoidal type inequalities with some examples are given. Furthermore, a trapezoidal type formula for definite integral approximation is presented.

(1) The class of Riemann-Liouville fractional integrals for $\mathcal{H} \in L^1[a,b]$ with $a \geqslant 0$, $\alpha > 0$ is defined as (see [9], [24]):

$$J^{\alpha}_{a^+}\mathcal{H}(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\mathcal{H}(t)\mathrm{d}t, \qquad (x > a),$$

and

$$J^{\alpha}_{b^-}\mathcal{H}(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\mathcal{H}(t)\mathrm{d}t, \qquad (b < x),$$

where Euler's gamma function (Euler's second integral function [3]) is considered:

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}\mathrm{d}t, \qquad (\mathrm{Re}(s) > 0).$$

Note that $\Gamma(s+1) = s\Gamma(s)$ for all s with $\mathrm{Re}(s) > 0$. Now, in Theorem 4, consider

$$\mathcal{W}(x) = \frac{(b-x)^{\alpha-1} + (x-a)^{\alpha-1}}{\Gamma(\alpha)}, \qquad x \in [a,b].$$

Since $\mathcal{W}(a+b-x) = \mathcal{W}(x)$, for all $x \in [a,b]$, $\mathcal{W}$ is symmetric to $\frac{a+b}{2}$. The calculations reveal that

$$\mathcal{M}(x) = \frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^x [(b-t)^{\alpha-1} + (t-a)^{\alpha-1}]\mathrm{d}t = \frac{(b-x)^{\alpha} - (x-a)^{\alpha}}{\Gamma(\alpha+1)},$$

$$\int_a^b \mathcal{W}(x)\mathrm{d}x = \frac{2(b-a)^{\alpha}}{\Gamma(\alpha+1)},$$

and

$$\int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x = \frac{1}{\Gamma(\alpha)}\Big[\int_a^b (b-x)^{\alpha-1}\mathcal{H}(x)\mathrm{d}x + \int_a^b (x-a)^{\alpha-1}\mathcal{H}(x)\mathrm{d}x\Big]$$
$$= J^{\alpha}_{a^+}\mathcal{H}(b) + J^{\alpha}_{b^-}\mathcal{H}(a).$$

On the other hand, instead of using (13), we give a straightforward calculation for the integral of $\mathcal{M}^2(x)$, which yields a better and accurate constant than the general bound:

$$\int_a^b \mathcal{M}^2(x)\mathrm{d}x = \frac{1}{\Gamma^2(\alpha+1)}\int_a^b \big[(b-x)^{\alpha} - (x-a)^{\alpha}\big]^2\mathrm{d}x$$
$$= \frac{1}{\Gamma^2(\alpha+1)}\Big[\int_a^b (b-x)^{2\alpha}\mathrm{d}x - 2\int_a^b (x-a)^{\alpha}(b-x)^{\alpha}\mathrm{d}x + \int_a^b (x-a)^{2\alpha}\mathrm{d}x\Big]$$
$$= \frac{4}{\Gamma^2(\alpha+1)}\Big[\frac{1}{2\alpha+1} - \frac{\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)} + \frac{1}{2\alpha+1}\Big](b-a)^{2\alpha+1}$$
$$= \frac{4}{\Gamma^2(\alpha+1)}\Big[\frac{\Gamma(2\alpha+2) - (2\alpha+1)\Gamma^2(\alpha+1)}{(4\alpha+2)\Gamma(2\alpha+2)}\Big](b-a)^{2\alpha+1}$$
$$= \frac{4}{\Gamma^2(\alpha+1)}\Big[\frac{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}{2\Gamma(2\alpha+2)}\Big](b-a)^{2\alpha+1}.$$

So we conclude that

$$\Big(\int_a^b \mathcal{M}^2(x)\mathrm{d}x\Big)^{\frac{1}{2}} = \frac{2}{\Gamma(\alpha+1)}\left[\frac{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}{2\Gamma(2\alpha+2)}\right]^{\frac{1}{2}}(b-a)^{\alpha+\frac{1}{2}}.$$

By using all the above results and simplifying, we get

$$\Big|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a^+}\mathcal{H}(b) + J^{\alpha}_{b^-}\mathcal{H}(a)\big]\Big| \tag{14}$$
$$\leqslant \left[\frac{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}{2\Gamma(2\alpha+2)}\right]^{\frac{1}{2}}(b-a)T^{\frac{1}{2}}(\mathcal{H}'),$$

which gives a sharp trapezoid type inequality related to Riemann-Liouville fractional integrals in the sense that for any $\alpha > 0$ the constant

$\Big(\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)\Big)\Big/\Big(2\Gamma(2\alpha+2)\Big)$ cannot be replaced by a smaller one. Note that a non-sharp kind of inequality (14) was given in [27] as:

$$\begin{aligned}&\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a^+}\mathcal{H}(b)+J^{\alpha}_{b^-}\mathcal{H}(a)\big]\right|\\&\leqslant\frac{b-a}{2(\alpha+1)}\Big(1-\frac{1}{2^{\alpha}}\Big)[|\mathcal{H}'(a)|+|\mathcal{H}'(b)|],\end{aligned}$$

where $|\mathcal{H}'|$ is assumed to be convex on $[a,b]$. Furthermore, the following result was obtained in [18], [19], which refined the Hermite-Hadamard inequality via Riemann–Liouville fractional integrals for monotone functions:

Theorem 6. *If $\mathcal{H}\colon[a,b]\to\mathbb{R}$ is a nonnegative, increasing (decreasing) convex function, then, for $\alpha\geqslant 1$, the following refinement of the Hermite-Hadamard inequality holds:*

$$\begin{aligned}\mathcal{H}\left(\frac{a+b}{2}\right)&\leqslant\frac{1}{b-a}\int_a^b\mathcal{H}(x)dx\\&\leqslant\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big[J^{\alpha}_{a^+}\mathcal{H}(b)+J^{\alpha}_{b^-}\mathcal{H}(a)\Big]\leqslant\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}.\end{aligned}\tag{15}$$

So, from (14) and (15), for $\alpha\geqslant 1$ and the monotone function $\mathcal{H}$, we obtain:

$$\begin{aligned}&\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}-\frac{1}{b-a}\int_a^b\mathcal{H}(x)dx\right|\\&\leqslant\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a^+}\mathcal{H}(b)+J^{\alpha}_{b^-}\mathcal{H}(a)\big]\right|\\&\leqslant\left[\frac{\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)}{2\Gamma(2\alpha+2)}\right]^{\frac{1}{2}}(b-a)T^{\frac{1}{2}}(\mathcal{H}').\end{aligned}$$

However, for the special case $\alpha=1$ in (14) we conclude that

$$\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}(b-a)-\int_a^b\mathcal{H}(x)\mathrm{d}x\right|\leqslant\frac{(b-a)^2T^{\frac{1}{2}}(\mathcal{H}')}{2\sqrt{3}},\tag{16}$$

which gives a new sharp Hermite-Hadamard trapezoidal type inequality. We can obtain this inequality by considering $\mathcal{W}\equiv 1$ in (12). Note that a

non-sharp well-known type of the above inequality was presented in [5] as follows:

Theorem 7. *Let $\mathcal{H}\colon I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|\mathcal{H}'|$ is convex on $[a, b]$, then*

$$\left|\frac{\mathcal{H}(a)+\mathcal{H}(b)}{2}(b-a)-\int_a^b \mathcal{H}(x)\mathrm{d}x\right| \leqslant \frac{(b-a)^2}{8}\Big(|\mathcal{H}'(a)|+|\mathcal{H}'(b)|\Big). \quad (17)$$

Here, by an example, we compare (16) and (17) to show that Theorem 4 is applicable to a wider range of functions and yields more optimized approximations than previous corresponding results, such as (17).

Example. For any $x > 0$, consider

$$\mathcal{H}(t) = \frac{1}{1+t^2}, \qquad t \in [0, x].$$

The first and second derivative tests [29] imply that

(i) $|\mathcal{H}'|$ is not convex on $[0, x]$, since

$$\frac{\mathrm{d}}{\mathrm{d}t^2}(|\mathcal{H}'|) \leqslant 0 \quad on \quad [0,1] \quad and \quad \frac{\mathrm{d}}{\mathrm{d}t^2}(|\mathcal{H}'|) \geqslant 0 \quad on \quad [1,\infty].$$

This means $|\mathcal{H}'|$ is concave on $[0, 1]$ and convex on $[1, \infty)$.

(ii) If $0 < x \leqslant \frac{1}{\sqrt{3}}$, then for any $t \in [0, x]$:

$$\min_{t\in[0,x]} \mathcal{H}(t) = \frac{-2x}{(1+x^2)^2} \leqslant \mathcal{H}'(t) \leqslant 0 = \max_{t\in[0,x]} \mathcal{H}(t).$$

If $x > \frac{1}{\sqrt{3}}$, then for any $t \in [0, x]$:

$$\min_{t\in[0,x]} \mathcal{H}(t) = -\frac{3\sqrt{3}}{8} \leqslant \mathcal{H}'(t) \leqslant 0 = \sup_{t\in[0,x]} \mathcal{H}(t).$$

(iii) The function $\mathcal{H}$ is continuous on $[0, x]$ and it is differentiable on $(0, x)$. Hence, for any $a, b \in [0, x]$ with $a < b$, by the mean-value theorem [29], there exists a $z \in (a, b)$, such that

$$\mathcal{H}'(z) = \frac{-2z}{(1+z^2)^2} = \frac{\mathcal{H}(b)-\mathcal{H}(a)}{b-a}.$$

This implies that

$$|\mathcal{H}(b)-\mathcal{H}(a)|=\Big|\frac{-2z}{(1+z^2)^2}\Big||b-a|\leqslant\max_{z\in[0,x]}\frac{2z|b-a|}{(1+z^2)^2}=\frac{3\sqrt{3}}{8}|b-a|.$$

So, $\mathcal{H}$ satisfies a Lipschitz condition with respect to the constant $\frac{3\sqrt{3}}{8}$ on $[0,x]$ and, consequently, it is absolutely continuous on $[0,x]$. Assertions (i) and (iii) indicate that inequality (17) is inapplicable whereas inequality (16) remains valid. Also, assertion (ii) along with Theorem 2 (Grüss inequality) shows that

$$T^{\frac{1}{2}}(\mathcal{H}')\leqslant\begin{cases}\dfrac{x}{(1+x^2)^2}, & 0\leqslant x\leqslant\dfrac{1}{\sqrt{3}};\\[2mm] \dfrac{3\sqrt{3}}{16}, & x\geqslant\dfrac{1}{\sqrt{3}}.\end{cases}$$

So, from (16) we obtain

$$\left|\frac{x(\mathcal{H}(0)+\mathcal{H}(x))}{2}-\int_0^x\frac{1}{1+t^2}\mathrm{d}t\right|\leqslant\frac{x^2T^{\frac{1}{2}}(\mathcal{H}')}{2\sqrt{3}}.\tag{18}$$

By replacing $\mathcal{H}(0)$, $\mathcal{H}(x)$, and $T^{\frac{1}{2}}(\mathcal{H}')$ in (18), we conclude that

$$\left|\frac{x(2+x^2)}{2(1+x^2)}-\mathrm{tg}^{-1}(x)\right|\leqslant\begin{cases}\dfrac{x^3}{2\sqrt{3}(1+x^2)^2}, & 0\leqslant x\leqslant\dfrac{1}{\sqrt{3}};\\[2mm] \dfrac{3x^2}{32}, & x\geqslant\frac{1}{\sqrt{3}}.\end{cases}$$

To examine how our method would work, for the special case $x=2-\sqrt{3}$ we have:

$$0.25615\leqslant\frac{\pi}{12}=\mathrm{tg}^{-1}(2-\sqrt{3})\leqslant 0.26381.$$

The direct calculation shows that $\frac{\pi}{12}\approx 0.26179$.

Remark. *For $\alpha>0$, consider the symmetric function*

$$\mathcal{W}(x)=\frac{1}{\alpha}\Big[\exp\Big(-\frac{1-\alpha}{\alpha}(b-x)\Big)+\exp\Big(-\frac{1-\alpha}{\alpha}(x-a)\Big)\Big],\quad x\in[a,b],$$

in (12), to obtain a sharp trapezoid type inequality for the following class of fractional integrals:

$$\mathcal{I}^{\alpha}_{a^+}[f](x)=\frac{1}{\alpha}\int_a^x\exp\Big(-\frac{1-\alpha}{\alpha}(x-u)\Big)f(u)du,\qquad x>a;$$

and

$$\mathcal{I}_{b^-}^{\alpha}[f](x) = \frac{1}{\alpha}\int_x^b \exp\Big(-\frac{1-\alpha}{\alpha}(u-x)\Big)f(u)du, \qquad x < b,$$

which was presented and discussed in [1]. Additionally, for $\alpha > 0$, $\gamma > b-a$, *and* $\beta \geqslant 0$, *consider*

$$\mathcal{W}(x) = \frac{\big(b-x\big)^{\alpha-1}\ln^{\beta}\Big(\frac{\gamma}{b-x}\Big) + \big(x-a\big)^{\alpha-1}\ln^{\beta}\Big(\frac{\gamma}{x-a}\Big)}{\Gamma(\alpha)}, \quad x \in [a,b],$$

to obtain a sharp trapezoid type inequality related to the following class of fractional integrals with power-logarithmic kernels presented and discussed in [24]:

$$\mathcal{K}^{\alpha,\beta}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x) = \frac{1}{\Gamma(\alpha)}\int_{m_t(\mathcal{L},\mathcal{R})}^{x}(x-u)^{\alpha-1}\ln^{\beta}\Big(\frac{\gamma}{x-u}\Big)f(u)du,\ x > m_t(\mathcal{L},\mathcal{R}),$$

and

$$\mathcal{K}^{\alpha,\beta}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{M_t(\mathcal{L},\mathcal{R})}(u-x)^{\alpha-1}\ln^{\beta}\Big(\frac{\gamma}{u-x}\Big)f(u)du, \quad x < M_t(\mathcal{L},\mathcal{R}).$$

We encourage the readers to study [9], [24] and the references therein to obtain more inequalities and results in this field.

(2) Define

$$\mathcal{W}(t) = [t(1-t)]^{x-2}, \qquad t \in [0,1],\ x \geqslant 2,$$

which is symmetric on $[0,1]$ with respect to $\frac{1}{2}$. So, $\|\mathcal{W}\|_\infty = \big(\frac{1}{4}\big)^{x-2}$. If we consider $\mathcal{H}(t) = t(1-t), t \in [0,1]$, then it is clear that $\mathcal{H}$ is absolutely continuous (Lipschitz with constant $L = 1$), $\mathcal{H}(0) = \mathcal{H}(1) = 0$ and $T^{\frac{1}{2}}(\mathcal{H}') = \frac{1}{\sqrt{3}}$. Also,

$$\int_0^1 \mathcal{H}(t)\mathcal{W}(t)\mathrm{d}t = \int_0^1 [t(1-t)]^{x-1}\mathrm{d}t = \beta(x,x),$$

where $\beta(.,.)$ is Euler's beta function [3](Euler's first integral function). Now, according to Theorem 4, we have the following upper bound for the beta function:

$$\beta(x,x) \leqslant \frac{\left(\frac{1}{4}\right)^{x-2}}{6}, \quad x \geqslant 2.$$

This is equivalent to

$$\frac{\Gamma^2(x)}{\Gamma(2x)} \leqslant \frac{2^{3-2x}}{3}, \quad x \geqslant 2.$$

Here $|\mathcal{H}'|$ is convex, and (3) and (4) result in

$$\beta(x,x) \leqslant \left(\frac{1}{4}\right)^{x-1}, \quad x \geqslant 2.$$

However, it is not hard to see that $\frac{\left(\frac{1}{4}\right)^{x-2}}{6} < (\frac{1}{4})^{x-1}$ for any $x \geqslant 2$, showing that (12) gives a more precise bound via trapezoidal type inequalities in comparison with previous corresponding results such as (3) and (4).

(3) For $a < b$, $m \in \mathbb{N}$ with $m \geqslant 2$, we have:

$$a \leqslant \frac{b+(m-1)a}{m} \leqslant \frac{a+(m-1)b}{m} \leqslant b,$$

and

$$\mathcal{W}\Big(\frac{a+(m-1)b}{m} + \frac{b+(m-1)a}{m} - x\Big) = \mathcal{W}(a+b-x) = \mathcal{W}(x),$$

showing that $\mathcal{W}$ is symmetric on $[\frac{b+(m-1)a}{m} + \frac{a+(m-1)b}{m}]$ with respect to $\frac{a+b}{2}$. Therefore, via (12), for $m \geqslant 3$ we obtain

$$\left|\frac{\mathcal{H}\Big(\frac{b+(m-1)a}{m}\Big) + \mathcal{H}\Big(\frac{a+(m-1)b}{m}\Big)}{2} \int_{\frac{b+(m-1)a}{m}}^{\frac{a+(m-1)b}{m}} \mathcal{W}(x)\mathrm{d}x - \int_{\frac{b+(m-1)a}{m}}^{\frac{a+(m-1)b}{m}} \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x\right|$$

$$\leqslant \frac{(m-2)^2(b-a)^2\|\mathcal{W}\|_{\infty,m}T_m^{\frac{1}{2}}(\mathcal{H}')}{2m^2\sqrt{3}}. \tag{19}$$

where $T_m(\mathcal{H}')$ means that $T(\mathcal{H}')$ is restricted to $[\frac{b+(m-1)a}{m} + \frac{a+(m-1)b}{m}]$, and the same happens for $\|\mathcal{W}\|_{\infty,m}$. Inequality (19) is, in fact, a special kind

of Fejér trapezoidal type inequality that is dependent on integer $m \geqslant 3$. In a special case, if we consider $\mathcal{W} \equiv 1$ and $m = 3$ simultaneously in (19), we have (see also [6], [31], [32] and references therein)

$$\left| \frac{f\left(\frac{b+2a}{3}\right) + f\left(\frac{a+2b}{3}\right)}{6}(b-a) - \int_{\frac{b+2a}{3}}^{\frac{a+2b}{3}} \mathcal{H}(x)\mathrm{d}x \right| \leqslant \frac{(b-a)^2 T_3^{\frac{1}{2}}(\mathcal{H}')}{18\sqrt{3}}.$$

Note that by letting $m \to \infty$ in (19), we recapture (12).

3.1 Trapezoidal formula. Here, by using Theorem 4, we provide an approximating rule for the value of a weighted Riemann integral on the basis of trapezoidal type inequalities. Consider a partition $\mathcal{P} = \{a = s_0, s_1, \ldots, s_{n-1}, s_n = b\}$ of interval $[a, b]$ and $d_i = s_{i+1} - s_i$, where $i \in \{0, 1, \ldots, n\}$. Consider the formula

$$\int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x = \mathcal{A}(\mathcal{H}, \mathcal{W}, \mathcal{P}) + E(\mathcal{H}, \mathcal{W}, \mathcal{P}),$$

where we have

$$\mathcal{A}(\mathcal{H}, \mathcal{W}, \mathcal{P}) = \sum_{i=0}^{n-1} \frac{\mathcal{H}(s_i) + \mathcal{H}(s_{i+1})}{2} \int_{s_i}^{s_{i+1}} \mathcal{W}(x)\mathrm{d}x,$$

and $E(\mathcal{H}, \mathcal{W}, \mathcal{P})$ as the approximation error. To find an upper bound for the approximation error, suppose that $\mathcal{H}$ is an absolutely continuous function on $[a, b]$, such that $\mathcal{H}' \in L^2([a, b])$ and that $\mathcal{W}$ is an integrable function defined on $[a, b]$ that is symmetric to $\frac{a+b}{2}$. For each $i \in \{0, 1, \ldots, n\}$, define

$$T_i(\mathcal{H}') = \frac{1}{d_i} \int_{s_i}^{s_{i+1}} (\mathcal{H}'(x))^2 \mathrm{d}x - \frac{1}{d_i^2}\Big(\int_{s_i}^{s_{i+1}} \mathcal{H}'(x)\mathrm{d}x \Big)^2.$$

Therefore, with the use of the triangle inequality,

$$\left| \int_a^b \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x - \sum_{i=0}^{n-1} \frac{\mathcal{H}(x_i) + \mathcal{H}(x_{i+1})}{2} \int_{s_i}^{s_{i+1}} \mathcal{W}(x)\mathrm{d}x \right|$$

$$\leqslant \sum_{i=0}^{n-1} \left| \int_{s_i}^{s_{i+1}} \mathcal{H}(x)\mathcal{W}(x)\mathrm{d}x - \frac{\mathcal{H}(x_i)+\mathcal{H}(x_{i+1})}{2} \int_{s_i}^{s_{i+1}} \mathcal{W}(x)\mathrm{d}x \right|$$
$$\leqslant \frac{\|\mathcal{W}\|_\infty}{\sqrt{12}} \sum_{i=0}^{n-1} d_i^2 T_i^{\frac{1}{2}}(\mathcal{H}').$$

This implies that

$$|E(\mathcal{H},\mathcal{W},\mathcal{P})| \leqslant \frac{\|\mathcal{W}\|_\infty}{\sqrt{12}} \sum_{i=0}^{n-1} d_i^2 T_i^{\frac{1}{2}}(\mathcal{H}').$$

On the other hand, by the Cauchy–Schwarz inequality we get

$$\sum_{i=0}^{n-1} T_i^{\frac{1}{2}}(\mathcal{H}') \leqslant \Big(n \sum_{i=0}^{n-1} T_i(\mathcal{H}')\Big)^{\frac{1}{2}},$$

which results in

$$|E(\mathcal{H},\mathcal{W},\mathcal{P})| \leqslant \frac{\|\mathcal{W}\|_\infty (b-a)^2}{\sqrt{12}n^{\frac{3}{2}}} \Big(\sum_{i=0}^{n-1} T_i(\mathcal{H}')\Big)^{\frac{1}{2}}.$$

It is not hard to find out that $\sum_{i=0}^{n-1} T_i\left(\mathcal{H}'\right) \leqslant nT\left(\mathcal{H}'\right)$, which implies that

$$|E(\mathcal{H},\mathcal{W},\mathcal{P})| \leqslant \frac{\|\mathcal{W}\|_\infty (b-a)^2}{\sqrt{12}n} T^{\frac{1}{2}}(\mathcal{H}'),$$

which gives another upper bound for the approximation error by the use of $T^{\frac{1}{2}}(\mathcal{H}')$.

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