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# ON KÖBE THEOREM FOR BIHOLOMORPHIC MAPPINGS OF A BALL IN $\mathbb{C}^N$

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In the paper there has been given an upper sharp estimation of the distance between the point $f(z)$ and the boundary of the image $f(\mathbb{B}^n)$ for biholomorphic mappings of the ball $\mathbb{B}^n$.

## Introduction

Let $\mathbb{C}^n$ be the space of $n$-complex variables with the Euclidean norm and the distance $\text{dist}(a,b) = ||a-b||$. Let $\mathbb{B}^n(a,r),\ a \in \mathbb{C}^n,\ r > 0$, denote the open ball $\{z \in \mathbb{C}^n : ||z-a|| < r\}$, (we write $\mathbb{B}^n$ for $\mathbb{B}^n(0,1)$). For a domain $G \subset \mathbb{C}^n$ the topological boundary of $G$ will be denoted by $\partial G$.

It is well-known (see [3]) that for every function, univalent and analytic in the open unit disc $\mathbb{B}^1$, the following Köbe inequality holds:

$$\frac{1}{4}(1-|z|^2)|f'(z)| \leq \text{dist}(f(z), \partial f(\mathbb{B}^1)) \leq (1-|z|^2)|f'(z)|, \ \ z \in \mathbb{B}^1. \quad (1)$$

Hence, if $f(0) = 0$ and $f'(0) = 1$, then

$$\frac{1}{4} \leq \text{dist}(0, \partial f(\mathbb{B}^1)) \leq 1. \quad (2)$$

The following example of biholomorphic mappings $f : \mathbb{B}^2 \to \mathbb{C}^2$

$$f(z) = (z_1 + cz_2^2, z_2),\ z = (z_1, z_2) \in \mathbb{B}^2,\ c \in \mathbb{C}$$

shows that the left-hand side of (2) is not true for $n > 1$. Moreover, it remains false even if the constant $\frac{1}{4}$ is replaced by a constant $d \in (0, \frac{1}{4})$.

In the paper [2] we have shown that for $n > 1$ the right-hand side of (2) is still true. Namely, if $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$ is a biholomorphic mapping, then

$$\mathrm{dist}(0, \partial f(\mathbb{B}^n)) \leq 1.$$

The equality holds for $f(z) \equiv z$.

In connection with the upper estimation from (1) we have proved in [2] that for every biholomorfic mapping $f : \mathbb{B}^n \to \mathbb{C}^n, n > 1$, there holds

$$\mathrm{dist}(f(z), \partial f(\mathbb{B}^n)) \leq (1 - \|z\|^2)^{\frac{1}{2}} \|Df(z)\|, \ z \in \mathbb{B}^n. \tag{3}$$

However, in the paper [2] the sharpness in this estimation has not been discussed.

In the next section of the paper we will give the sharp estimation of $\mathrm{dist}(f(z), \partial f(\mathbb{B}^n))$ for all $n$ from the set $\mathbb{B}N$ of all positive integers.

## Main result

By $J_f(z)$ let us denote the complex Jacobian of the mapping $f$ at the point $z \in \mathbb{B}^n$.

We will prove the following theorem.

THEOREM 1. *If $f : \mathbb{B}^n \to \mathbb{C}^n$, $n \in \mathbb{N}$, is a biholomorphic mapping, then*

$$\mathrm{dist}(f(z), \partial f(\mathbb{B}^n)) \leq (1 - \|z\|^2)^{\frac{n+1}{2n}} \left|J_f(z)\right|^{\frac{1}{n}}, \ z \in \mathbb{B}^n. \tag{4}$$

*The estimation is sharp.*

PROOF. By $G$ let us denote the image $f(\mathbb{B}^n)$ of the ball $\mathbb{B}^n$ in the mapping $f$. We will consider the Bergman space $A^2(G)$ of all holomorphic functions $g : G \to \mathbb{C}$ such that

$$\left[\int_G |g(w)|^2 dV(w)\right]^{\frac{1}{2}} \equiv \|g\|_{A^2(G)} < \infty.$$

Let us fix $z \in \mathbb{B}^n$ and let $w \in G$ be such that $w = f(z)$.

It is well-known (see [1]) that for every $g \in A^2(G)$ and $r > 0$ such that $\mathbb{B}^n(w, r) \subset G$

$$|g(w)|^2 \leq \left[V(\mathbb{B}^n(w, r))\right]^{-1} \left[\|g\|_{A^2(G)}\right]^2, \tag{5}$$

where $V(\mathbb{B}^n(w,r))$ denotes the volume (in $\mathbb{R}^{2n}$) of the ball $\mathbb{B}^n(w,r)$.

Applying inequality (5) to the function $\hat{g} \in A^2(G)$ such that

$$||\hat{g}||_{A^2(G)} = \inf_{g \in A^2(G),\, g(w)=1} ||g||_{A^2(G)},$$

(see [S]), we obtain

$$1 \le [V(\mathbb{B}^n(w,r))]^{-1} \left[||\hat{g}||_{A^2(G)}\right]^2 .$$

Since

$$V(\mathbb{B}^n(w,r)) = \frac{r^{2n}\pi^n}{n!}$$

and

$$\left[||\hat{g}||_{A^2(G)}\right]^2 = \frac{1}{K_G(w,w)},$$

where $K_G$ denotes the Bergman kernel function for $G$, we have

$$\frac{r^{2n}\pi^n}{n!} \le [K_G(w,w)]^{-1} .$$

From this, in view of the transformation formula for Bergman kernel function, we get

$$\frac{r^{2n}\pi^n}{n!} \le |J_f(z)|^2 \, [K_{\mathbb{B}^n}(z,z)]^{-1} .$$

Therefore,

$$r^{2n} \le |J_f(z)|^2 \, (1 - ||z||^2)^{n+1},$$

because (see [1])

$$K_{\mathbb{B}^n}(z,z) = \frac{n!}{\pi^n(1-||z||^2)^{n+1}}.$$

To prove inequality (4) it is sufficient to use the fact that the last inequality holds for every $r > 0$ such that $\mathbb{B}^n(w,r) \subset f(\mathbb{B}^n)$.

Finally, we prove that the equality in (4) also holds.

To this purpose let $q_a$ be the mapping defined as follows:

$$q_a(z) = \frac{a - s_a z - (1-s_a)P_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B}^n,$$

where

$$s_a = (1 - ||a||^2)^{\frac{1}{2}}, \quad \langle z, a\rangle = \sum_{j=1}^{n} z_j \bar{a}_j, \quad P_a(z) = \begin{cases} \dfrac{\langle z, a\rangle}{||a||^2} a & \text{for } a \neq 0, \\ 0 & \text{for } a = 0. \end{cases}$$

It is well-known (see [4]) that $q_a$ is a holomorphic automorphism of the ball $\mathbb{B}^n$, $q_a(a) = 0$ and $|J_{q_a}(a)| = (1 - ||a||^2)^{-\frac{n+1}{2}}$. Then, for $z = a$ and $f = q_a$ we have

$$\operatorname{dist}(q_a(a), \partial q_a(\mathbb{B}^n)) = \operatorname{dist}(0, \partial\mathbb{B}^n) = 1$$

and

$$(1 - ||a||^2)^{\frac{n+1}{2n}} |J_{q_a}(a)|^{\frac{1}{n}} = 1.$$

This proves the sharpness of estimation (4). □

REMARK 1. *Estimation* (3) *follows from estimation* (4). *For $z \neq 0$ estimation* (3) *was not sharp.*

Indeed, for $n > 1$ and $z \neq 0$ we have

$$(1 - ||a||^2)^{\frac{n+1}{2n}} < (1 - ||a||^2)^{\frac{1}{2}}$$

and

$$|J_f(z)| \leq ||Df(z)||^n.$$

Now, we consider the case of the maximum norm in $\mathbb{C}^n$. Let $\mathbb{D}^n(a, r)$, $a \in \mathbb{C}^n$, $r > 0$, denote the open polydisc

$$\{z \in \mathbb{C}^n : \operatorname{dist}(z, a) = \max_{1 \leq j \leq n} |z_j - a_j| < r\},$$

(we write $\mathbb{D}^n$ for $\mathbb{D}^n(0, 1)$).

We have proved the following theorem, similar to Theorem 1.

THEOREM 2. *If $f : \mathbb{D}^n \to \mathbb{C}^n$, $n \in \mathbb{B}N$, is a biholomorphic mapping, then*

$$\operatorname{dist}(f(z), \partial f(\mathbb{D}^n)) \leq \left[\prod_{j=1}^{n} (1 - |z_j|^2)\right]^{\frac{1}{n}} |J_f(z)|^{\frac{1}{n}}. \tag{6}$$

*The estimation is sharp. The equality in* (6) *is achieved in any point $a \in \mathbb{D}^n$ by the holomorphic automorphism $q_a = [q_{1a}, \ldots, q_{na}]$*

$$q_{ja}(z) = \frac{a_j - z_j}{1 - z_j a_j}, \quad z = (z_1, \ldots, z_n) \in \mathbb{D}^n,$$

*of the polydisc $\mathbb{D}^n$.*

The proof of Theorem 2 runs similarly to the proof of Theorem 1. It is sufficient to replace (5) by the following formula

$$|g(w)|^2 \leq V(\mathbb{D}^n(w,r)) \left[\|g\|_{A^2(G)}\right]^2, \quad w \in G = f(\mathbb{D}^n), \quad \mathbb{D}^n(w,r) \subset G,$$

(see [S]) and observe that for the Bergman kernel function $K_{\mathbb{D}^n}$ of the polydisc $\mathbb{D}^n$

$$K_{\mathbb{D}^n}(z,z) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1-|z_j|^2)^2}, \quad z \in \mathbb{D}^n.$$

REMARK 2. *In the paper [2] it has been proved that for every biholomorphic mapping $f : \mathbb{D}^n \to \mathbb{C}^n$ the following estimation*

$$\operatorname{dist}(f(z), \partial f(\mathbb{D}^n)) \leq \|Df(z)\| \max_{1\leq j\leq n} (1-|z_j|^2) \tag{7}$$

*holds. Moreover, if $a = \|a\|(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{D}^n$ and $f = q_a$, defined as above, then in* (7) *the equality also holds.*

REMARK 3. *We do not know, whether it is possible to obtain inequality* (7) *from inequality* (6).

Although

$$\left[\prod_{j=1}^{n}(1-|z_j|^2)\right]^{\frac{1}{n}} \leq \max_{1\leq j\leq n}(1-|z_j|^2),$$

but we have only

$$|J_f(z)| \leq \sqrt{n}\,\|Df(z)\|^n$$

in the case of maximum norm in $\mathbb{C}^n$.

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