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ACCELERATION CONVERGENCE OF SEQUENCE OF BICOMPLEX NUMBERS

Abstract. In this article, we explore three fundamental concepts: acceleration convergence, logarithmic convergence, and subsequence transformation in the context of bicomplex number sequences. Acceleration convergence is a technique for speeding up the convergence of slowly convergent sequences, particularly in iterative methods. We examine how this method can be adapted for sequences of bicomplex numbers by leveraging their unique algebraic structure. Logarithmic convergence, in which a sequence converges logarithmically, is analyzed in the bicomplex framework. Additionally, subsequence transformations are introduced as tools for manipulating and extracting useful convergent subsequences from a given bicomplex sequence.

Key words: *bicomplex number, acceleration convergence, logarithmic convergence, subsequence transformation*

2020 Mathematical Subject Classification: *40A05, 40C05, 40A25, 40D10.*

1. Introduction and Preliminaries. In the study of bicomplex numbers, sequences and their convergence properties are of significant interest due to their applications in complex analysis, quantum mechanics, and other fields. Dawson [7] has described the summability field of a matrix A as being convergence-preserving over the set of all sequences that converge more quickly than a fixed sequence (p_n). We look for an analogy of this conclusion that deals with the acceleration field of a subsequence transformation.

Let $p = (p_n)$ be a sequence that converges to λ more quickly than the sequence $q = (q_n)$ converges to σ $(p < q)$ if

$$\lim_n \frac{p_n - \lambda}{q_n - \sigma} = 0.$$

In this instance, we also say that the sequence $q = (q_n)$ converges to σ more slowly than the sequence $p = (p_n)$ converges to λ. The sequence $p = (p_n)$ converges to λ at the same rate as the sequence $q = (q_n)$ converges to σ $(p \approx q)$ if

$$0 < \underline{\lim}_n \frac{|p_n - \lambda|}{|q_n - \sigma|} \leqslant \overline{\lim}_n \frac{|p_n - \lambda|}{|q_n - \sigma|} < +\infty.$$

Let $A = (a_{mn})$ be an infinite matrix. The A transform of the sequence (p_n) is $Ap = (A_n p)$, where

$$(A_n p) = \sum_{m=1}^{\infty} a_{mn} p_m$$

for each $n \in \mathbb{N}$, provided the summation converges for each fixed $n \in \mathbb{N}$. The matrix $A = (a_{mn})$ accelerates the convergence of p if $Ap < p$. The acceleration field of A is $\{p : Ap < p\}$. For details of acceleration convergence, refer to Keagy and Ford [9].

The concept of a bicomplex number is not new; however, investigation of bicomplex numbers has continued in recent years, and much remains to be explored. Segre [11] investigates these numbers by identifying them as bicomplex numbers. Subsequently, Price [14] conducted a thorough analysis of derivatives, integrals, holomorphic functions of bicomplex numbers, and their generalizations to higher dimensions. A bicomplex number is defined as follows:

$$z = z_1 + jz_2 = x + iy + ju + ijv,$$

where $z_1 = x + iy$ and $z_2 = u + iv, z_1, z_2 \in \mathbb{C}$; $x, y, u, v \in \mathbb{R}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers. Independent units i, j are such that $i^2 = j^2 = -1$ and $ij = ji$.

We denote the set of bicomplex numbers as $\mathbb{BC}$. The set of bicomplex numbers $\mathbb{BC}$ has two distinguished zero divisors $e_1 = \frac{1 + ij}{2}$ and $e_2 = \frac{1 - ij}{2}$. All bicomplex numbers $z = z_1 + jz_2$ can be expressed as

$$z = \beta_1 e_1 + \beta_2 e_2,$$

where $\beta_1 = z_1 - iz_2$ and $\beta_2 = z_1 + jz_2$. For any two bicomplex numbers $z = \beta_1 e_1 + \beta_2 e_2$ and $w = \gamma_1 e_1 + \gamma_2 e_2$, we have the following:

$$z \pm w = (\beta_1 \pm \gamma_1)e_1 + (\beta_2 \pm \gamma_2)e_2,$$

and

$$zw = (\beta_1\gamma_1)e_1 + (\beta_2\gamma_2)e_2.$$

The Euclidean norm $\|z\|$ on $\mathbb{BC}$ is defined as

$$\|z\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|\beta_1|^2 + |\beta_2|^2}{2}}.$$

A sequence (p_n) in $\mathbb{BC}$ is considered convergent to a limit p if and only if for each $\varepsilon > 0$ there corresponds an $n(\varepsilon)$, such that

$$\|p_n - p\| < \varepsilon \text{ for all } n \geqslant n(\varepsilon).$$

Remark 1. *A sequence (p_n) in $\mathbb{BC}$, defined as $p_n = x_n + iy_n + ju_n + ijv_n$, converges to $p = x + iy + ju + ijv$ in $\mathbb{BC}$, if and only if the sequences (x_n), (y_n), (u_n), and (v_n) converge to x, y, u, and v, respectively. That is,*

$$\lim_{n\to\infty} p_n = p$$

if and only if

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} u_n = u, \text{ and } \lim_{n\to\infty} v_n = v.$$

For details on bicomplex numbers, one may refer to Price [14]. Bicomplex numbers have been studied and successfully applied by Beg et al. [1], Sager, and Sağır [10]. Different classes of sequences of bicomplex numbers have been introduced and investigated by Srivastava and Srivastava [12], Wagh [13], Bera and Tripathy [2] – [5], Değirmen and Sağır [8], and others.

Definition 1. [9] *The subsequence $(p_{n(i)})$ of the sequence $p = (p_n)$ can be represented as a regular transformation A applied to p by letting $a_{i,n(i)} = 1$ and $a_{mn} = 0$, otherwise. A subsequence can never converge more slowly than the original sequence. Some subsequences converge at the same rate, and some converge faster than the sequence p, but in every case*

$$\underline{\lim}_{i} \frac{|p_{n(i)}|}{|p_i|} \leqslant 1.$$

Lemma 1. [9] *Suppose that A is a subsequence transformation that accelerates $p \in S_0$, and q is a sequence in S_0, such that $p \approx q$. Then A accelerates q.*

Lemma 2. [9] *Suppose that A is a subsequence transformation and $p \in S_0$. Then there exist $q, z \in S_0$, such that $q < p < z$ and A does not accelerate neither q nor z.*

Lemma 3. [9] *Suppose that A is a subsequence transformation and $p \in S_0$. Then there exists $q \in S_0$, such that $q < p$ and A accelerates q.*

Definition 2. [9] *Let $p \in S_0$, such that*

$$\lim_n \frac{p_{n+1} - p}{p_n - p} = \rho.$$

The sequence $p = (p_n)$ is said to converge linearly if $\rho \neq 0$ and $\rho \neq 1$ and to converge logarithmically if $\rho = 1$.

The collection of logarithmically convergent sequences is difficult to accelerate. Brezinski et al. [6] discovered the following result, which leads to an acceleration algorithm for a proper subset of logarithmically convergent sequences.

Lemma 4. [9] *Let p be a monotone logarithmically convergent element of S_0 and*

$$\Gamma(p) = \left\{ q \in S_0 \colon \text{there exist } \lambda \neq 0 \text{ such that } \lim_n \frac{q_n}{p_n} = \lambda \right\}.$$

For every $q \in \Gamma(p)$, there is an increasing sequence of positive integers $(n(i))$, such that $(q_{n(i)})$ is linearly convergent.

Throughout this article, c_0 denotes the set of all null sequences, and S_0 denotes the set of all sequences in c_0 that have at most a finite number of zero terms.

2. Main Results. In this section, we introduce some concepts of acceleration convergence and acceleration convergence of sub-sequential transformation in the setting of bicomplex numbers. Also, we generalize the definition of logarithmic convergence and introduce related theorems in the setting of bicomplex numbers.

Definition 3. *Let $c_0(\mathbb{BC})$ denote the set of all bicomplex null sequences, and let S_0' denote the set of all sequences in $c_0(\mathbb{BC})$ that have at most a finite number of zero terms. Let (p_n) and (q_n) be sequences in $\mathbb{BC}$. The sequence $p = (p_n)$ converges to λ faster than the sequence $q = (q_n)$ converges to σ $(p < q)$ in $\mathbb{BC}$ if*

$$\lim_n \frac{\|p_n - \lambda\|}{\|q_n - \sigma\|} = 0.$$

Remark 2. *In this instance, we can also say that the sequence $q = (q_n)$ converges to σ more slowly than the sequence $p = (p_n)$ converges to λ in $\mathbb{BC}$.*

The sequence $p = (p_n)$ converges to λ at the same rate as the sequence $q = (q_n)$ converges to σ $(p \approx q)$ if

$$0 < \underline{\lim}_n \frac{\|p_n - \lambda\|}{\|q_n - \sigma\|} \leqslant \overline{\lim}_n \frac{\|p_n - \lambda\|}{\|q_n - \sigma\|} < +\infty.$$

The matrix $A = (a_{mn})$ is said to accelerate the convergence of p in $\mathbb{BC}$ if $Ap < p$. The acceleration field of A is defined as $\{p\colon Ap < p\}$.

Definition 4. *A subsequence $(p_{n(i)})$ of the bicomplex sequence (p_n) can be represented as a regular matrix transformation A applied to p by defining $a_{i,n(i)} = 1$ and $a_{mn} = 0$, otherwise. A subsequence never converges more slowly than the original sequence. Some subsequences converge at the same rate, while others converge faster than the sequence (p_n), but in all cases*

$$\underline{\lim}_i \frac{\|p_{n(i)}\|}{\|p_i\|} \leqslant 1.$$

Definition 5. *An operator A is said to preserve the order of convergence, if for a sequence $(p_n) \in \mathbb{BC}$ that converges to $\lambda \in \mathbb{BC}$, the sequence $(A(p_n)) \in \mathbb{BC}$ converges to $\lambda \in \mathbb{BC}$. Formally,*

$$\lim_{n\to\infty} p_n = \lambda \implies \lim_{n\to\infty} Ap_n = \lambda.$$

This property is called the preservation of the limit under a regular transformation.

Theorem 1. *Let $p = (p_n)$, where $p_n = x_n + iy_n + ju_n + ijv_n$, and $q = (q_n)$, where $q_n = \alpha_n + i\beta_n + j\gamma_n + ij\delta_n$, be two sequence of bicomplex numbers. Then $p < q$ if and only if $x < \alpha$, $y < \beta$, $u < \gamma$, and $v < \delta$, where*

$x = (x_n)$, $y = (y_n)$, $u = (u_n)$, $v = (v_n)$, $\alpha = (\alpha_n)$, $\beta = (\beta_n)$, $\gamma = (\gamma_n)$, and $\delta = (\delta_n)$ are sequences of real numbers.

Proof. Suppose that $x < \alpha$, $y < \beta$, $u < \gamma$, and $v < \delta$, and that the sequences (x_n), (y_n), (u_n), (v_n), (α_n), (β_n), (γ_n), and (δ_n) converge to λ_1, λ_2, λ_3, λ_4, σ_1, σ_2, σ_3, and σ_4, respectively. Then

$$x < \alpha \Longrightarrow \lim_n \frac{x_n - \lambda_1}{\alpha_n - \sigma_1} = 0,$$

$$y < \beta \Longrightarrow \lim_n \frac{y_n - \lambda_2}{\beta_n - \sigma_2} = 0,$$

$$u < \gamma \Longrightarrow \lim_n \frac{u_n - \lambda_3}{\gamma_n - \sigma_3} = 0,$$

$$v < \delta \Longrightarrow \lim_n \frac{v_n - \lambda_4}{\delta_n - \sigma_4} = 0.$$

Now,

$$\begin{aligned}
&\lim_n \frac{x_n + iy_n + ju_n + ijv_n - (\lambda_1 + i\lambda_2 + j\lambda_3 + ij\lambda_4)}{\alpha_n + i\beta_n + j\gamma_n + ij\delta_n - (\sigma_1 + i\sigma_2 + j\sigma_3 + ij\sigma_4)} \\
&= \lim_n \frac{(x_n - \lambda_1) + i(y_n - \lambda_2) + j(u_n - \lambda_3) + ij(v_n - \lambda_4)}{(\alpha_n - \sigma_1) + i(\beta_n - \sigma_2) + j(\gamma_n - \sigma_3) + ij(\delta_n - \sigma_4)} \\
&= \lim_n \frac{\frac{x_n - \lambda_1}{\alpha_n - \sigma_1}}{1 + i\frac{\beta_n - \sigma_2}{\alpha_n - \sigma_1} + j\frac{\gamma_n - \sigma_3}{\alpha_n - \sigma_1} + ij\frac{\delta_n - \sigma_4}{\alpha_n - \sigma_1}} \\
&+ i \lim_n \frac{\frac{y_n - \lambda_2}{\beta_n - \sigma_2}}{i + \frac{\alpha_n - \sigma_1}{\beta_n - \sigma_2} + j\frac{\gamma_n - \sigma_3}{\beta_n - \sigma_2} + ij\frac{\delta_n - \sigma_4}{\beta_n - \sigma_2}} \\
&+ j \lim_n \frac{\frac{u_n - \lambda_3}{\gamma_n - \sigma_3}}{j + i\frac{\beta_n - \sigma_2}{\gamma_n - \sigma_3} + \frac{\alpha_n - \sigma_1}{\gamma_n - \sigma_3} + ij\frac{\delta_n - \sigma_4}{\gamma_n - \sigma_3}} \\
&+ ij \lim_n \frac{\frac{v_n - \lambda_4}{\delta_n - \sigma_4}}{ij + i\frac{\beta_n - \sigma_2}{\delta_n - \sigma_4} + j\frac{\gamma_n - \sigma_3}{\delta_n - \sigma_4} + \frac{\alpha_n - \sigma_1}{\delta_n - \sigma_4}}
\end{aligned}$$

This shows, that

$$\lim_n \frac{x_n + iy_n + ju_n + ijv_n - (\lambda_1 + i\lambda_2 + j\lambda_3 + ij\lambda_4)}{\alpha_n + i\beta_n + j\gamma_n + ij\delta_n - (\sigma_1 + i\sigma_2 + j\sigma_3 + ij\sigma_4)} = 0.$$

Therefore, $p < q$.

Conversely, suppose that $p = (p_n)$ converges faster than $q = (q_n)$. Then

$$\lim_n \frac{x_n + iy_n + ju_n + ijv_n - (\lambda_1 + i\lambda_2 + j\lambda_3 + ij\lambda_4)}{\alpha_n + i\beta_n + j\gamma_n + ij\delta_n - (\sigma_1 + i\sigma_2 + j\sigma_3 + ij\sigma_4)} = 0$$
$$\Rightarrow \lim_n \frac{(x_n - \lambda_1) + i(y_n - \lambda_2) + j(u_n - \lambda_3) + ij(v_n - \lambda_4)}{(\alpha_n - \sigma_1) + i(\beta_n - \sigma_2) + j(\gamma_n - \sigma_3) + ij(\delta_n - \sigma_4)} = 0$$
$$\Rightarrow \lim_n \frac{\frac{x_n-\lambda_1}{\alpha_n-\sigma_1}}{1 + i\frac{\beta_n-\sigma_2}{\alpha_n-\sigma_1} + j\frac{\gamma_n-\sigma_3}{\alpha_n-\sigma_1} + ij\frac{\delta_n-\sigma_4}{\alpha_n-\sigma_1}}$$
$$+ i\lim_n \frac{\frac{y_n-\lambda_2}{\beta_n-\sigma_2}}{i + \frac{\alpha_n-\sigma_1}{\beta_n-\sigma_2} + j\frac{\gamma_n-\sigma_3}{\beta_n-\sigma_2} + ij\frac{\delta_n-\sigma_4}{\beta_n-\sigma_2}}$$
$$+ j\lim_n \frac{\frac{u_n-\lambda_3}{\gamma_n-\sigma_3}}{j + i\frac{\beta_n-\sigma_2}{\gamma_n-\sigma_3} + \frac{\alpha_n-\sigma_1}{\gamma_n-\sigma_3} + ij\frac{\delta_n-\sigma_4}{\gamma_n-\sigma_3}}$$
$$+ ij\lim_n \frac{\frac{v_n-\lambda_4}{\delta_n-\sigma_4}}{ij + i\frac{\beta_n-\sigma_2}{\delta_n-\sigma_4} + j\frac{\gamma_n-\sigma_3}{\delta_n-\sigma_4} + \frac{\alpha_n-\sigma_1}{\delta_n-\sigma_4}} = 0.$$

This shows, that

$$\lim_n \frac{x_n - \lambda_1}{\alpha_n - \sigma_1} = 0,$$
$$\lim_n \frac{y_n - \lambda_2}{\beta_n - \sigma_2} = 0,$$
$$\lim_n \frac{u_n - \lambda_3}{\gamma_n - \sigma_3} = 0,$$
$$\lim_n \frac{v_n - \lambda_4}{\delta_n - \sigma_1} = 0.$$

Therefore, the sequences x, y, u, and v converge faster than α, β, γ, and δ, respectively. This completes the proof. □

Theorem 2. *If* (p_n) *and* (q_n) *converge to the same limit in* $\mathbb{BC}$ *and* $(q_n) \approx (p_n)$*, then convergence preservation of* A *for* (p_n) *implies convergence preservation for* (q_n).

Proof. Let (p_n) and (q_n) converge to the same limit λ, and suppose that they converge at the same rate. Let A be a convergence preserving matrix for (p_n); that is,

$$\lim_{n\to\infty} \frac{p_n - \lambda}{q_n - \lambda} = k, \tag{1}$$

where k is a non-zero positive constant.

Since A preserves convergence for (p_n), we have

$$\lim_{n\to\infty}\{A(p_n)-\lambda\}=0. \tag{2}$$

Then, by the above relations, it follows that

$$\lim_{n\to\infty}\{A(q_n)-\lambda\}=\lim_{n\to\infty}\frac{\{A(q_n)-\lambda\}}{\{A(p_n)-\lambda\}}\times\lim_{n\to\infty}\{A(p_n)-\lambda\}=\frac{1}{k}\times 0=0.$$

So, $Aq_n\to\lambda$ as $n\to\infty$. Therefore, A is convergence preserving for (q_n). □

Theorem 3. *If (p_n), (q_n), and (r_n) converge to the same limit, A is convergence preserving for all $(q_n)\approx(p_n)$, and $(p_n)<(r_n)$ in $\mathbb{BC}$, then A is convergence preserving for all $(q_n)\approx(r_n)$.*

Proof. Suppose (p_n), (q_n), and (r_n) converge to the same limit λ. Since A preserves the convergence for all $q\approx p$, it follows that

$$Ap_n\to\lambda \text{ and } Aq_n\to\lambda \quad \text{as } n\to\infty.$$

So, (p_n) converges to λ faster than (r_n). By the hypothesis, (p_n) preserves the convergence to the limit λ, hence, (r_n) also preserves the convergence to the limit λ. That is $Ar_n\to\lambda$ as $n\to\infty$.
Therefore, A is convergence preserving for all $(q_n)\approx(r_n)$.

Theorem 4. *Suppose that A is a subsequence transformation that accelerates $p\in S_0'$, and let $q\in S_0'$ be such that $p\approx q$. Then A accelerates q in $\mathbb{BC}$.*

Proof. Let $p=(p_n)$ and $q=(q_n)$ be the sequences in S_0' defined as

$$p_n=x_n+iy_n+ju_n+ijv_n,$$
$$q_n=\alpha_n+i\beta_n+j\gamma_n+ij\delta_n.$$

Let $(p_{n(i)})$ and $(q_{n(i)})$ be subsequences of (p_n) and (q_n), respectively. Since A is a subsequence transformation that accelerates $p\in S_0'$, and $q\in S_0'$ is such that $p\approx q$, we have

$$\varliminf_i\frac{\|p_{n(i)}\|}{\|p_i\|}\leqslant 1.$$

Moreover, since p and q converge at the same rate, it follows that

$$\varliminf_i\frac{\|q_{n(i)}\|}{\|p_{n(i)}\|}\leqslant 1,$$

and

$$\underline{\lim}_{i} \frac{\|p_i\|}{\|q_i\|} \leqslant 1.$$

Therefore,

$$\frac{\|q_{n(i)}\|}{\|q_i\|} = \frac{\|q_{n(i)}\|}{\|p_{n(i)}\|} . \frac{\|p_i\|}{\|q_i\|} . \frac{\|p_{n(i)}\|}{\|p_i\|} \leqslant 1.$$

This shows that A accelerates q. □

Theorem 5. *Suppose that A is a subsequence transformation and $p \in S_0'$. Then there exist q, $r \in S_0'$, such that $q < p < r$ and A does not accelerate q or r in $\mathbb{BC}$.*

Proof. Let $p = (p_n)$, where $p_n = x_n + iy_n + ju_n + ijv_n$ for $n \in \mathbb{N}$, be a sequence of bicomplex numbers. First, consider the sequence of real parts (x_n) of the sequence (p_n). Let A represent the subsequence transformation as defined by

$$(Ax)_i = x_{n(i)}.$$

Case 1. Construct a sequence $\alpha = (\alpha_n)$ of real numbers as follows:

$$\alpha_1 = \alpha_{n(1)} = \min\{|x_1|^2, |x_{n(1)}|^2\},$$

and

$$\alpha_i = |x_i|^2 \text{ for } 1 < i < n(1).$$

Next, define

$$\alpha_{n(1)+1} = \alpha_{n(n(1)+1)} = \min\{|x_{n(1)+1}|^2, |x_{n(n(1)+1)}|^2\},$$

and

$$\alpha_i = |x_i|^2 \text{ for } n(1) + 1 < i < n(n(1) + 1).$$

Similarly, define

$$\alpha_{n(n(1)+1)+1} = \alpha_{n(n(n(1)+1)+1)},$$

and continue this process inductively.
Then $\alpha < x$, and

$$\overline{\lim}_{i} \frac{|\alpha_{n(i)}|}{|\alpha_i|} \geqslant 1.$$

Hence, A does not accelerate α.

Similarly, we can construct sequences $\beta = (\beta_n)$, $\gamma = (\gamma_n)$, and $\delta = (\delta_n)$ of real numbers, such that $\beta < y$, $\gamma < u$, and $\delta < v$. Hence, A does not accelerate β, γ, or δ, respectively.

Let

$$q = (q_n) = \alpha_n + i\beta_n + j\gamma_n + ij\delta_n.$$

By Theorem 1, we can conclude that A does not accelerate q.

Case 2. Construct a sequence $g = (g_n)$ of real numbers as follows:

$$g_1 = g_{n(1)} = \max\{|x_1|^{1/2}, |x_{n(1)}|^2\},$$

and

$$g_i = |x_i|^{1/2}, \text{ for } 1 < i < n(1).$$

Similarly, define

$$g_{n(1)+1} = g_{n(n(1)+1)} = \max\{|x_{n(1)+1}|^{1/2}, |x_{n(n(1)+1)}|^{1/2}\},$$

and continue this process inductively.
Then $x < g$, and

$$\overline{\lim_i} \frac{|g_{n(i)}|}{|g_i|} \geqslant 1.$$

Hence, A does not accelerate g.

Similarly, we can construct sequences $h = (h_n)$, $t = (t_n)$, and $s = (s_n)$ of real numbers, such that $y < h$, $u < t$, and $v < s$. Hence, A does not accelerate h, t, or s, respectively.
Let

$$r = (r_n) = g_n + ih_n + jt_n + ijs_n.$$

Hence, by Theorem 1, we conclude that A does not accelerate r. Hence, A does not accelerate q or r. This completes the proof. □

Theorem 6. *Suppose that A is a subsequence transformation and $p \in S_0'$. Then there exists $q \in S_0'$, such that $q < p$ and A accelerates q.*

Proof. Let $p = (p_n)$, where $p_n = x_n + iy_n + ju_n + ijv_n$, for $n \in \mathbb{N}$, be a sequence of bicomplex numbers. First, consider the sequence of real parts (x_n) of the sequence (p_n). Let A represent the subsequence transformation as defined by

$$(A(x_i)) = x_{n(i)}.$$

Construct a sequence $\alpha = (\alpha_n)$ as follows: define

$$\alpha_i = |x_i|^2 \text{ for } 1 < i < n(1),$$

and

$$\alpha_1 = \alpha_{n(1)} = \min\{\alpha_1, |x_{n(1)}|^2\}.$$

Next, define

$$\alpha_i = |x_i|^2 \quad \text{for } n(1) < i < n(2),$$

and

$$\alpha_{n(2)} = \frac{\min\{\alpha_2, |x_{n(2)}|^2\}}{2}.$$

Continuing this process, and defining at each stage, we obtain

$$\alpha_{n(k)} = \frac{\min\{\alpha_k, |x_{n(k)}|^2\}}{k}.$$

It follows that $\alpha < x$ and

$$\underline{\lim}_{i} \frac{|\alpha_{n(i)}|}{|\alpha_i|} \leqslant 1.$$

Hence, A accelerates α.

Similarly, we can construct sequences $\beta = (\beta_n)$, $\gamma = (\gamma_n)$, and $\delta = (\delta_n)$ of real numbers, such that $\beta < y$, $\gamma < u$, and $\delta < v$. Hence, A accelerates β, γ, and δ, respectively.
Let

$$q = (q_n) = \alpha_n + i\beta_n + j\gamma_n + ij\delta_n.$$

Hence, by Theorem 1, we conclude that $q < p$. Therefore, A accelerates q. This completes the proof. □

Definition 6. *Let $p = (p_n)$ be any sequence of bicomplex numbers, such that*

$$\lim_{n} \frac{p_{n+1} - p}{p_n - p} = \rho.$$

The sequence $p = (p_n)$ is said to converge logarithmically if $\rho = 1$, and converge linearly if $0 < \|\rho\| < 1$.

Accelerating the set of logarithmically convergent sequences is challenging. We develop a bicomplex version of Lemma 4, which yields an acceleration method for a proper subset of logarithmically convergent sequences in $\mathbb{BC}$.

Theorem 7. *Let $p = (p_n)$ and $q = (q_n)$ be sequences of bicomplex numbers, such that $\|p_n\|$ is a monotone logarithmically convergent element of S'_0, and*

$$\Gamma(p) = \left\{ q \in S'_0 \colon \textit{there exist } \lambda \neq 0 \textit{ such that } \lim_n \frac{\|q_n\|}{\|p_n\|} = \lambda \right\}.$$

Then there exists an increasing sequence of positive integers $(n(i))$, such that, for each $q \in \Gamma(p)$, the subsequence $(q_{n(i)})$ is linearly convergent.

Proof. Let $(p_{n(i)})$ be a subsequence of the sequence (p_n). Then, by the subsequence transformation of (p_n), we have:

$$\frac{\|p_{n(i)}\|}{\|p_i\|} \leqslant 1. \tag{3}$$

Moreover,

$$\frac{\|p_{n(i)+1}\|}{\|p_{n(i)}\|} < 1. \tag{4}$$

Let $(q_{n(i)})$ be a subsequence of the sequence (q_n). For each $q \in \Gamma(p)$, we have

$$\lim_i \frac{\|q_{n(i)}\|}{\|p_{n(i)}\|} = \lambda \text{ and } \lim_i \frac{\|q_i\|}{\|p_i\|} = \lambda. \tag{5}$$

Also,

$$\lim_i \frac{\|q_{n(i)+1}\|}{\|q_{n(i)}\|} > 0. \tag{6}$$

Using the inequalities (3), (4), (5), we obtain

$$\lim_i \frac{\|q_{n(i)+1}\|}{\|q_{n(i)}\|} < 1. \tag{7}$$

Combining(6) and (7), we conclude that

$$0 < \lim_i \frac{\|q_{n(i)+1}\|}{\|q_{n(i)}\|} < 1.$$

This shows that $(q_{n(i)})$ is linearly convergent. □

Theorem 8. *Let (p_n) be a sequence of bicomplex numbers, and suppose that $\|p_n\|$ is a monotone logarithmic sequence. For every $\rho \in (0,1)$, there exists $a\colon \mathbb{N} \to \mathbb{N}$ strictly increasing sequence, such that*

$$\lim_{i\to\infty} \frac{\|(p_{a(i+1)} - p)\|}{\|(p_{a(i)} - p)\|} = \rho.$$

Proof. Let

$$a(i) = \min\{j > a(i-1)\colon \|p_j - p\| < \rho^i \|p_0 - p\|\}$$

with $a(0) = 0$.
Let

$$\varepsilon_i = \frac{\|\rho^i(p_0 - p) - (p_{a(i)} - p)\|}{\|(p_{a(i)} - p)\|} = \rho^i \frac{\|p_0 - p\|}{\|p_{a(i)} - p\|} - 1 > 0.$$

Since (p_n) is logarithmic and $\rho < 1$, for i sufficiently large,

$$\|p_{a(i)-1} - p\| \geqq \rho^i \|p_0 - p\|.$$

Thus,

$$\varepsilon_i \leqq \frac{\|(p_{a(i)-1} - p) - (p_{a(i)} - p)\|}{\|(p_{a(i)} - p)\|} = \frac{\|(p_{a(i)-1} - p)\|}{\|(p_{a(i)} - p)\|} - 1 \to 0 \text{ as } i \to \infty.$$

And

$$\frac{\|(p_a(i+1) - p)\|}{\|(p_{a(i)-p})\|} = \frac{\rho^{i+1}\|(p_0 - p)\|}{1 + \varepsilon_{i+1}} \frac{1 + \varepsilon_i}{\rho^i \|(p_0 - p)\|} \to \rho \text{ as } i \to \infty.$$

Remark 3. *The assumption on the monotonicity of $\|(p_n)\|$ can be removed because it can be easily shown that if (p_n) is a logarithmic sequence, then there exists $a\colon \mathbb{N} \to \mathbb{N}$ strictly increasing sequence, such that $\|(p_{a(i)})\|$ is monotone and logarithmic.*

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