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TOPOLOGICAL PROPERTY AND OPERATOR IDEAL IN BRONZE LEONARDO SEQUENCE SPACES

Abstract. In this study, we introduce new Banach sequence spaces $\ell_p(\Psi), \ell_\infty(\Psi), c_0(\Psi)$, and $c(\Psi)$, defined via a regular infinite matrix $\Psi = (\Psi_{nk})$, where

$$\Psi = \begin{cases} \dfrac{3\Psi_k}{4\Psi_n + \Psi_{n-1} - n - 1}, & 1 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases}$$

and $\{\Psi_k\}$ denotes the Bronze Leonardo number sequence. We examine the fundamental properties of these spaces, particularly their topological structure. In addition, we construct Schauder bases for these new sequence spaces and present several results concerning their operator ideals.

Key words: *Bronze Leonardo numbers, sequence space, Schauder basis, operator ideals*

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1. Introduction. Matrix operators play a fundamental role in the analysis of linear mappings on sequence spaces, particularly in understanding how infinite-dimensional structures can be studied through discrete representations. Among their many properties, compactness has proven especially significant in functional analysis and spectral theory. A compact matrix operator, typically represented by an infinite matrix acting on spaces such as ℓ_p or c_0, exhibits behavior analogous to finite-dimensional operators in that it can be approximated by operators of finite rank. This property is central to the study of eigenvalue distributions, as compact operators possess a discrete spectrum with zero as the only possible accumulation point. Furthermore, compactness conditions are closely tied to summability theory, where infinite matrices define transformations that preserve or improve convergence of sequences. These connections make compact matrix operators indispensable in approximation methods and in the analysis of operator equations, particularly within the framework of Fredholm theory.

Let $\mathbb{N}$ be the set of natural numbers, and let ω denote the space of all real-valued sequences. The spaces ℓ_∞, c_0, and c are, respectively, the sets of bounded sequences, sequences that converge to zero, and sequences that converge. Their norm is given by $\|x\|_\infty = \sup_k |x_k|$. The space ℓ_p is the set of sequences whose p-th powers are summable, for $1 \leqslant p < \infty$, with norm $\|x\|_{\ell_p} = \left(\sum_k |x_k|^p\right)^{1/p}$. We will also use the notation $e = (1, 1, 1, \ldots)$, and denote by $e^{(k)}$ the sequence whose only nonzero term is 1 in k-th position.

A Banach space $\mathfrak{X}$ is called a BK-space if the map that takes a sequence to its n-th term, $x \mapsto x_n$, is continuous for every n. Examples include ℓ_p and ℓ_∞. Given two sequence spaces $\mathfrak{X}$ and $\mathfrak{Y}$, and an infinite real matrix $A = (a_{nk})$, we denote the n-th row as A_n. The matrix A maps $\mathfrak{X}$ to $\mathfrak{Y}$ if for every sequence $x = (x_k)$, the sequence

$$Ax = \{A_n x\}_{n=0}^{\infty} \quad \text{with} \quad A_n x = \sum_k a_{nk} x_k \tag{1}$$

belongs to $\mathfrak{Y}$. The domain of A is the set $\mathfrak{X}_A = \{x \in \omega : Ax \in \mathfrak{Y}\}$. The notation $(\mathfrak{X}, \mathfrak{Y})$ designates the family of all matrices A mapping from $\mathfrak{X}$ into $\mathfrak{Y}$. Thus, $A \in (\mathfrak{X}, \mathfrak{Y})$ precisely when the series in equation (1) converges for every $n \in \mathbb{N}$ and each $x \in \mathfrak{X}$, which guarantees that $Ax \in \mathfrak{Y}$ for all $x \in \mathfrak{X}$. Yaying and Hazarika [15] defined the Tribonacci sequence spaces $\ell_p(T)$ $(1 \leqslant p \leqslant \infty)$ as the domain corresponding to a newly introduced regular Tribonacci matrix.

Recent work by Demiriz et al. [5] introduced new BK-spaces named $\ell_p(G)$ and $\ell_\infty(G)$ using generalized Motzkin matrices, uncovering their basis properties in the process. Later on, Erdem et al. [6] developed the Motzkin matrix spaces $c(\mathcal{M})$ and $c_0(\mathcal{M})$, revealing their inner structure and introducing the fresh idea of the Motzkin core. Building upon these findings, Erdem [7] explored $\ell_p(\mathcal{M})$ spaces along with compact operators, while in [8] Erdem extended the theory to Schröder–Catalan matrix spaces with similar results.

2. Bronze Leonardo matrix Operator and Bronze Leonardo Sequence Spaces. The Bronze Leonardo sequence $\{\Psi_n\}_{n\geqslant 0}$ is a non-homogeneous Leonardo-type integer sequence associated with the bronze ratio (see [9]). It is defined by the recurrence relation

$$\Psi_{n+2} = 3\Psi_{n+1} + \Psi_n + 1, \quad n \geqslant 0,$$

with the initial conditions; $\Psi_0 = 1$, $\Psi_1 = 1$. This recurrence produces the sequence

$$1, 1, 5, 17, 57, 189, 625, 2065, 6821, 22529, \ldots$$

Equivalently, the sequence $\{\Psi_n\}$ satisfies the third-order homogeneous recurrence relation

$$\Psi_{n+3} = 4\Psi_{n+2} - 2\Psi_{n+1} - \Psi_n,$$

with initial values $\Psi_0 = 1$, $\Psi_1 = 1$, and $\Psi_2 = 5$. The associated characteristic equation is

$$x^3 - 4x^2 + 2x + 1 = 0,$$

whose three roots are

$$\lambda = \frac{3+\sqrt{13}}{2}, \quad \psi = \frac{3-\sqrt{13}}{2}, \quad \delta = 1.$$

Thus, λ, ψ, and δ denote the three roots of the above characteristic equation. In particular, $\delta = 1$ is the third root, whereas λ, commonly known as the bronze ratio, is the dominant positive root. The generating function of the Bronze Leonardo sequence is

$$\sum_{n=0}^{\infty} \Psi_n x^n = \frac{1 + x - 3x^2}{1 - 4x + 2x^2 + x^3}.$$

We can easily derive the relation

$$\sum_{s=1}^{k} \Psi_s = \frac{4\Psi_k + \Psi_{k-1} - n - 1}{3}.$$

For a nonnegative integer k, let Ψ_k represent the k-th Bronze Leonardo number. Consider the matrix $\Psi = (\Psi_{nk})$, defined by

$$\Psi = \begin{cases} \dfrac{3\Psi_k}{4\Psi_n + \Psi_{n-1} - n - 1}, & 1 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases}$$

where $n, k = 1, 2, \ldots$.

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \frac{3}{18} & \frac{15}{18} & 0 & 0 & 0 & 0 & \ldots \\ \frac{3}{69} & \frac{15}{69} & \frac{51}{69} & 0 & 0 & 0 & \ldots \\ \frac{3}{240} & \frac{15}{240} & \frac{51}{240} & \frac{171}{240} & 0 & 0 & \ldots \\ \frac{3}{807} & \frac{15}{807} & \frac{51}{807} & \frac{567}{807} & \frac{1182}{807} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is obvious that Ψ is triangular. Therefore, the Ψ-transform of a sequence $b' = (b'_k)$ is expressed as

$$\Omega'_n = (\Psi b')_n = \frac{3 \sum\limits_{k=1}^{n} \Psi_k b'_k}{4\Psi_n + \Psi_{n-1} - n - 1} \tag{2}$$

Lemma 1. [14] *An infinite matrix $\mathcal{B} = (b_{n,k})$ qualifies as regular if, and only if, each of the conditions listed below is satisfied.*

(i) $\sup\limits_{n \in \mathbb{N}} \sum\limits_k |b_{nk}| < \infty$

(ii) $\lim\limits_{n \to \infty} \sum\limits_k b_{nk} = 1$

(iii) $\lim\limits_{n \to \infty} b_{nk} = 0.$

Corollary. *Bronze Leonardo matrix $\Psi = \Psi_{n,k}$ is regular.*

3. Bronze Leonardo sequence spaces.

We now introduce the Bronze Leonardo sequence spaces $\ell_p(\Psi)$ $(1 \leqslant p < \infty)$, $\ell_\infty(\Psi)$, $c(\Psi)$, and $c_0(\Psi)$. A sequence belongs to one of these spaces if and only if its Ψ-transform lies in the respective classical sequence space ℓ_p, ℓ_∞, c_0, or c.

$$\ell_p(\Psi) = \left\{ h = (h_k) \in \omega \colon \sum_{n=1}^{\infty} \left| \frac{3 \sum\limits_{k=1}^{n} \Psi_k h_k}{4\Psi_n + \Psi_{n-1} - n - 1} \right|^p < \infty \right\} (1 \leqslant p < \infty).$$

$$\ell_\infty(\Psi) = \left\{ h = (h_k) \in \omega \colon \sup_{n \in \mathbf{N}} \left| \frac{3 \sum\limits_{k=1}^{n} \Psi_k h_k}{4\Psi_n + \Psi_{n-1} - n - 1} \right| < \infty \right\}.$$

$$c_0(\Psi) = \left\{h = (h_k) \in \omega\colon \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{3\sum\limits_{k=1}^{n} \Psi_k h_k}{4\Psi_n + \Psi_{n-1} - n - 1} = 0\right\}.$$

$$c(\Psi) = \left\{h = (h_k) \in \omega\colon \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{3\sum\limits_{k=1}^{n} \Psi_k h_k}{4\Psi_n + \Psi_{n-1} - n - 1} \text{ exists}\right\}.$$

We can express $\mathcal{G}(\Psi)$ as $\mathcal{G}_{(\Psi)}$, where $\mathcal{G}$ denotes any of the spaces ℓ_p, ℓ_∞, c_0, or c, where $p \in [1,\infty)$.

4. Topological Property.

Theorem 1. *The sequence spaces $\ell_p(\Psi)$ and $\ell_\infty(\Psi)$ are BK-spaces normed, respectively, by*

$$\|h\|_{\ell_p(\Psi)} = \left(\sum_{n=1}^{\infty} \left|\frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \sum_{k=1}^{n} \Psi_k\, h_k\right|^p\right)^{\frac{1}{p}}, \tag{3}$$

and

$$\|h\|_{\ell_\infty(\Psi)} = \sup_{n\in\mathbb{N}} \left|\frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \sum_{k=1}^{n} \Psi_k\, h_k\right|. \tag{4}$$

Moreover, both spaces are of non-absolute type.

Proof. Clearly, the matrix Ψ is triangular. By the definition of matrix domains, we have

$$(\ell_p)_\Psi = \ell_p(\Psi) \quad \text{and} \quad (\ell_\infty)_\Psi = \ell_\infty(\Psi).$$

Since the classical spaces ℓ_p $(1 \leqslant p < \infty)$ and ℓ_∞ are BK-spaces, and since Ψ is triangular, it follows from Theorem 4.3.12 in [14] that the matrix domains $\ell_p(\Psi)$ and $\ell_\infty(\Psi)$ are also BK-spaces endowed with the norms defined above.

To show that $\ell_p(\Psi)$ is of non-absolute type, consider the sequence

$$a = (1, -1, 0, 0, \ldots).$$

Using (2), we obtain $\Psi_n(a) = \left(1, -\frac{12}{18}, -\frac{12}{69}, \ldots\right)$. On the other hand, for $|a| = (1, 1, 0, 0, \ldots)$, we have $\Psi_n(|a|) = (1, 1, 1, \ldots)$. Hence, $|\Psi_n(a)| \neq \Psi_n(|a|)$, which implies that

$$\|a\|_{\ell_p(\Psi)} \neq \||a|\|_{\ell_p(\Psi)}.$$

Therefore, $\ell_p(\Psi)$ is not of absolute type. The same argument applies to $\ell_\infty(\Psi)$. □

Theorem 2. *The spaces $c_0(\Psi)$ and $c(\Psi)$ are BK-spaces, where the norm is given by*

$$\|h\|_{c_0(\Psi)} = \|h\|_{c(\Psi)} = \sup_{k\in\mathbb{N}} |(\Psi h)_k|.$$

Proof. According to Theorem 4.3.12 in Wilansky (1984, p. 63) [14], it is clear that $c_0(\Psi)$ and $c(\Psi)$ are BK-spaces, equipped with the given norm. □

Theorem 3. *The spaces $\ell_p(\Psi)$ and $\ell_\infty(\Psi)$ are linearly isomorphic to the classical sequence spaces ℓ_p and ℓ_∞, respectively.*

Proof. In this section, we establish the result for the space $\ell_p(\Psi) \cong \ell_p$; the proof for $\ell_\infty(\Psi) \cong \ell_\infty$ follows analogously.
Consider a mapping

$$\mathcal{H} : \ell_p(\Psi) \to \ell_p \text{ s.t. } \mathcal{H}(b') = \Psi b'.$$

From the result $\mathcal{H}(b') = 0 \implies b' = 0$, the injection property of $\mathcal{H}$ follows. Suppose $\Omega' = (\Omega'_k)$ lies in the sequence space ℓ_p, where $1 \leqslant p \leqslant \infty$. Then we define another sequence $b' = (b'_k)$ given by

$$b'_k = \sum_{l=k-1}^{k} (-1)^{k-l} \frac{4\Psi_l + \Psi_{l-1} - n - 1}{3\Psi_k} \Omega'_l, \quad (k \in \mathbb{N}). \tag{5}$$

Then, for $1 \leqslant p < \infty$, we have

$$\begin{aligned} \|b'\|_{\ell_p(\Psi)} &= \left(\sum_{k=1}^{\infty} |\Psi_k b'|^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty} \left(\sum_{l=1}^{k} \frac{3\Psi_l}{4\Psi_k + \Psi_{k-1} - n - 1} b'_l \right)^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{\infty} \left(\sum_{l=1}^{k} \frac{3\Psi_l \sum\limits_{j=l-1}^{l} (-1)^{l-j} \frac{4\Psi_k + \Psi_{k-1} - n - 1}{3\Psi_l} \Omega'_j}{4\Psi_k + \Psi_{k-1} - n - 1} \right)^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{\infty} |\Omega'_k|^p \right)^{\frac{1}{p}} = \|\Omega'\|_{\ell_p} < \infty. \end{aligned}$$

For $p = \infty$, we have

$$\|b'\|_{\ell_\infty(\Psi)} = \sup_{k\in\mathbb{N}} |\Psi_k b'| = \sup_{k\in\mathbb{N}} |\Omega'_k| = \|\Omega'\|_\infty < \infty.$$

This indicates that $b' \in \ell_p(\Psi)$ (for $1 \leqslant p \leqslant \infty$). Thus, $\mathcal{H}$ is both surjective and norm-preserving. As a result, $\ell_p(\Psi) \cong \ell_p$ for $1 \leqslant p \leqslant \infty$. □

Theorem 4. *The sequence spaces $c_0(\Psi)$ and $c(\Psi)$ are linearly isomorphic to the classical spaces c_0 and c, respectively.*

Proof. We define the mapping

$$\mathcal{T}\colon c_0(\Psi) \to c_0 \text{ s.t. } \mathcal{T}(b') = \Psi b'.$$

From the result $\mathcal{T}(b') = 0 \implies b' = 0$, the injection property of $\mathcal{T}$ follows. Furthermore, let $\Omega' \in c_0$ and define the sequence $b' = (b'_k)$ by

$$b'_k = \sum_{l=k-1}^{k} (-1)^{k-l} \frac{4\Psi_l + \Psi_{l-1} - n - 1}{3\Psi_k} \Omega'_l, \quad (k \in \mathbb{N}). \tag{6}$$

Then

$$\begin{aligned}\lim_{k\to\infty} (\Psi b')_k &= \lim_{k\to\infty} \left(\sum_{l=1}^{k} \frac{3\Psi_l}{4\Psi_k + \Psi_{k-1} - n - 1} b'_l \right)\\ &= \lim_{k\to\infty} \left(\sum_{l=1}^{k} \frac{3\Psi_l}{4\Psi_k + \Psi_{k-1} - n - 1} \sum_{j=l-1}^{l} (-1)^{l-j} \frac{4\Psi_l + \Psi_{l-1} - n - 1}{3\Psi_l} \Omega'_l \right)\\ &= \lim_{k\to\infty} \Omega'_k = 0.\end{aligned}$$

Therefore, $b' \in c_0(\Psi)$. Thus, $\mathcal{T}$ is surjective and preserves the norm. Consequently, $c_0(\Psi) \cong c_0$. The other can be done in a similar way. □

Theorem 5. *Inclusion $\ell_p \subset \ell_p(\Psi)$ holds.*

Proof. Let $b = (b'_k) \in \ell_p$ with $1 < p < \infty$. Then, applying Hölder's inequality for each $n \in \mathbb{N}$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} |\Psi_n b'|^p &\leqslant \sum_{n} \left(\sum_{k=1}^{n} \frac{3\Psi_k |b'_k|}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^p\\ &\leqslant \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{3\Psi_k |b'_k|^p}{4\Psi_n + \Psi_{n-1} - n - 1} \right) \left(\sum_{k=1}^{n} \frac{3\Psi_k}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^{p-1}\\ &= \sum_{n=1}^{\infty} \frac{3 \sum\limits_{k=1}^{n} \Psi_k |b'_k|^p}{4\Psi_n + \Psi_{n-1} - n - 1} = \sum_{k=1}^{\infty} |b'_k|^p \Psi_k \sum_{n=k}^{\infty} \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1}.\end{aligned}$$

Therefore,

$$\|b'\|^p_{\ell_p(\Psi)} \leqslant D\,\|b'\|^p_{\ell_p} < \infty, \text{ where } D = \sup_k \left(\Psi_k \sum_{n=k}^{\infty} \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \right).$$

This ensures that $b' \in \ell_p(\Psi)$, and hence we conclude that $\ell_p \subseteq \ell_p(\Psi)$. By the same reasoning, one also obtains $\ell_1 \subseteq \ell_1(\Psi)$; the proof is analogous and is therefore omitted. □

Theorem 6. *The inclusion $\ell_\infty \subset \ell_\infty(\Psi)$ holds.*

Proof. Let $b' = (b'_k) \in \ell_\infty$. Then there exists a constant $S > 0$ with $|b'_k| \leqslant S$ for all $k \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, we obtain

$$|\Psi_n b'| \leqslant \frac{3\sum_{k=1}^{n} \Psi_k |b'_k|}{4\Psi_n + \Psi_{n-1} - n - 1} \leqslant \frac{2S\sum_{k=1}^{n} \Psi_k}{4\Psi_n + \Psi_{n-1} - n - 1} = S.$$

Thus, $\Psi_n b' \in \ell_\infty$ for $n \in \mathbb{N}$, which means $b' \in \ell_\infty(\Psi)$. Consequently, we have $\ell_\infty \subset \ell_\infty(\Psi)$. □

Theorem 7. *The inclusions $c_0 \subset c_0(\Psi)$ and $c \subset c(\Psi)$ are strict.*

Proof. Since the matrix Ψ is regular, the inclusions follow naturally. To demonstrate strictness, consider the sequence $b' = (1, 0, 1, 0, \dots)$. We can compute the following for this sequence:

$$(\Psi b')_n = \sum_{k=1}^{n} \frac{3\Psi_k}{4\Psi_n + \Psi_{n-1} - n - 1} b'_k = \frac{3\left(\Psi_1 + \cdots + \Psi_n\right)}{4\Psi_n + \Psi_{n-1} - n - 1}, \text{ where } n \in \mathbb{N}.$$

This expression converges, which implies that $b' \in c(\Psi)\backslash c$. A similar approach can be used to prove the other case. □

Theorem 8. *The inclusion $c_0(\Psi) \subset c(\Psi)$ holds strictly.*

Proof. To illustrate that the inclusion $c_0(\Psi) \subset c(\Psi)$ holds, consider the sequence $b' = (b'_k)$ defined by $b'_k = 1$ for every k. In this case, we obtain

$$(\Psi b')_n = \sum_{k=1}^{n} \frac{3\Psi_k}{4\Psi_n + \Psi_{n-1} - n - 1} = 1.$$

Since $\Psi b' \in c$ and $\Psi b' \notin c_0$, so, $b' \in c(\Psi)\backslash c_0(\Psi)$, which proves the result. □

Theorem 9. *For $1 \leqslant p < \infty$, the space $\ell_p(\Psi)$ is not a Hilbert space unless $p = 2$.*

Proof. Let us define the sequences $b' = (b'_k) = \left(1, 1, -\frac{48}{111}, 0, \dots\right)$ and $\Omega' = (\Omega'_k) = \left(1, -\frac{60}{36}, \frac{48}{111}, 0, \dots\right)$. Applying the operator Ψ, we get

$$\Psi_n b' = (1, 1, 0, 0, \dots) \quad \text{and} \quad \Psi_n \Omega' = (1, -1, 0, 0, \dots).$$

Next, we calculate

$$\|b' + \Omega'\|^2_{\ell_p(\Psi)} + \|b' - \Omega'\|^2_{\ell_p(\Psi)} = 8 \neq 2\left(\|b'\|^2_{\ell_p(\Psi)} + \|\Omega'\|^2_{\ell_p(\Psi)}\right) \quad \text{for } p \neq 2.$$

This result indicates that the parallelogram law is not fulfilled for $\ell_p(\Psi)$ when $p \neq 2$. Consequently, $\ell_p(\Psi)$ is not a Hilbert space unless $p = 2$. □

Theorem 10. *For $1 \leqslant p < r$, the inclusion $\ell_p(\Psi) \subsetneq \ell_r(\Psi)$ holds.*

Proof. Let $1 \leqslant p < r$. Since $\ell_p \subset \ell_r$, it follows that $\ell_p(\Psi) \subset \ell_r(\Psi)$. To show this inclusion is strict, consider $b' = (b'_k) \in \ell_r(\Psi) \backslash \ell_p(\Psi)$ and define $\Omega = (\Omega_k)$ by

$$\Omega_k = \frac{b'_k(4\Psi_k + \Psi_{k-1} - n - 1) - b'_{k-1}(4\Psi_k + \Psi_{k-1} - n - 1)}{3\Psi_k}, \quad k \in \mathbb{N}.$$

Now, for $\Psi_n \Omega$, we compute

$$\Psi_n \Omega = \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \sum_{k=1}^{n} \Psi_k \Omega_k.$$

Substituting Ω_k into the sum gives

$$\Psi_n \Omega = \frac{3 \sum_{k=1}^{n} \left[b'_k(4\Psi_n + \Psi_{n-1} - n - 1) - b'_{k-1}(4\Psi_n + \Psi_{n-1} - n - 1)\right]}{4\Psi_n + \Psi_{n-1} - n - 1}.$$

Simplifying, we find $\Psi_n \Omega = b'_n, \quad \forall n \in \mathbb{N}$. Thus, $\Psi_n \Omega = b'$, so $\Psi\Omega \in \ell_r \backslash \ell_p$. Hence, $\Omega \in \ell_r(\Psi) \backslash \ell_p(\Psi)$, proving $\ell_p(\Psi) \subsetneq \ell_r(\Psi)$. □

Definition 1. *Let $(Z, \| \cdot \|_Z)$ be a normed space. A sequence $z = (z_j)$ constitutes a Schauder basis provided that, for each $w \in Z$, there is a unique sequence of coefficients (c_j) with the property that*

$$\lim_{n \to \infty} \left\| w - \sum_{j=0}^{n} c_j z_j \right\| = 0.$$

The mapping $\mathcal{H}\colon \mathcal{G}(\Psi) \to \mathcal{G}$, introduced in the proof of the preceding Theorem 3 and Theorem 4 is an isomorphism between the two spaces. Consequently, the inverse image under $\mathcal{H}$ of the canonical basis $\{e^{(k)}\}_{k\in\mathbb{N}}$ of the space $\mathcal{G}$ forms a Schauder basis for the newly defined function space $\mathcal{G}(\Psi)$. Therefore, we obtain the following result:

Theorem 11. *Consider the sequence* $b^{(k)} = (b_n^{(k)})$ *defined for each fixed* $k \in \mathbb{N}$ *as follows:*

$$b_n^{(k)} = \begin{cases} \dfrac{(-1)^{n-k}(4\Psi_n + \Psi_{n-1} - n - 1)}{3\Psi_k}, & \text{if } 1 \leqslant k \leqslant n), \\ 0, & \text{if } k > n. \end{cases}$$

Then we have the following results:

(i) The family of sequences $\{b^{(k)}\}_{k\in\mathbb{N}}$ *constitutes a basis for the space* $\ell_p(\Psi)$*. Furthermore, each element* $b' \in \ell_p(\Psi)$ *admits a unique expansion of the form*

$$b' = \sum_k \alpha_k b^{(k)},$$

where $\alpha_k = (\Psi b')_k$ *for each* $k \in \mathbb{N}$*.*

(ii) The family of sequences $\{b^{(k)}\}_{k\in\mathbb{N}}$ *constitutes a basis of the space* $c_0(\Psi)$*. Furthermore, any sequence* $b' \in c_0(\Psi)$ *admits a unique expansion in terms of this basis.*

$$b' = \sum_k \alpha_k b(k),$$

where $\alpha_k = (\Psi b')_k$ *for each* $k \in \mathbb{N}$*.*

(iii) The set of sequences $\{e, b^{(0)}, b^{(1)}, \ldots\}$ *serves as a basis for the space* $c(\Psi)$*. Moreover, every element* $b' \in c(\Psi)$ *can be written uniquely as a linear combination of these basis sequences*

$$b' = fe + \sum_k (\alpha_k - f) b(k),$$

where $\alpha_k = (\Psi b')_k$ *for all* $k \in \mathbb{N}$*, and* $f = \lim_{k\to\infty} (\Psi b')_k$*.*

Corollary. *For* $1 \leqslant p < \infty$*, the space* $\ell_p(\Psi)$ *is separable.*

Proof. This follows directly from Theorems 1 and 11. □

Corollary. *The sequence spaces* $c_0(\Psi)$ *and* $c(\Psi)$ *are separable.*

Proof. The claim is an immediate consequence of Theorems 2 and 11. $\square$

5. Operator ideal $\ell_p^{(s)}(\Psi)$. In this section, we analyze the characteristics of s-type $\ell_p(\Psi)$ operators concerning the Bronze Leonardo sequence space. The symbol $\mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$, or simply $\mathcal{L}$ refers to the collection of all bounded linear maps from $\acute{\mathfrak{A}}$ to $\acute{\mathfrak{B}}$. The dual space $\acute{\mathfrak{A}}'$ consists of continuous linear functionals on $\acute{\mathfrak{A}}$. Given $\xi' \in \acute{\mathfrak{A}}'$ and $\zeta \in \acute{\mathfrak{B}}$, the mapping $\xi' \otimes \zeta : \acute{\mathfrak{A}} \rightarrow \acute{\mathfrak{B}}$ is defined by $(\xi' \otimes \zeta)(\xi) = \xi'(\xi)\zeta$ for each ξ in $\acute{\mathfrak{A}}$.

Definition 2. [1, 2] *If a mapping $s\colon \mathcal{L} \rightarrow \omega^+$, where ω^+ is a class of positive real numbers, satisfies the following criteria, it is said to be an s-number sequence.*

1) ***Monotonicity:*** $\|s\| = s_1(\mathfrak{K}) \geqslant s_2(\mathfrak{K}) \geqslant \ldots \geqslant 0$, *for* $\mathfrak{K} \in \mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$.
2) ***Additivity:*** $s_{n+k-1}(\mathfrak{K} + \mathfrak{R}) \leqslant s_n(\mathfrak{K}) + s_k(\mathfrak{R})$ *for* $\mathfrak{K}, \mathfrak{R} \in \mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$ *and* $n, k \in \mathbb{N}$.
3) ***Ideal Property:*** $s_n(\mathfrak{K}\mathfrak{H}\mathfrak{R}) \leqslant \|\mathfrak{K}\| |s_n(\mathfrak{H})| \|\mathfrak{R}\|$ *for* $\mathfrak{K} \in \mathcal{L}(\acute{\mathfrak{A}}_0, \acute{\mathfrak{A}})$, $\mathfrak{H} \in \mathcal{L}(\acute{\mathfrak{A}}, W)$, $\mathfrak{R} \in \mathcal{L}(W, W_0)$, *and* $n \in \mathbb{N}$.
4) ***Rank Property:*** *If rank* $(\mathfrak{K}) < n$, *then* $s_n(\mathfrak{K}) = 0$.
5) ***Norming Property:*** $s_n\big(I_2\colon \ell_2^{(n)} \rightarrow \ell_2^{(n)}\big) = 1$, *where* I_2 *denotes the identity operator on the* $i-$*dimensional Hilbert space.*

Definition 3. [11] *Suppose $\acute{\mathfrak{A}}$ and $\acute{\mathfrak{B}}$ are Banach spaces. For a subset P of $\mathcal{L}$, set $P(\acute{\mathfrak{A}}, \acute{\mathfrak{B}}) = P \cap \mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$. Then P is called an operator ideal if the following properties are fulfilled:*

(i) *If* $\xi' \in \acute{\mathfrak{A}}', \zeta \in B$, *then* $\xi' \odot \zeta \in P(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$.
(ii) $\mathfrak{K} + \mathfrak{R} \in P(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$ *for* $\mathfrak{K}, \mathfrak{R} \in P(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$.
(iii) *If* $\mathfrak{H} \in p(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$, $\mathfrak{R} \in \mathcal{L}(\acute{\mathfrak{A}}_0, \acute{\mathfrak{A}})$, *and* $\mathfrak{K} \in P(\acute{\mathfrak{B}}, \acute{\mathfrak{B}}_0)$, *then* $\mathfrak{K}\mathfrak{H}\mathfrak{R} \in P(\acute{\mathfrak{A}}_0, \acute{\mathfrak{B}}_0)$.

For a chosen pair of Banach spaces $\acute{\mathfrak{A}}$ and $\acute{\mathfrak{B}}$, the set $P(\acute{\mathfrak{A}},\acute{\mathfrak{B}})$ is called a component of P.

Definition 4. [11, 12] *The term ideal quasi-norm implies a real-valued function $\mathfrak{X}\colon \acute{\mathfrak{A}} \rightarrow \mathbb{R}_+$ that satisfies the following requirements:*

(i) *If* $\xi' \in \acute{\mathfrak{A}}', \zeta \in \acute{\mathfrak{B}}$, *then* $\mathfrak{X}(\xi' \odot \zeta) = \|\xi'\| \|\zeta\|$.
(ii) *There exists a constant $M \geqslant 1$, such that*

$$\mathfrak{X}(\mathfrak{K} + \mathfrak{R}) \leqslant M[\mathfrak{X}(\mathfrak{K}) + \mathfrak{X}(\mathfrak{R})] \quad \textit{for } \mathfrak{K}, \mathfrak{R} \in P(\acute{\mathfrak{A}}, \acute{\mathfrak{B}}).$$

(iii) If $\mathfrak{H} \in P(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$, $\mathfrak{R} \in \mathcal{L}(\acute{\mathfrak{A}}_0, \acute{\mathfrak{A}})$, *and* $\mathfrak{K} \in \mathcal{L}(\acute{\mathfrak{B}}, \acute{\mathfrak{B}}_0)$, *then* $\mathfrak{K}\mathfrak{H}\mathfrak{R} \in P(\acute{\mathfrak{A}}_0, \acute{\mathfrak{B}}_0)$.

Lemma 2. [10] *Let* $\mathfrak{K}, \mathfrak{R} \in \mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$. *Then*

$$|s_n(\mathfrak{K}) - s_n(\mathfrak{R})| \leqslant \|\mathfrak{K} - \mathfrak{R}\| \quad \textit{for } i \in \mathbb{N}.$$

We define an operator $\mathfrak{K} \in \mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$ *as an s-type* $\ell_p(\Psi)$ *operator whenever*

$$\sum_{n=0}^{\infty} \left| \frac{3 \sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{K})}{4\Psi_n + \Psi_{n-1} - n - 1} \right|^p < \infty, 1 < p < \infty.$$

We denote by $\ell_p^{(s)}(\Psi)$ *the collection of all s-type* $\ell_p(\Psi)$ *operators.*

Theorem 12. *For* $1 < p < \infty$, *the class* $\ell_p^{(s)}(\Psi)$ *is an operator ideal.*

Proof. Consider Banach spaces $\acute{\mathfrak{A}}$ and $\acute{\mathfrak{B}}$. If $\xi' \in \acute{\mathfrak{A}}'$ and $\zeta \in \acute{\mathfrak{B}}$, then $\xi' \odot \zeta$ defines a rank-one operator. It follows that $s_n(\xi' \odot \zeta) = 0$ for all $n \geqslant 2$. Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left| \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \sum_{k=1}^{n} \Psi_k s_k(\xi' \odot \zeta) \right|^p \\
&= \sum_{n=0}^{\infty} \left| \frac{3\ s_1(\xi' \odot \zeta)}{4\Psi_n + \Psi_{n-1} - n - 1} \right|^p \\
&= \left[s_1(\xi' \odot \zeta)\right]^p \sum_{n=0}^{\infty} \left| \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \right|^p \\
&= \|\xi' \odot \zeta\|^p \sum_{n=0}^{\infty} \left| \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \right|^p < \infty.
\end{aligned}$$

Thus $\xi' \odot \zeta \in \ell_p^{(s)}(\Psi)$.

Let $\mathfrak{K}$ and $\mathfrak{R}$ be elements of $\ell_p^{(s)}(\Psi)$. Considering the non-negativity and non-increasing characteristics of s-numbers, we can utilize Minkowski's inequality to obtain

$$\left(\sum_{n=1}^{\infty} \left(\frac{3 \sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{K} + \mathfrak{R})}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^p \right)^{1/p}$$

$$\leqslant \left(\left(\sum_{n=1}^{\infty}\frac{3\sum\limits_{k=1}^{n}\Psi_{2k}\ s_{2k-1}(\mathfrak{K}+\mathfrak{N})}{3\Psi_n+\Psi_{n-1}-n-1}+\sum_{n=1}^{\infty}\frac{3\sum\limits_{k=1}^{n}\Psi_{2k+1}\ s_{2k}(\mathfrak{K}+\mathfrak{N})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{1/p}$$

$$\leqslant \left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}(\Psi_{2k}+\Psi_{2k+1})s_{2k-1}(\mathfrak{K}+\mathfrak{N})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{1/p}$$

$$\leqslant M\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K})}{3\Psi_n+\Psi_{n-1}-n-1}+\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{N})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{1/p}$$

$$\leqslant M\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{\frac{1}{p}}+\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{N})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{\frac{1}{p}}<\infty.$$

Thus, $\mathfrak{K}+\mathfrak{N}\in\ell_p^{(s)}(\Psi)$.

Let $\mathfrak{K}\in\mathcal{L}(\acute{\mathfrak{A}}_0,\acute{\mathfrak{A}})$, $\mathfrak{N}\in\mathcal{L}(\acute{\mathfrak{B}},\acute{\mathfrak{B}}_0)$ and $\mathfrak{H}\in\ell_p^{(s)}(\Psi)$. Using the property (3) in Definition 2, we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{N}\mathfrak{H}\mathfrak{K})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{\frac{1}{p}}$$

$$\leqslant \|\mathfrak{N}\|\|\mathfrak{K}\|\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{j=1}^{n}\Psi_k s_k(\mathfrak{H})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{\frac{1}{p}}<\infty.$$

Therefore, $\mathfrak{N}\mathfrak{H}\mathfrak{K}$ lies in $\ell_p^{(s)}(\Psi)$. This shows that $\ell_p^{(s)}(\Psi)$ forms an operator ideal. □

Theorem 13. *For $1<p\leqslant r<\infty$, it holds that $\ell_p^{(s)}(\Psi)$ is contained in $\ell_r^{(s)}(\Psi)$.*

Proof. This result is a direct consequence of the inclusion $\ell_p(\Psi)\subseteq\ell_r(\Psi)$ for $1<p\leqslant r<\infty$. □

Assume that $\ell_p^{(s)}(\Psi)$ is an operator ideal. For $1<p<\infty$, define the mapping $Q^{(s)}\colon \ell_p^{(s)}(\Psi)\to\omega^{+}$,

$$Q^{(s)}(\mathfrak{K})=\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^{p}\right)^{\frac{1}{p}},$$

where $\mathfrak{K} \in \ell_p^{(s)}(\Psi)$.

Theorem 14. *For values of p with $1 < p < \infty$, the operator ideal $\ell_p^{(s)}(\Psi)$ becomes a quasi-normed space under the functional $\widetilde{Q}^{(s)}$, where*

$$\widetilde{Q}^{(s)}(\mathfrak{K}) = \frac{Q^{(s)}(\mathfrak{K})}{\left[\left(\sum\limits_{n=1}^{\infty} \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right]^{\frac{1}{p}}}.$$

Proof. Consider Banach spaces $\acute{\mathfrak{A}}$ and $\acute{\mathfrak{B}}$. The operator $\xi' \odot \zeta \colon \acute{\mathfrak{A}} \to \acute{\mathfrak{B}}$ is rank one, and therefore $s_n(\xi' \odot \zeta) = 0$ holds for all $n \geqslant 2$. Thus, it can be represented as

$$\begin{aligned} Q^{(s)}(\xi' \odot \zeta) &= \left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n} \Psi_k s_k(\xi \odot \zeta)}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty}\left(\frac{3\ s_1(\xi' \odot \zeta)}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}} \\ &= \|\xi' \odot \zeta\|\left(\sum_{n=1}^{\infty}\left(\frac{3}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}}. \end{aligned}$$

Since $\|\xi' \odot \zeta\| = \|\xi'\|\|\zeta\|$, we have

$$\widetilde{Q}^{(s)}(\xi' \odot \zeta) = \|\xi'\|\|\zeta\|.$$

Let $\mathfrak{K}, \mathfrak{N} \in \ell_p^{(s)}(\Psi)$. Then

$$\begin{aligned} &Q^{(s)}(\mathfrak{K} + \mathfrak{N}) = \left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{K} + \mathfrak{N})}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}} \\ &\leqslant M\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{K})}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{N})}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}} \\ &\leqslant M\left(Q^{(s)}(\mathfrak{K}) + Q^{(s)}(\mathfrak{N})\right). \end{aligned}$$

Thus,

$$\widetilde{Q}^{(s)}(\mathfrak{K} + \mathfrak{N}) \leqslant M\left(\widetilde{Q}^{(s)}(\mathfrak{K}) + \widetilde{Q}^{(s)}(\mathfrak{N})\right).$$

Finally, let $\mathfrak{H} \in \ell_p^{(s)}(\Psi)(\acute{\mathfrak{A}} \to \acute{\mathfrak{B}})$, $\mathfrak{K} \in \mathcal{L}(\acute{\mathfrak{B}}, \acute{\mathfrak{B}}_0)$, and $\mathfrak{R} \in \mathcal{L}(\acute{\mathfrak{A}}_0, \acute{\mathfrak{A}})$. Then

$$\begin{aligned} Q^{(s)}(\mathfrak{K}\mathfrak{H}\mathfrak{R}) &= \left(\sum_{n=1}^{\infty} \left(\frac{3 \sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{K}\mathfrak{H}\mathfrak{R})}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^p \right)^{\frac{1}{p}} \\ &\leqslant \|\mathfrak{K}\| \|\mathfrak{R}\| \left(\sum_{n=1}^{\infty} \left(\frac{3 \sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{H})}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^p \right)^{\frac{1}{p}} \\ &\leqslant \|\mathfrak{K}\| \widetilde{Q}^{(s)}(\mathfrak{H}) \|\mathfrak{R}\|. \end{aligned}$$

Thus,

$$\widetilde{Q}^{(s)}(\mathfrak{K}\mathfrak{H}\mathfrak{R}) \leqslant \|\mathfrak{K}\| \widetilde{Q}^{(s)}(\mathfrak{H}) \|\mathfrak{R}\|.$$

Hence, $\widetilde{Q}^{(s)}$ indeed defines a quasi-norm on the operator ideal $\ell_p^{(s)}(\Psi)$. $\square$

Theorem 15. *For $1 < p < \infty$, the operator ideal $\ell_p^{(s)}(\Psi)$ becomes a complete space when equipped with the quasi-norm $\tilde{Q}^{(s)}$.*

Proof. Let us consider the case when $1 < p < \infty$. Then we have

$$\begin{aligned} \tilde{Q}^{(s)}(\mathfrak{K}) &= \left(\sum_{n=1}^{\infty} \left(\frac{3 \sum\limits_{k=1}^{n} \Psi_k s_k(\mathfrak{K})}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^p \right)^{\frac{1}{p}} \\ &= \left(\left(\sum_{n=1}^{\infty} \frac{3 s_1(\mathfrak{K})}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^p \right)^{\frac{1}{p}} \\ &\leqslant \|\mathfrak{K}\| \left(\left(\sum_{n=1}^{\infty} \frac{3}{4\Psi_n + \Psi_{n-1} - n - 1} \right)^p \right)^{\frac{1}{p}}. \end{aligned}$$

From this, we can conclude that

$$\|\mathfrak{K}\| \leqslant \tilde{Q}^{(s)}(\mathfrak{K}) \quad \text{for all } \mathfrak{K} \in \ell_p^{(s)}(\Psi)(\acute{\mathfrak{A}} \to B). \tag{7}$$

Now, let $(\mathfrak{K}_n)$ denote a Cauchy sequence in $\ell_p^{(s)}(\Psi)(\acute{\mathfrak{A}} \to \acute{\mathfrak{B}})$. For every $\epsilon > 0$, there exists some $\kappa \in \mathbb{N}$ satisfying

$$\tilde{Q}^{(s)}(\mathfrak{K}_i - \mathfrak{K}_j) < \epsilon \quad \text{for all } n, k \geqslant \epsilon. \tag{8}$$

From (7), we deduce that

$$\|\mathfrak{K}_n - \mathfrak{K}_m\| \leqslant Q^{(s)}(\mathfrak{K}_n - \mathfrak{K}_m).$$

Applying (8), we obtain

$$\|\mathfrak{K}_n - \mathfrak{K}_m\| \leqslant Q^{(s)}(\mathfrak{K}_n - \mathfrak{K}_m) \quad \text{for all } n, m \geqslant \epsilon.$$

Therefore, the sequence $(\mathfrak{K}_n)$ is Cauchy in the space $\mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$. Since $\mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$ is a Banach space, we can conclude that $\mathfrak{K}_n \to \mathfrak{K}$ as $n \to \infty$ in $\mathcal{L}(\acute{\mathfrak{A}}, \acute{\mathfrak{B}})$.
Utilizing Lemma 2, we have

$$|s_n(\mathfrak{K}_n - \mathfrak{K}_m) - s_n(\mathfrak{K} - \mathfrak{K}_m)| \leqslant \|\mathfrak{K}_n - \mathfrak{K}\|.$$

Taking the limit as $n \to \infty$ gives us

$$s_n(\mathfrak{K}_n - \mathfrak{K}_m) \to s_n(\mathfrak{K} - \mathfrak{K}_m). \tag{9}$$

Now, from (8), we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K}_n - \mathfrak{K}_m)}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}} < \varepsilon\left(\sum_{n=1}^{\infty}\left(\frac{3}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}}$$

For all $m, n \geqslant \varepsilon$, by fixing m and letting $n \to \infty$, we can apply (9) to derive the following inequality:

$$\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K} - \mathfrak{K}_m)}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}} < \varepsilon\left(\sum_{n=1}^{\infty}\left(\frac{3}{4\Psi_n + \Psi_{n-1} - n - 1}\right)^p\right)^{\frac{1}{p}},$$

which leads to $\tilde{Q}^{(s)}(\mathfrak{K} - \mathfrak{K}_m) < \varepsilon \quad$ for all $m \geqslant \varepsilon$. It follows that $\mathfrak{K}_m \to \mathfrak{K}$ with respect to the quasi-norm $\tilde{Q}^{(s)}$.

The final verification is that $\mathfrak{K}$ indeed belongs to the operator ideal $\ell_p^{(s)}(\Psi)(\acute{\mathfrak{A}} \to \acute{\mathfrak{B}})$. To see this,

$$\begin{aligned}\sum_{k=1}^{n} s_k(\mathfrak{K}) &\leqslant \sum_{k=1}^{n}\Psi_{2k-1}s_{2k-1}(\mathfrak{K}) + \sum_{k=1}^{n}\Psi_{2k}s_{2k}(\mathfrak{K}) \leqslant \sum_{k=1}^{n}(\Psi_{2k-1} + \Psi_{2k})s_{2k-1}(\mathfrak{K}). \\ &\leqslant M\left(\sum_{k=1}^{n}\Psi_k s_k(\mathfrak{K} - \mathfrak{K}_m) + \sum_{k=1}^{n}\Psi_k s_k(\mathfrak{K}_m)\right)\end{aligned}$$

Consequently,

$$\left(\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K})}{4\Psi_n+\Psi_{n-1}-n-1}\right)^p\right)^{\frac{1}{p}} \leqslant M\left(\sum_{n=1}^{\infty}\left(\frac{4\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K}-\mathfrak{K}_m)}{4\Psi_n+\Psi_{n-1}-n-1}\right)^p\right)^{\frac{1}{p}}$$

$$+ M\left(\sum_{n=1}^{\infty}\left(\frac{4\sum\limits_{k=1}^{n}\Psi_k s_k(\mathfrak{K}_m)}{4\Psi_n+\Psi_{n-1}-n-1}\right)^p\right)^{\frac{1}{p}} < \infty$$

which is finite. Since $\tilde{Q}^{(s)}(\mathfrak{K}-\mathfrak{K}_m)\to 0$ as $m\to\infty$ and $\mathfrak{K}_m\in\ell_p^{(s)}(\Psi)(\acute{\mathfrak{A}}\to\acute{\mathfrak{B}})$. Hence, we conclude that $\mathfrak{K}\in\ell_p^{(s)}(\Psi)(\acute{\mathfrak{A}}\to\acute{\mathfrak{B}})$. □

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