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ESTIMATION OF THE DEVIATION OF THE RATIO OF LOGARITHMS OF THE MAXIMUM TERMS OF DIRICHLET SERIES FROM UNITY

Abstract. We study the question of estimating the degree of stability of the maximal term of the Dirichlet series with positive exponents, the sum of which is an entire function.

This problem is of interest because the Leont'ev formulas for coefficients calculated using a biorthogonal system of functions play a key role in obtaining asymptotic estimates for the entire Dirichlet series on various continua going to infinity (for example, curves). This fact naturally leads to the need to study the behavior of the logarithm of the maximum term also for the Hadamard composition of the corresponding Dirichlet series. The issues discussed here have important applications in complex dynamics, namely, in problems related to the structure of the Fatou set of entire transcendental functions. For the wide class of entire Dirichlet series determined by a convex growth majorant, we establish an estimate of the degree for the equivalence of the logarithms of maximal terms of the original series and a modified Dirichlet series outside some exceptional c_q–set.

Key words: *Dirichlet series, Hadamard composition, stability of the maximal term, Borel–Nevanlinna lemma, convex function,* c_q*–set*

2020 Mathematical Subject Classification: *30D40*

1. Introduction. The equivalence problem of logarithms of maximal terms of entire Dirichlet series $\sum_n a_n e^{\lambda_n s}$ and $\sum_n a_n b_n e^{\lambda_n s}$, $0 < \lambda_n \uparrow \infty$, was first studied in [3]. This important property, called the stability of the maximal term, turned out to be very useful for obtaining asymptotic estimates of the sum of the Dirichlet series on curves going to infinity, namely, for proving the well-known Polya conjecture (1929). Similar research was later carried out for Dirichlet series of a given growth, for example, of a finite Ritt order [6]. The key role in such research is played by Borel–Nevanlinna type lemmas (see, for example, in [3], [2]). However, in the mentioned works [3], [6], a rather strong, although natural in the main problems considered there, restriction was required on the exponents λ_n of the Dirichlet series

$$\varlimsup_{n\to\infty} \frac{\ln n}{\ln \lambda_n} < \infty. \tag{1}$$

In other words, λ_n are zeros of some entire function of finite order.

The study of the stability of the maximal term is also of independent interest. In this situation, as shown in [8], it is enough to assume that

$$\sum_{n=1}^{\infty} \frac{1}{n\lambda_n} < \infty \tag{2}$$

or, what is the same,

$$\int_1^{\infty} \frac{\ln n(t)}{t^2}\, dt < \infty, \quad n(t) = \sum_{\lambda_n \leqslant t} 1. \tag{3}$$

It is clear that condition (2) is weaker than (1).

The papers [3], [8] study the stability of maximal terms of entire Dirichlet series without restriction on the growth of the sum of the series. Similar questions are studied in [1], [2] for classes of Dirichlet series defined by a certain convex majorant. In [1], [2], relations of the type

$$\ln \mu(\sigma) = (1 + o(1)) \ln \mu_b^*(\sigma), \tag{4}$$

are obtained, which hold for $\sigma \to \infty$ outside a certain set $E \subset \mathbb{R}_+$ of zero lower density. Here $\mu(\sigma)$ and $\mu_b^*(\sigma)$ are the maximal terms of the original and modified Dirichlet series (see above). However, these works did not discuss at all the question of the behavior of the infinitesimal quantity $o(1)$ from equality (4), which by definition means the stability of the maximum term $\mu(\sigma)$ («stability by Gaisin» according to the terminology of the authors of the work [8]).

The goal of this paper is to obtain a quantitative estimate of the equivalence measure in (4) outside some c_q–set $E_{\alpha\beta\gamma} \subset \mathbb{R}_+$, i.e., an estimate of the form

$$\left|1 - \frac{\ln \mu_b^*(\sigma)}{\ln \mu(\sigma)}\right| \leqslant \frac{\text{const}}{\sigma^{\gamma+\mu}}, \quad \mu \in (0, \beta),$$

for given α, β, γ, $0 < \alpha \leqslant \alpha_0 < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, $\nu = 1-\beta-\gamma > 0$.

In the article [2], we consider the entire Dirichlet series in the class $\underline{D}(\Phi)$ defined by some convex majorant Φ (dual class $D(\Phi)$ has been previously considered by us in [1]). As in works [1]–[6], the stability criterion of the maximal term of the Dirichlet series of class $\underline{D}(\Phi)$ is proven in terms of the sequence $\{b_n\}$. In the situation we are considering here, the convergence of the series (2) (of the integral (3)) is not assumed at all.

2. The Main Result. Let $\Lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow \infty$, be a sequence, such that

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0. \tag{5}$$

We denote by $D(\Lambda)$ the class of all functions F that can be represented by the Dirichlet series in the whole plane:

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it. \tag{6}$$

By (5), if the series (6) converges in the whole plane, then it absolutely converges in the plane and its sum F is an entire function [7]. We denote by L the class of all continuous infinitely increasing positive functions on $\mathbb{R}_+ = [0, \infty)$. Let Φ be a convex function of class L, and let

$$\underline{D}_m(\Phi) = \{F \in D(\Lambda)\colon \exists\, \{\sigma_n\},\, 0 < \sigma_n \uparrow \infty,\ \ln M_F(\sigma_n) \leqslant \Phi(m\sigma_n)\}\,, m \geqslant 1,$$

where $M_F(\sigma) = \sup\limits_{|t|<\infty} |F(\sigma + it)|$. We set

$$\underline{D}(\Phi) = \bigcup_{m=1}^{\infty} \underline{D}_m(\Phi).$$

We note that in [1] we talked about the class

$$D(\Phi) = \bigcup_{m=1}^{\infty} D_m(\Phi),$$

where

$$D_m(\Phi) = \{F \in D(\Lambda)\colon\ \ln M_F(\sigma) \leqslant \Phi(m\sigma)\}\,, \quad m \geqslant 1.$$

Together with the series (6), we consider the series

$$F_b^*(s) = \sum_{n=1}^{\infty} a_n b_n e^{\lambda_n s}, \tag{7}$$

where the sequence $b = \{b_n\}$ of complex numbers b_n ($b_n \neq 0$ for $n \geqslant N$) satisfies the condition
$$\overline{\lim_{n\to\infty}} \frac{|\ln|b_n||}{\lambda_n} < \infty. \tag{8}$$
In this case, $F_b^* \in \underline{D}(\Phi)$ if and only if $F \in \underline{D}(\Phi)$ (see [2]).

Let $E \subset [0, \infty)$ be a Lebesgue measurable set. By the upper DE and lower dE densities of the set E, we mean the quantities
$$DE = \overline{\lim_{\sigma\to\infty}} \frac{\operatorname{mes}(E \cap [0, \sigma])}{\sigma}, \quad dE = \underline{\lim_{\sigma\to\infty}} \frac{\operatorname{mes}(E \cap [0, \sigma])}{\sigma}.$$
In what follows, we assume that all exceptional sets $E \subset [0, \infty)$ outside of which asymptotic estimates will be obtained are represented by the unions of segments of the form $[A_n, A_n']$, where
$$0 < A_1 < A_1' \leqslant A_2 < A_2' \leqslant \ldots \leqslant A_n < A_n' \leqslant \ldots .$$
By definition, the exceptional set $E \subset \mathbb{R}_+$ is called a C_q–set if $DE \leqslant q < 1$, a c_q–set if $dE \leqslant q < 1$.

Let φ be the inverse of Φ, such that
$$\overline{\lim_{x\to\infty}} \frac{\varphi(x^2)}{\varphi(x)} < \infty. \tag{9}$$

From (9) it follows that $\Phi \in M$, where M is the class of convex functions Φ, such that $x\Phi(x) < \Phi(Kx)$ for $x \geqslant x_0$, where K is some constant. Indeed, due to (9), there exists a constant $K > 0$, such that $\varphi(t^2) \leqslant K\varphi(t)$, and hence $t^2 \leqslant \Phi(K\varphi(t))$, $t \geqslant 0$. Denoting $x = \varphi(t)$ and using the fact that $x < \Phi(x)$ for $x \geqslant x_0$, we have: $x\Phi(x) < \Phi^2(x) \leqslant \Phi(Kx)$. So $\Phi \in M$.

Assuming that
$$\inf_{x\geqslant e} \frac{\ln w(x)}{\ln x} > 0,$$
we introduce the class of functions
$$\underline{W}(\varphi) = \left\{ w \in L\colon \lim_{x\to\infty} \frac{w(x)}{x\varphi(x)} = 0, \ \underline{\lim_{x\to\infty}} \frac{1}{\varphi(x)} \int_1^x \frac{w(t)}{t^2} dt = 0 \right\}.$$
Note that for any $\varphi \in L$, the function $w(x) = x^\mu$, $0 < \mu < 1$, belongs to the class
$$W(\varphi) = \left\{ w \in L\colon \lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{w(t)}{t^2}\, dt = 0 \right\}.$$

Without loss of generality of reasoning, we can assume that $w(x) \geqslant \sqrt{x}$, $x \geqslant 1$.

We denote by $\mu(\sigma)$ and $\mu_b^*(\sigma)$ the maximal terms of the series (6) and (7), respectively, i.e.,

$$\mu(\sigma) = \max_{n\geqslant 1} \left\{|a_n|\, e^{\lambda_n \sigma}\right\}, \quad \mu_b^*(\sigma) = \max_{n\geqslant 1} \left\{|a_n|\,|b_n|\, e^{\lambda_n \sigma}\right\}.$$

Let $n(t) = \sum_{\lambda_n \leqslant t} 1$ be the counting functions of the sequence Λ, and let $n_l(t)$ be the least concave majorant of $\ln n(t)$. It is well defined because of the condition (5).

The following theorem is proved in [2]:

Theorem 1. *Let $\{b_n\}$ be a sequence of complex numbers ($b_n \neq 0$, $n \geqslant N$), satisfying (8), and let Φ be a convex function of class L. We assume that the inverse φ of Φ satisfies (9) and the function $n_l(t)$ satisfies*

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{n_l(t)}{t^2} dt = 0.$$

For any $F \in \underline{D}(\Phi)$ for $\sigma \to \infty$, for the asymptotic equality

$$\ln \mu(\sigma) = (1 + o(1)) \ln \mu_b^*(\sigma), \tag{10}$$

to be valid outside some set $E \subset [0, \infty)$ of zero lower density, it is sufficient and necessary that there exists a function $w \in W(\varphi)$, such that

$$|\ln |b_n|| \leqslant w(\lambda_n), \quad n \geqslant N. \tag{11}$$

In this theorem, the function $n_l(t)$ can be replaced by $\ln n(t)$. Proving this statement requires a slightly different approach. This case is considered in [4], [5].

Let p, $p > 0$, be a given number, and let $w = w(x)$ be a majorant from class L:

$$d(w) = \inf_{x\geqslant e} \frac{\ln w(x)}{\ln x}, \quad d_p(w) = \varlimsup_{x\to\infty} \frac{w(x)}{x\varphi^p(x)}, \tag{12}$$

where $\varphi(x)$ is some fixed concave majorant from L satisfying condition (9). We introduce the following classes of majorants:

$$W^p(\varphi) = \left\{w \in L\colon d(w) > 0, \varlimsup_{x\to\infty} \frac{1}{\varphi^p(x)} \int_1^x \frac{w(t)}{t^2} dt < \infty\right\}, \tag{13}$$

$$\underline{W}^p(\varphi) = \left\{w \in L\colon d(w) > 0,\, d_p(w) < \infty,\ \varliminf_{x\to\infty} \frac{1}{\varphi^p(x)} \int_1^x \frac{w(t)}{t^2}dt < \infty\right\},$$

where the numbers $d(w)$ and $d_p(w)$ are defined in (12).

The main result of the paper is the following: let α, β, γ, q be fixed numbers, $0 < \beta < 1$, $0 < \gamma < 1$, $\nu = 1 - \beta - \gamma$, $\nu \in (0, 1)$, $q \in [0, 1)$, $0 < \alpha < 1$. Suppose that $n_l(t)$ belongs to the class $W^{p\nu}(\varphi)$, and the sequence $b = (b_1, b_2, \dots, b_n, \dots)$ is subject to condition (11) with majorant $w \in W^{p\nu}(\varphi)$, $p > 0$. Obviously, $d_{p\nu}(w) < \infty$ for $w \in W^{p\nu}(\varphi)$.

Let $F \in D(\Lambda)$ and for some sequence $\{\sigma_n\}$, $0 < \sigma_n \uparrow \infty$, $\ln M_F(\sigma_n) \leqslant \Phi(m\sigma^{1/p})$, $n = 1, 2, \dots$, for some $m \in \mathbb{N}$. Here Φ is a convex function inverse to φ. It is shown that for $\sigma \to \infty$ outside some exceptional c_q–set $E_{\alpha\beta\gamma} \subset \mathbb{R}_+$, $0 < \alpha \leqslant \alpha_0 < 1$, the estimate

$$\left|1 - \frac{\ln \mu^*(\sigma)}{\ln \mu(\sigma)}\right| \leqslant \frac{\text{const}}{\sigma^{\gamma+\mu}}, \quad 0 < \mu < \beta < 1, \quad \gamma \in (0, 1), \quad \gamma + \beta < 1,$$

is true.

3. Evidence of the main results. The proof of Theorem 1 is based on the following accessory statement – a new version of the Borel–Nevanlinna type theorem.

Theorem 2. *Let $\Phi \in L$, and the inverse φ of Φ satisfy (9). Let $u(\sigma)$ be a nondecreasing positive continuous function on $[r_0, \infty)$; moreover*

$$\lim_{\sigma\to\infty} u(\sigma) = \infty, \qquad \varliminf_{\sigma\to\infty} \frac{u(\sigma)}{\ln \Phi(\sigma^{1/p})} < \infty, \quad p > 0. \tag{14}$$

Let $\{x_n\}$ be a sequence chosen so that

$$u(x_n) \leqslant C \ln \Phi(x_n^{1/p}), \quad 0 < C < \infty.$$

Suppose that $w \in W^{p\nu}(\varphi)$, $\nu \in (0, 1)$. If $v = v(\sigma)$ is a solution to the equation

$$w(v) = e^{u(\sigma)}, \tag{15}$$

then for the same sequence $\{x_n\}$ for $\sigma \to \infty$ outside a set $E \subset [0, \infty)$, $E = E^p(\alpha, \beta, \gamma)$, $0 < \alpha < \alpha_0 < 1$,

$$\overline{\lim_{x\to\infty}} \frac{\text{mes}(E \cap [0, x_n])}{x_n} \leqslant q < 1,$$

the following estimate holds:

$$u\Big(\sigma+\sigma^{\gamma}\frac{w(v(\sigma))}{v(\sigma)}\Big)<u(\sigma)+\frac{1}{\alpha\sigma^{\beta}},$$

where $0<\beta<1$, $0<\gamma<1$, *and* $\nu=1-\gamma-\beta>0$.

Proof. We show that for any $\gamma\in(0,1)$, $\beta\in(0,1)$, $\gamma+\beta<1$, $0<\alpha\leqslant\alpha_0<1$,

$$u\Big(\sigma+\sigma^{\gamma}\frac{w(v(\sigma))}{v(\sigma)}\Big)<u(\sigma)+\frac{1}{\alpha\sigma^{\beta}},$$

outside a set $E\subset[0,\infty)$, $E=E^p(\alpha,\beta,\gamma)$, $dE\leqslant q<1$.

Indeed, let $E\subset[0,\infty)$ be a set on which

$$u\Big(\sigma+\sigma^{\gamma}\frac{w(v(\sigma))}{v(\sigma)}\Big)\geqslant u(\sigma)+\frac{1}{\alpha\sigma^{\beta}}.$$

Then, in the same way as in [1], [2], the sequences $\{\sigma_n\}$ and $\{\sigma'_n\}$ are constructed, such that $E\subset\bigcup\limits_{n=1}^{\infty}[\sigma_n,\sigma'_n]$, and

$$0<\sigma'_n-\sigma_n\leqslant\sigma_n^{\gamma}\frac{w(v(\sigma_n))}{v(\sigma_n)},\quad u(\sigma_{n+1})-u(\sigma_n)\geqslant\frac{1}{\alpha\sigma_n^{\beta}}. \tag{16}$$

Note that in [1], [2] these estimates did not contain the quantities σ_n^{γ}, $\alpha\sigma_n^{\beta}$, and on the right-hand side of the second inequality there was simply an undefined infinitesimal quantity.

Let $v_n=v(\sigma_n)$, $\delta_n=w(v_n)/v_n$, $n\geqslant 1$. If $2v_n\leqslant v_{n+1}$, then

$$\delta_n\leqslant 2w(v_n)\int\limits_{v_n}^{v_{n+1}}\frac{dt}{t^2}\leqslant 2\int\limits_{v_n}^{v_{n+1}}\frac{w(t)}{t^2}\,dt. \tag{17}$$

If $2v_n>v_{n+1}$, then from (15), (16) and the monotonicity of $w=w(t)$

$$\begin{aligned}\delta_n&\leqslant\alpha\sigma_n^{\beta}\frac{w(v_n)}{v_n}\,[u(\sigma_{n+1})-u(\sigma_n)]\leqslant 2\alpha\sigma_n^{\beta}\int\limits_{v_n}^{v_{n+1}}\frac{w(t)}{t}\,d\ln w(t)\\&=2\alpha\sigma_n^{\beta}\Big[\frac{w(v_{n+1})}{v_{n+1}}-\frac{w(v_n)}{v_n}+\int\limits_{v_n}^{v_{n+1}}\frac{w(t)}{t^2}\,dt\Big].\end{aligned} \tag{18}$$

Since
$$\int\limits_{v_n}^{v_{n+1}} \frac{dw(t)}{t} \geqslant 0,$$
then, obviously,
$$\int\limits_{v_n}^{v_{n+1}} \frac{w(t)dt}{t^2} \leqslant \frac{w(v_{n+1})}{v_{n+1}} - \frac{w(v_n)}{v_n} + 2\int\limits_{v_n}^{v_{n+1}} \frac{w(t)}{t^2}\,dt.$$
Consequently, from (17), (18) we conclude that
$$\delta_n \leqslant 2\max(1, \alpha\sigma_n^{\beta})\Big[\frac{w(v_{n+1})}{v_{n+1}} - \frac{w(v_n)}{v_n} + 2\int\limits_{v_n}^{v_{n+1}} \frac{w(t)}{t^2}\,dt\Big]. \tag{19}$$

It follows from the condition of the theorem that there exists a sequence $\{x_n\}$, $0 < x_n \uparrow \infty$, such that $u(x_n) \leqslant C\Phi(x_n^{1/p})$, $0 < C < \infty$, $p > 0$. It is clear that for any $n \geqslant 0$ there exists $k \geqslant 0$, such that $\sigma_{k-1} \leqslant x_n < \sigma_k$. Consequently, we have
$$\operatorname{mes}(E \cap [0, x_n]) \leqslant \sum_{m=1}^{k-1} \sigma_m^{\gamma}\delta_m \leqslant \sigma_{k-1}^{\gamma}\delta_{k-1} + x_n^{\gamma}\sum_{m=1}^{k-2} \delta_m. \tag{20}$$
Then, if $\sigma_{k-1} \leqslant x_n < \sigma_k$, then for $n \geqslant n_0$, taking into account (19), (20), we have:
$$\frac{\operatorname{mes}(E \cup [0, x_n])}{x_n} \leqslant \frac{w(v_{k-1})}{x_n^{1-\gamma}v_{k-1}} + \frac{2\alpha}{x_n^{\nu}}\Big[\frac{w(v_{k-1})}{v_{k-1}} + 2\int\limits_{v_1}^{v_{k-1}} \frac{w(t)}{t^2}dt\Big], \tag{21}$$
where α, β are the same numbers from (0,1), such that $\nu \in (0, 1)$, $\nu = 1 - \gamma - \beta$, and $x_n \in [\sigma_{k-1}, \sigma_k)$.

Further, taking into account equality (15) and conditions (12), (14), we have:
$$v(x_n)^{1/m} \leqslant w(v(x_n)) = e^{u(x_n)} \leqslant \Phi^K(x_n^{1/p}), \quad n \geqslant n_1, \quad m, K \in \mathbb{N}, \quad p > 0.$$
Hence, if we take into account (9),
$$\varphi(v(x_n)) \leqslant \varphi(\Phi^{mK}(x_n^{1/p})) \leqslant c_0 x_n^{1/p}, \quad c_0 > 0, \quad n \geqslant n_1.$$

Hence, for $x_n \in [\sigma_{k-1},\sigma_k)$

$$\frac{1}{x_n} \leqslant \frac{Ac_0^p}{\varphi^p(v(x_n))} \leqslant \frac{Ac_0^p}{\varphi^p(v_{k-1})}, \qquad n \geqslant n_1,\ p > 0,\ A > 0. \tag{22}$$

Thus, from (21), (22) we obtain the estimate

$$\overline{\lim_{n\to\infty}} \frac{\operatorname{mes}(E \cap [0, x_n])}{x_n} \leqslant A^{1-\gamma} c_0^{(1-\gamma)p} \overline{\lim_{k\to\infty}} \frac{w(v_{k-1})}{\varphi^{(\nu+\beta)p}(v_{k-1})v_{k-1}} +$$

$$+2Ac_0^{p\nu}\alpha \overline{\lim_{k\to\infty}} \left[\frac{w(v_{k-1})}{v_{k-1}\varphi^{p\nu}(v_{k-1})} + \frac{2}{\varphi^{p\nu}(v_{k-1})} \int\limits_{v_1}^{v_{k-1}} \frac{w(t)}{t^2} dt\right] \leqslant$$

$$\leqslant 2A^{\nu} c_0^{p\nu} \alpha (d_{p\nu}(w) + 2T_{p\nu}^{-}),$$

where $E = E^p(\alpha,\beta,\gamma)$, $p > 0$, $\nu = 1-\gamma-\beta$, $\nu \in (0,1)$. Since $w \in W^{p\nu}(\varphi)$, then $T_{p\nu}^{-} < \infty$, $d_{p\nu}(w) < \infty$. Therefore, for $0 < \alpha \leqslant \alpha_0 < 1$

$$\overline{\lim_{x_n\to\infty}} \frac{\operatorname{mes}(E \cap [0, x_n])}{x_n} \leqslant 4\alpha A^{\nu} c_0^{p\nu} (d_{p\nu}(w) + 2T_{p\nu}^{-}) \leqslant q < 1. \tag{23}$$

Hence, as can be seen from the last estimate, everything follows.

Theorem 2 is proved. □

To formulate the main theorem, we introduce some notation.

Let Φ be a convex function from class L, $p > 0$,

$$\underline{D}_m^p(\Phi) = \left\{F \in D(\Lambda) \colon \ \exists \{\sigma_n\}, 0 < \sigma_n \uparrow \infty, \ln M_F(\sigma_n) \leqslant \Phi(m\sigma_n^{1/p})\right\}, m \geqslant 1.$$

We set $\underline{D}^p(\Phi) = \cup_{m=1}^{\infty} \underline{D}_m^p(\Phi)$, $p > 0$. The dual class $D^p(\Phi)$ is defined naturally.

We will also need the following simple lemma. In the case $p = 1$ it is proved in [2]. The case $p > 0$ is similar. Therefore, we will not provide a proof of this lemma.

Lemma. *Let $F \in \underline{D}^p(\Phi)$, $p > 0$, where $\Phi \in L$. Then there exists a sequence of numbers σ_j, $\sigma_j \uparrow \infty$, such that for $\sigma = \sigma_j$ and some $m \in \mathbb{N}$*

$$\ln \mu(\sigma) \leqslant \Phi(m\sigma^{1/p}), \quad \ln \mu_b^*(\sigma) \leqslant \Phi(m\sigma^{1/p}),$$

where $\mu(\sigma)$, $\mu_b^*(\sigma)$ are the maximal members of the series (6) and (7) accordingly.

The following theorem is true:

Theorem 3. *Let* $\{b_n\}$ $(b_n \neq 0,\ n \geqslant N)$ *be a sequence of complex numbers satisfying condition (8), and* Φ *be a convex function from the class* L. *Let* α, β, γ *be arbitrary parameters,* $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, $\nu \in (0, 1)$, $\nu = 1 - \gamma - \beta$. *Suppose that the function* $n_l(t)$ *belongs to the class* $W^{p\nu}(\varphi)$*, and the function* φ*, the inverse of* Φ*, satisfies condition (9). If there is a function* $w \in W^{p\nu}(\varphi)$*, such that*

$$|\ln|b_n|| \leqslant w(\lambda_n), \quad n \geqslant N, \tag{24}$$

then for any function $F \in \underline{D}^p(\Phi)$ *with* $\sigma \to \infty$ *outside some exceptional* c_q*–set* $E \subset [0, \infty)$, $E = E^p(\alpha, \beta, \gamma)$, $0 < \alpha \leqslant \alpha_0 < 1$*, the estimate*

$$\left|1 - \frac{\ln \mu_b^*(\sigma)}{\ln \mu^*(\sigma)}\right| < \frac{2}{\sigma^{\gamma+\mu}}, \tag{25}$$

is correct. Here μ *is any number from* $(0, \beta)$*, arbitrarily close to* β.

The corresponding theorem also holds for the class $D^p(\Phi)$, but we do not present it here.

Let us prove the Theorem 3.

Proof. Let (11) hold with $w \in W^{p\nu}(\varphi)$, i.e., the condition (24) is fulfilled.

Let $v = v(\sigma)$, $p = p(\sigma)$ be solutions to the equations

$$w(v) = \sigma^{-\gamma-\mu} \ln \mu(\sigma), \quad w(p) = \sigma^{-\gamma-\mu} \ln \mu_b^*(\sigma), \tag{26}$$

where $\gamma \in (0, 1)$, $\mu \in (0, \beta)$, $\gamma + \beta < 1$. Set

$$R_v = \sum_{\lambda_j > v} |a_j|\, e^{\lambda_j \sigma}, \quad h = 3\sigma^\gamma \frac{w(v)}{v}, \quad v = v(\sigma).$$

Next, $\ln n = \ln n(\lambda_n) \leqslant n_l(\lambda_n)$. Since a function $n_l(t)$ is concave, and $n_l(t) \leqslant w(t)$, then, in addition,

$$n_l(\lambda_n) \leqslant \frac{w(v)}{v} \lambda_n$$

for $\lambda_n \geqslant v$. Consequently,

$$R_v \leqslant \mu(\sigma + h) \sum_{\lambda_n > v} e^{-\lambda_n h} \leqslant \mu(\sigma + h) c_0 \exp(\max_{t \geqslant v} \psi(t)),$$

where

$$\psi(t) = 2n_l(t) - ht, \quad c_0 = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence, if we take into account the previous estimate for $n_l(t)$, when

$$\max_{t \geqslant v}(\psi(t)) \leqslant 2\frac{w(v)}{v}t - ht \leqslant -3\sigma^\gamma(1+o(1))w(v)$$
$$\leqslant -2\sigma^\gamma w(v), \quad v = v(\sigma), \quad \sigma \geqslant \sigma_0. \quad (27)$$

Thus,

$$R_v \leqslant c_0\mu(\sigma+h)\exp[-2\sigma^\gamma w(v)], \quad v = v(\sigma), \quad \sigma \geqslant \sigma_0. \quad (28)$$

Set

$$u(\sigma) = \ln\left[\frac{\ln\mu(\sigma)}{\sigma^{\gamma+\mu}}\right], \quad u^*(\sigma) = \ln\left[\frac{\ln\mu_b^*(\sigma)}{\sigma^{\gamma+\mu}}\right], \quad \mu \in (0,\beta), \quad \gamma+\beta < 1.$$

As is well-known, $u(\sigma) \uparrow \infty$, $u^*(\sigma) \uparrow \infty$ as $s \to +\infty$. Since $F \in \underline{D}^p(\Phi)$, then, according to the lemma, there exists the sequence $\{\tau_j\}$, $0 < \tau_j \uparrow \infty$, such that for some $m \in \mathbb{N}$

$$u(\sigma) \leqslant \ln\Phi(m\sigma^{1/p}), \quad u^*(\sigma) \leqslant \ln\Phi(m\sigma^{1/p}), \quad \sigma = \tau_j.$$

Therefore, taking into account (26) for $\sigma = \tau_j$, $j \geqslant 1$, we have

$$\ln w(v(\sigma)) = u(\sigma) \leqslant \ln\Phi(m\sigma^{1/p}), \quad \ln w(p(\sigma)) = u^*(\sigma) \leqslant \ln\Phi(m\sigma^{1/p}).$$

Hence, we get that for $\sigma = \tau_j$, $j \geqslant 1$,

$$\varphi(w(v(\sigma))) \leqslant m\sigma^{1/p}, \quad \varphi(w(v(p(\sigma)))) \leqslant m\sigma^{1/p}.$$

Thus,

$$\frac{1}{\sigma} \leqslant \frac{m^p}{\varphi^p(w(v(\sigma)))}, \quad \frac{1}{\sigma} \leqslant \frac{m^p}{\varphi^p(w(p(\sigma)))}, \quad \sigma = \tau_j, \quad j \geqslant 1. \quad (29)$$

Taking into account (9) and that $\sqrt[n]{x} \leqslant w(x)$, $n \geqslant n_0$, we have

$$\varphi(x) \leqslant c_1\varphi(w(x)), \quad x \geqslant x_0, \quad 0 < c_1 < \infty. \quad (30)$$

Thus, from (29) and (30) we obtain the estimates

$$\frac{1}{\sigma} \leqslant \frac{c_2}{\varphi^p(v(\sigma))}, \quad \frac{1}{\sigma} \leqslant \frac{c_2}{\varphi^p(p(\sigma))}, \qquad \sigma = \tau_j, \quad 0 < c_2 < \infty. \tag{31}$$

It is obvious that the change of the condition $u(\sigma) \leqslant C\Phi(\sigma^{1/p})$, $\sigma = \tau_j$, with $u(\sigma) \leqslant \Phi(m\sigma^{1/p})$, $\sigma = \tau_j$, $j \geqslant 1$, the conclusion of Theorem 2 remains the same provided that the other assumptions remain unchanged. Therefore, applying Theorem 2 with the functions u, w and taking into account (31), outside some set $E_1 \subset [0, \infty)$, $E_1 = E_1^p(\alpha, \beta, \gamma)$, $0 < \alpha < \alpha_1 < 1$,

$$\varlimsup_{\tau_j \to \infty} \frac{\operatorname{mes}(E_1 \cap [0, \tau_j])}{\tau_j} \leqslant q_1 < \frac{1}{2}, \tag{32}$$

from (26) we have

$$u(\sigma + h) < u(\sigma) + \frac{1}{\alpha\sigma^\beta},$$

i.e.,

$$\ln\ln\mu(\sigma + h) - (\gamma + \mu)\ln(\sigma + h) < \ln\ln\mu(\sigma) - (\gamma + \mu)\ln\sigma + \frac{1}{\alpha\sigma^\beta}.$$

From this we obtain that for $\sigma \to \infty$ outside E_1

$$\begin{aligned}\ln\ln\mu(\sigma + h) &< \ln\ln\mu(\sigma) + \frac{1}{\alpha\sigma^\beta} + (\gamma + \mu)\ln\left(1 + \frac{h}{\sigma}\right) \\ &\leqslant \ln\ln\mu(\sigma) + \frac{1}{\alpha\sigma^\beta}\left[1 + 3(\gamma + \beta)\alpha d_{p\nu}(w)\right].\end{aligned}$$

Therefore, taking into account the elementary inequality $e^\varepsilon < 1 + 2\varepsilon$, $\varepsilon \in (0, \ln 2)$, from (26), (28), we obtain

$$\begin{aligned}R_v &\leqslant c_0\mu(\sigma)^{1+2a/(\alpha\sigma^\beta)}\exp[-2\sigma^\gamma w(v)] \\ &= c_0(\mu(\sigma))^{1-2(1+o(1))\sigma^{-\mu}} = o(1)\mu(\sigma), \quad \sigma \geqslant \sigma_1,\end{aligned}$$

where $a = 1 + 3(\gamma + \beta)\alpha d_{p\nu}(w)$, $0 < \mu < \beta$. Hence, for $\sigma \geqslant \sigma_1$, but for $\sigma \notin E_1$ we find that $\lambda_{\nu(\sigma)} \leqslant v(\sigma)$, where $\nu = \nu(\sigma)$ is the central index of the series (6). Then, by (11), (26), for $\sigma \to \infty$ outside the set E_1, we have

$$\mu(\sigma) = |a_\nu|\, e^{\lambda_\nu\sigma} = |a_\nu b_\nu|\, e^{\lambda_\nu\sigma}|b_\nu|^{-1} \leqslant \mu_b^*(\sigma)e^{w(\lambda_\nu)} \leqslant \mu_b^*(\sigma)e^{w(v)} =$$

$$= \mu_b^*(\sigma)\mu(\sigma)^{1/\sigma^{\gamma+\mu}}, \quad 0 < \gamma < 1, \quad 0 < \mu < \beta.$$

This means that when $\sigma \to \infty$

$$(1 - \sigma^{-\gamma-\mu}) \ln \mu(\sigma) \leqslant \ln \mu_b^*(\sigma) \tag{33}$$

outside the set E_1.

Since $|b_n| \leqslant e^{w(\lambda_n)}$, $n \geqslant N$, then for $k \geqslant N$ we have

$$\mu_b^*(\sigma) = |a_k b_k|\, e^{\lambda_k \sigma} \leqslant \mu(\sigma) e^{w(\lambda_k)}, \tag{34}$$

where $k = k(\sigma)$ is the central index of series (7).

Let

$$R_p^* = \sum_{\lambda_n > p} |a_n b_n| e^{\lambda_n \sigma}, \quad p = p(\sigma).$$

But $u^*(\sigma) \leqslant \Phi(m\sigma^{1/p})$ for $\sigma = \tau_j$ ($\{\tau_j\}$ is a sequence introduced above), where

$$u^*(\sigma) = \ln \left[\frac{\ln \mu_b^*(\sigma)}{\sigma^{\gamma+\mu}} \right], \quad \gamma \in (0,1), \quad \mu \in (0, \beta).$$

So, applying Theorem 2 and arguing in the same way as in the proof of estimate for R_v, we find

$$R_p^* \leqslant c_0 (\mu_b^*(\sigma))^{1-2(1+o(1))\sigma^{-\mu}} = o(1)\mu_b^*(\sigma), \quad c_0 = \sum_{n=1}^{\infty} \frac{1}{n^2}, \tag{35}$$

for $\sigma \to \infty$ outside some set $E_2 \subset [0, \infty)$, $E_2 = E_2(\alpha, \beta, \gamma)$, $0 < \alpha < \alpha_2 < 1$,

$$\overline{\lim_{\tau_j \to \infty}} \frac{\mathrm{mes}(E_2 \cap [0,\tau_j])}{\tau_j} \leqslant q_2 < \frac{1}{2}. \tag{36}$$

Thus, if $\sigma \geqslant \sigma_2$, $\sigma \notin E_2$, we find that $\lambda_{k(\sigma)} \leqslant p(\sigma)$. Consequently, by (26), (34), we have for $\sigma \to \infty$

$$\mu_b^*(\sigma) \leqslant \mu(\sigma) e^{w(p(\sigma))} = \mu(\sigma)\mu_b^*(\sigma)^{1/\sigma^{\gamma+\mu}}, \quad \gamma \in (0,1), \quad \mu \in (0, \beta),$$

outside the set E_2; so,

$$(1 - \sigma^{-\gamma-\mu}) \ln \mu_b^*(\sigma) \leqslant \ln \mu(\sigma). \tag{37}$$

Let $E = E_1 \cup E_2$. From (32), (36), for $\tau_j \to \infty$, we get that

$$\overline{\lim_{\tau_j\to\infty}} \frac{\mathrm{mes}(E\cap[0,\tau_j])}{\tau_j} \leqslant$$
$$\leqslant \overline{\lim_{\tau_j\to\infty}} \left[\frac{\mathrm{mes}(E_1\cap[0,\tau_j])}{\tau_j} + \frac{\mathrm{mes}(E_2\cap[0,\tau_j])}{\tau_j}\right] \leqslant$$
$$\leqslant q_1 + q_2 \leqslant q < 1, \quad 0 < \alpha < \alpha_3 < 1. \quad (38)$$

Thus, from (33), (37), taking into account (38), we finally get that for $\sigma\to\infty$

$$\left|1 - \frac{\ln \mu_b^*(\sigma)}{\ln \mu^*(\sigma)}\right| < \frac{2}{\sigma^{\gamma+\mu}}, \quad \gamma\in(0,1), \quad \mu\in(0,\beta),$$

outside the set $E = E_1\cup E_2$, such that $dE \leqslant q < 1$.

The Theorem 3 is proved. □

Question: Is condition (24) also necessary for estimate (25) to hold for any function $F \in \underline{D}^p(\Phi)$ outside some c_q–set $E^p(\alpha,\beta,\gamma)$?

For any sequence $\{b_n\}$ for which condition (24) is not satisfied, there exists an example of a function $F \in \underline{D}^{p\nu}(\Phi)$, $0<\nu<1$, for which estimate (25) does not hold. However, the question of the existence of a corresponding example for the class $\underline{D}^p(\Phi)$ remains open.

Note. The introductory part of the article (paragraphs 1, 2) belongs to A. M. Gaisin, and the main part (paragraph 3) belongs to G. A. Gaisina.

References

[1] Gaisin A. M., Aitkuzhina N. N. *Stability criterion for maximal terms of Dirichlet series.* Journal of Math. Sciences., 2022, vol. 260, no. 6, pp. 715 – 724. DOI: https://doi.org/10.1007/s10958-022-05723-0

[2] Gaisin A. M., Aitkuzhina N. N. Stability-preserving perturbation of the maximal terms of Dirichlet series. Probl. Anal. Issues. Anal., 2022, vol. 11, no. 3, pp. 1–15. DOI: https://doi.org/10.15393/j3.art.2022.12370

[3] Gaisin A. M. *Estimates of the growth and decrease on curves of an entire function of infinite order.* Sb. Math., 2002, vol. 194, no. 8, pp. 55 – 82. (in Russian). Translated in Sbornik: Mathematics, 2003, vol 194, no. 8, pp. 1167 – 1194. DOI: https://doi.org/10.4213/sm761

[4] Gaisin A. M., Gaisina G. A. *On the stability of the maximum term of the Dirichlet series.* Iz. VUZ, 2023, no. 1, pp. 25 – 35 (in Russian). Translated in Sbornik: Russian Math. (Iz. VUZ), 2023, vol. 67, no. 1, pp. 20 – 29. DOI: https://doi.org/10.3103/S1066369X23010024

[5] Gaisina G. A. *Stability of the Maximal Term of a Dirichlet Series of a Given Growth on a Discrete Set.* Math. Notes, 2025, vol. 118, no. 1, pp. 31 – 43. (in Russian). Translated in Sbornik: Math. Notes, 2025, vol. 118, no. 1, pp. 31–43. DOI: https://doi.org/10.4213/mzm14413

[6] Gaisin A. M., Latipov I. D. *Asymptotics of the logarithm of maximal term of the modified Dirichlet series.* Iz. VUZ, 2002, vol. 46, no. 9, pp. 15 – 24. (in Russian). Translated in Russian Math. (Iz. VUZ), 2002, vol. 46, no. 9, pp. 13 – 22.

[7] Leont'ev A. F. *Exponential Series.* Moscow, Nauka, 1976. (in Russian)

[8] Skaskiv O. B., Trakalo O. M. *On the stability of the maximum term of the entire Dirichlet series.* Ukr.Math. J., 2005, vol. 57, no. 4, pp. 571 – 576. (in Ukraine). Translated in Ukr Math J., 2005, vol. 57, pp. 686 — 693.
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