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ON SOME CLASSES OF HARMONIC FUNCTIONS WITH CONDITIONS IMPOSED ON COEFFICIENTS AND THEIR ARGUMENTS

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In this paper we consider a few classes of functions f harmonic in the unit disc Δ of the form $f = h + \overline{g}$, where h, g are suitably normalized functions holomorphic in Δ. Our special attention is drawn to some classes generated by respective coefficient conditions and to classes of functions with conditions imposed on coefficient arguments. We examine relationships between these conditions and some analytic conditions of stalikeness or convexity of considered functions.

1. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We consider complex functions harmonic in the disc Δ of the form

$$f = h + \overline{g}, \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \ |b_1| < 1. \tag{1}$$

It is known ([1]) that functions f of the form (1) are locally univalent and sense-preserving if and only if

$$|g'(z)| < |h'(z)|, \qquad z \in \Delta. \tag{2}$$

In 1984 J. Clunie and T. Sheil-Small ([2]) published their studies of some geometric properties of univalent functions f of the form (1) that are sense-preserving in Δ. In paper [2] the authors examined, among others, convexity and close-to-convexity of such functions. They pointed to the

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fact that if a complex harmonic univalent and sense-preserving function of the form (1) maps the disc Δ onto a convex domain, then not all images $f(\Delta_r)$, where $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$, $r \in (0,1)$, need to be convex. As an example there was given the function $f_0 = h_0 + \overline{g_0}$ of the form

$$f_0(z) = \mathrm{Re}\frac{z}{1-z} + i\ \mathrm{Im}\frac{z}{(1-z)^2},$$

where

$$h_0(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2}\ , \qquad g_0(z) = -\frac{\frac{1}{2}z^2}{(1-z)^2}\ , \quad z \in \Delta.$$

We have $f_0(\Delta) = \{w \in \mathbb{C} : \mathrm{Re}\ w > -\frac{1}{2}\}$ and for every $r \in (\sqrt{2}-1, 1)$ the set $f_0(\Delta_r)$ is not convex ([2], Remark 5.6, see also [3], pp. 40-41, 46-48).

It is known that this situation is different from the case of holomorphic univalent functions h, where $h(\Delta)$ is convex if and only if for each $r \in (0,1)$ the set $h(\Delta_r)$ is convex.

For holomorphic functions we know (see [4]) coefficient conditions, which imply the univalence of functions and the starlikeness or convexity of the image of Δ. In consequence, in this case the image of every disc Δ_r, $r \in (0,1)$, is starlike or convex, respectively.

In many papers we can find studies concerning influence of appropriate coefficient conditions on geometric properties of complex harmonic functions (e.g. [5], [6], [7], [8], [9], [10]). Various authors considered some classes of functions of the form (1) where the signs of coefficients of h and g are fixed (e.g. [11], [12]). We recall some results connected with these problems.

In 1990 Y. Avci and E. Złotkiewicz proved the following theorems.

THEOREM A ([5]). *If a function f of the form (1) satisfies the condition*

$$\sum_{n=2}^{+\infty} n\,(|a_n| + |b_n|) \le 1 - |b_1|, \tag{3}$$

then f is univalent and sense-preserving in the disc Δ.

THEOREM B ([5]). *Let f be a function of the form (1) such that $b_1 = 0$. If*

$$\sum_{n=2}^{+\infty} n\,(|a_n| + |b_n|) \le 1,$$

then $f(\Delta)$ is a domain starlike with respect to the point 0.

The result contained in Theorem B we can find also in paper [12] published by H. Silverman in 1998. Moreover, H. Silverman considered the case when for a function f of the form (1) we have

$$a_n \leq 0, \quad n = 2, 3, \ldots, \qquad b_n \leq 0, \quad n = 1, 2, \ldots. \tag{4}$$

Among others, he proved

THEOREM C ([12]). *Let f be a function of the form (1) such that the inequalities (4) hold and $b_1 = 0$. The function f is sense-preserving, univalent and maps the disc Δ onto a domain starlike with respect to the point 0 if and only if it satisfies the condition (3).*

In papers [7], [10] and [11] one can find a generalization of Theorems B and C where the restriction $b_1 = 0$ is omitted.

In the mentioned papers the conclusion that the set $f(\Delta)$ is starlike follows from the starlikeness of $f(\Delta_r)$ for every $r \in (0, 1)$, i.e. from the fact that if a function f of the form (1) satisfies the condition (3) then

$$\frac{\partial}{\partial \theta}\left(arg\, f(re^{i\theta})\right) = \frac{re^{i\theta}h'(re^{i\theta}) - \overline{re^{i\theta}g'(re^{i\theta})}}{h(re^{i\theta}) + \overline{g(re^{i\theta})}} \geq 0 \tag{5}$$

for any $\theta \in \langle 0, 2\pi)$, $r \in (0, 1)$.

Inspirations for these results were the appropriate theorems concerning functions holomorphic in the disc Δ. Among the oldest papers there should be mentioned e.g. [4], [13], [14], [15], [16], [17], [18], [19].

2. Let V_H denote the class of functions f of the form (1) such that $b_1 \in \langle 0, 1)$ and

$$a_n = -|a_n|e^{-i(n-1)\varphi}, \quad b_n = |b_n|e^{-i(n-1)\varphi}, \quad n = 2, 3, \ldots, \tag{6}$$

where $\varphi \in \langle 0, 2\pi)$, $\varphi = \varphi(f)$.

In 2002 J. M. Jahangiri and H. Silverman ([20]) published some consideration on starlike functions of the class V_H. One of the main theorems of this paper is given below.

Theorem D ([20]). *Let $f \in V_H$. The function f is sense-preserving, univalent in Δ and maps each disc Δ_r, $r \in (0,1)$, onto a starlike domain if and only if the condition (3) holds.*

We will prove a slightly more general theorem.

Theorem 1. *If a function f of the form (1) belongs to the class V_H and satisfies the condition (2), i.e. is locally univalent and sense-preserving in Δ, then it satisfies the condition (3).*

Proof. Let a function f of the form (1) belong to the class V_H. Then $b_1 \in \langle 0,1)$ and there exists $\varphi \in \langle 0, 2\pi)$ such that

$$h(z) = z - \sum_{n=2}^{+\infty} |a_n| e^{-i(n-1)\varphi} z^n = z - e^{i\varphi} \sum_{n=2}^{+\infty} |a_n| \left(e^{-i\varphi} z\right)^n, \quad z \in \Delta,$$

$$g(z) = b_1 z + \sum_{n=2}^{+\infty} |b_n| e^{-i(n-1)\varphi} z^n = b_1 z + e^{i\varphi} \sum_{n=2}^{+\infty} |b_n| \left(e^{-i\varphi} z\right)^n, \quad z \in \Delta.$$

Hence we obtain

$$h'(z) = 1 - \sum_{n=2}^{+\infty} n|a_n| e^{-i(n-1)\varphi} z^{n-1} = 1 - \sum_{n=2}^{+\infty} n|a_n| \left(e^{-i\varphi} z\right)^{n-1}, \quad z \in \Delta,$$

$$g'(z) = b_1 + \sum_{n=2}^{+\infty} n|b_n| e^{-i(n-1)\varphi} z^{n-1} = b_1 + \sum_{n=2}^{+\infty} n|b_n| \left(e^{-i\varphi} z\right)^{n-1}, \quad z \in \Delta.$$

Assume that the condition (2) holds, i.e. for each $z \in \Delta$ we have $|h'(z)| > |g'(z)|$, so

$$\left|1 - \sum_{n=2}^{+\infty} n|a_n| \left(e^{-i\varphi} z\right)^{n-1}\right| > \left|b_1 + \sum_{n=2}^{+\infty} n|b_n| \left(e^{-i\varphi} z\right)^{n-1}\right|, \quad z \in \Delta.$$

In particular, setting $z = re^{i\varphi}$, where $r \in (0,1)$, we obtain

$$\left|1 - \sum_{n=2}^{+\infty} n|a_n| r^{n-1}\right| > \left|b_1 + \sum_{n=2}^{+\infty} n|b_n| r^{n-1}\right|.$$

The expressions in the modulus signs are real and the sum on the right-hand side of the above inequality is nonnegative. Therefore

$$\left|1-\sum_{n=2}^{+\infty} n|a_n|r^{n-1}\right| > b_1+\sum_{n=2}^{+\infty} n|b_n|r^{n-1} \geq 0, \quad r\in(0,1).$$

From this fact and according to the continuity of power series we get either

$$\text{(a)} \qquad 1-\sum_{n=2}^{+\infty} n|a_n|r^{n-1} < -b_1-\sum_{n=2}^{+\infty} n|b_n|r^{n-1}, \quad r\in(0,1),$$

or

$$\text{(b)} \qquad 1-\sum_{n=2}^{+\infty} n|a_n|r^{n-1} > b_1+\sum_{n=2}^{+\infty} n|b_n|r^{n-1}, \quad r\in(0,1).$$

In the case (a) letting $r\to 0^+$ we obtain $b_1<-1$, which contradicts the assumption that $b_1\in\langle 0,1)$.

In the case (b) we have

$$\sum_{n=2}^{+\infty} n\left(|a_n|+|b_n|\right)r^{n-1} < 1-b_1, \qquad r\in(0,1).$$

Consequently, if $r\to 1^-$, then we get the condition (3), which completes the proof. □

If $f\in V_H$ satisfies the condition (3) then, according to Theorem A and the Lewy's result ([1]), it satisfies the condition (2). By Theorem 1 we have

Corolary 1. *For functions $f\in V_H$ the conditions (2) and (3) are equivalent.*

Let $\mathcal{H}^*$ denote the class of functions f of the form (1) satisfying the condition (2), which are univalent in the disc Δ and map Δ onto domains starlike with respect to the point 0.

From the presented facts we have the next corollary (see [20]).

Corolary 2. *Let $f\in V_H$. Then the following conditions are equivalent:*
i) f satisfies the condition (2);

ii) f *satisfies the condition (3);*
iii) $f \in \mathcal{H}^*$.

The implication iii) $\Rightarrow$ i) is a direct consequence of the definition of the class $\mathcal{H}^*$. The implication ii) $\Rightarrow$ iii) follows from the mentioned results contained in papers [5] and [10].

It is worth mentioning that if a function f of the form (1) belongs to the class V_H and its coefficients are real, then either $a_n \leq 0$ and $b_n \geq 0$ for $n = 2, 3, \dots$ ($\varphi = 0$) or $a_n = (-1)^n |a_n|$ and $b_n = (-1)^{n+1}|b_n|$ for $n = 2, 3, \dots$ ($\varphi = \pi$) (see [10]). The class V_H does not contain functions with coefficient satisfying the inequalities (4).

3. We know the following properties of holomorphic functions.

Theorem E ([4]). *If a holomorphic function* h *of the form*

$$h(z) = z + \sum_{n=2}^{+\infty} a_n z^n \ , \qquad z \in \Delta, \tag{7}$$

satisfies the condition

$$\sum_{n=2}^{+\infty} n|a_n| \leq 1, \tag{8}$$

then

$$\frac{h(z)}{z} \neq 0, \quad \left| \frac{zh'(z)}{h(z)} - 1 \right| < 1 \quad for \quad z \in \Delta. \tag{9}$$

Corollary A ([4]). *If a function* h *of the form (7) satisfies the condition (8), then* h *is univalent and starlike in* Δ. *Moreover, we have*

$$0 < \frac{\partial}{\partial \theta} \left(arg\, h(re^{i\theta}) \right) < 2 \ , \qquad \theta \in \langle 0, 2\pi), \ r \in (0,1). \tag{10}$$

We also have

THEOREM F ([19]). *If a function h is of the form*

$$h(z) = z - \sum_{n=2}^{+\infty} a_n z^n \ , \qquad z \in \Delta, \quad a_n \geq 0, \ n = 2, 3, \ldots, \tag{11}$$

and the conditions (9) hold, then we have (8).

COROLLARY B ([19]). *A function h of the form (11) satisfies the condition (8) if and only if it satisfies (9).*

In [12] H. Silverman observed that these properties, especially (9), cannot be directly extended to harmonic functions $f = h + \overline{g}$ of the form (1) or respectively such that

$$h(z) = z - \sum_{n=2}^{+\infty} a_n z^n \ , \quad g(z) = -\sum_{n=2}^{+\infty} b_n z^n, \quad a_n, b_n \geq 0, \ n = 2, 3, \ldots .$$

An extention of (9) can be obtained on an additional assumption.

For a function f of the form (1) we put

$$f^\bullet(z) := zh'(z) - \overline{zg'(z)} = z\left(1 + \sum_{n=2}^{+\infty} n a_n z^{n-1}\right) - \overline{z\left(b_1 + \sum_{n=2}^{+\infty} n b_n z^{n-1}\right)} \tag{12}$$

for $z \in \Delta$.

It is clear that if J_f denotes the Jacobian of f, i.e. $J_f = |h'|^2 - |g'|^2$ for $f = h + \overline{g}$, then the condition (2) is equivalent to the condition

$$J_f(z) > 0 \ , \qquad z \in \Delta. \tag{13}$$

By (12) we have $|f^\bullet(z)| \geq |z| \left(|h'(z)| - |g'(z)|\right)$, $z \in \Delta$. We can observe that if the condition (13) holds, then

$$|f^\bullet(z)| > 0, \qquad z \in \Delta \setminus \{0\}.$$

If $f^\bullet(z_0) = 0$ for a $z_0 \in \Delta \setminus \{0\}$, then by (12) we have $z_0 h'(z_0) = \overline{z_0 g'(z_0)}$, so

$$f^\bullet(z_0) = 0 \implies J_f(z_0) = 0, \qquad z_0 \in \Delta \setminus \{0\}.$$

Let us note that the converse property does not hold.

Set

$$f_1 = h_1 + \overline{g_1}, \quad h_1(z) = \frac{z}{1-z}, \quad g_1(z) = b_1 z, \quad z \in \Delta,\ |b_1| < 1.$$

Then for $z \in \Delta$ we have

$$J_{f_1}(z) = \left(\frac{1}{|1-z|^2} + |b_1|\right)\left(\frac{1}{|1-z|^2} - |b_1|\right), \quad f_1^{\bullet}(z) = \frac{z}{(1-z)^2} - \overline{b_1 z}.$$

If $b_1 \in \left(\frac{1}{4}, 1\right)$, then there exists exactly one point $z_{b_1} \in \Delta$ such that $\frac{1}{(1-z_{b_1})^2} = b_1$, $z_{b_1} \in (-1,0)$. Hence

$$J_{f_1}(z_{b_1}) = 0 \qquad \text{and} \qquad f_1^{\bullet}(z_{b_1}) = 0.$$

If $\frac{1}{4} < |b_1| < 1$, $b_1 \neq \overline{b_1}$, $b_1 \in D$, where $D = \omega(\Delta)$, $\omega(z) = \frac{1}{(1-z)^2}$, $z \in \Delta$, then there exists also exactly one point $z_{b_1} \neq \overline{z_{b_1}}$ such that $\frac{1}{(1-z_{b_1})^2} = b_1$. Therefore

$$J_{f_1}(z_{b_1}) = 0 \qquad \text{and} \qquad f_1^{\bullet}(z_{b_1}) \neq 0,$$

which follows from the fact that $f_1^{\bullet}(z_{b_1}) = 2i\,\mathrm{Im}(b_1 z_{b_1}) = 2i\,\mathrm{Im}\left(\frac{z_{b_1}}{(1-z_{b_1})^2}\right)$.

Let f be a function of the form (1). As it has been known (e.g. [20], [10]), if $f(z) \neq 0$ for $z \in \Delta \setminus \{0\}$, then from (5) and (12) we have

$$\frac{\partial}{\partial\theta}\left(arg\, f(re^{i\theta})\right) = \mathrm{Re}\frac{f^{\bullet}(re^{i\theta})}{f(re^{i\theta})}, \tag{14}$$

where

$$\frac{f^{\bullet}(re^{i\theta})}{f(re^{i\theta})} = \frac{1 - \overline{b_1}e^{-2i\theta} + \sum_{n=2}^{+\infty}\left(na_n e^{i(n-1)\theta} - n\overline{b_n}e^{-i(n+1)\theta}\right)r^{n-1}}{1 + \overline{b_1}e^{-2i\theta} + \sum_{n=2}^{+\infty}\left(a_n e^{i(n-1)\theta} + \overline{b_n}e^{-i(n+1)\theta}\right)r^{n-1}}, \tag{15}$$

$\theta \in \langle 0, 2\pi), \ r \in (0,1)$.

THEOREM 2. *If a function f of the form (1) satisfies the condition*

$$\sum_{n=1}^{+\infty}\left(n|a_n| + (n+2)|b_n|\right) \leq 2, \qquad a_1 = 1, \tag{16}$$

then it satisfies the condition (3) and

$$\left| \frac{f^{\bullet}(z)}{f(z)} - 1 \right| < 1 \ , \qquad z \in \Delta \setminus \{0\}. \tag{17}$$

Proof. Let f be a function of the form (1) satisfying the condition (16). Obviously, then the condition (3) holds and therefore

$$\sum_{n=1}^{+\infty} (|a_n| + |b_n|) \leq 2, \qquad a_1 = 1,$$

Consequently, for $z \in \Delta \setminus \{0\}$ we have

$$|f^{\bullet}(z)| \geq |z| \left(1 - \sum_{n=2}^{+\infty} n|a_n||z|^{n-1} - |b_1| - \sum_{n=2}^{+\infty} n|b_n||z|^{n-1} \right) > 0 \ ,$$

$$|f(z)| \geq |z| \left(1 - \sum_{n=2}^{+\infty} |a_n||z|^{n-1} - |b_1| - \sum_{n=2}^{+\infty} |b_n||z|^{n-1} \right) > 0 \ .$$

Hence

$$f^{\bullet}(z) f(z) \neq 0 \qquad \text{for} \quad z \in \Delta \setminus \{0\}. \tag{18}$$

According to (16), for $z = re^{i\theta}$, $r \in (0,1)$, $\theta \in \langle 0, 2\pi)$, we obtain

$$\begin{aligned}
&|f(z)| - |f^{\bullet}(z) - f(z)| = \\
&= r \left| 1 + \overline{b_1} e^{-2i\theta} + \textstyle\sum_{n=2}^{+\infty} \left(a_n e^{i(n-1)\theta} + \overline{b_n} e^{-i(n+1)\theta} \right) r^{n-1} \right| + \\
&- r \left| -2\overline{b_1} e^{-2i\theta} + \textstyle\sum_{n=2}^{+\infty} \left((n-1) a_n e^{i(n-1)\theta} - (n+1) \overline{b_n} e^{-i(n+1)\theta} \right) r^{n-1} \right| \geq \\
&\geq r \left(1 - |b_1| - \textstyle\sum_{n=2}^{+\infty} (|a_n| + |b_n|) r^{n-1} \right) + \\
&- r \left(2|b_1| + \textstyle\sum_{n=2}^{+\infty} ((n-1)|a_n| + (n+1)|b_n|) r^{n-1} \right) = \\
&= r \left(1 - 3|b_1| - \textstyle\sum_{n=2}^{+\infty} (n|a_n| + (n+2)|b_n|) r^{n-1} \right) > 0.
\end{aligned}$$

Thus, by (15) and (18), we get (17). □

By (14) and from Theorem 2 we have

Corolary 3. *If a function f of the form (1) satisfies the condition (16), then*

$$0 < \frac{\partial}{\partial \theta} \left(arg\, f(re^{i\theta}) \right) < 2 \ , \qquad r \in (0,1), \ \theta \in \langle 0, 2\pi). \tag{19}$$

In paper [12] H. Silverman gave an example of a function f of the form (1) such that the condition (3) holds for it, but the condition (19) is not satisfied ($f_2(z) = z - \frac{1}{2}\overline{z}^2$, $z \in \Delta$).

Let us consider the function $f_3(z;b) = z + b\overline{z}^n$, $z \in \Delta$, $b \in \mathbf{R}$, $n \in \{2,3,\ldots\}$. If $|b| \leq \frac{1}{n}$, then the condition (3) holds, so f_3 is univalent and sense-preserving and for each $r \in (0,1)$ the set $f_3(\Delta_r;b)$ is starlike with respect to the origin. From (12) we obtain $\left|\frac{f_3^\bullet(z;b)}{f_3(z;b)} - 1\right| = \left|\frac{-(n+1)b\overline{z}^n}{z+b\overline{z}^n}\right|$, $z \in \Delta \setminus \{0\}$, and after some computations we conclude that the condition (17) holds only if $|b| \leq \frac{1}{n+2}$. Thus we get

Corollary 4. *The function $f_3(\cdot;b)$ satisfies the conditions (16) and (17) if and only if $|b| \leq \frac{1}{n+2}$.*

It is evident that the mentioned function f_2 does not satisfy this assumption.

Next we consider the function $f_4(\cdot;q)$ of the form

$$f_4(z;q) = z + \sum_{n=1}^{+\infty} \overline{q}^n \overline{z}^n = z + \overline{\frac{qz}{1-qz}}\,, \qquad z \in \Delta, \quad |q| < 1.$$

For $|q| < \frac{5-\sqrt{13}}{6}$ we get

$$\sum_{n=2}^{+\infty} n|a_n| + \sum_{n=1}^{+\infty} (n+2)|b_n| = \sum_{n=1}^{+\infty} (n+2)|q|^n = \frac{3|q| - 2|q|^2}{(1-|q|)^2} \leq 1,$$

so then the condition (16) is fulfilled. In consequence, by Theorem 2 and Corollary 3 the conditions (3), (17) and (19) hold.

Now we turn our attention to the function f_5 of the form (1) with $a_n = 0$, $n = 2,3,\ldots$, $b_n = -\beta_n$, $\beta_n \geq 0$, $n = 1,2,\ldots$, $\beta_1 < 1$, i.e.

$$f_5(z) = z - \sum_{n=1}^{+\infty} \beta_n \overline{z^n}\,, \quad z \in \Delta,\ \beta_n \geq 0,\ n = 1,2,\ldots,\ \beta_1 < 1. \tag{20}$$

Assume that the function f_5 of the form (20) satisfies the condition (17). Note that such functions exist (e.g. $\tilde{f}_5(z) = z - \beta_1\overline{z}$, $z \in \Delta$, $0 \leq$

$\beta_1 < \frac{1}{3}$). Consequently, the function f_5 satisfies the condition (18). Then we have

$$\frac{f_5(r)}{r} = 1 - \beta_1 - \sum_{n=2}^{+\infty} \beta_n r^{n-1} \ , \qquad r \in (0,1).$$

Observe that $\lim_{r\to 0^+} \frac{f_5(r)}{r} = 1-\beta_1 > 0$. By the continuity of f_5 and from (18) we conclude that

$$f_5(r) > 0, \qquad r \in (0,1). \tag{21}$$

According to (17) we have

$$|f_5(r)| - |f_5^\bullet(r) - f_5(r)| > 0 \ , \qquad r \in (0,1).$$

Hence, by (12), (20) and (21), we obtain

$$1 - \beta_1 - \sum_{n=2}^{+\infty} \beta_n r^{n-1} > 2\beta_1 + \sum_{n=2}^{+\infty} (n+1)\beta_n r^{n-1} \ , \qquad r \in (0,1),$$

and thus

$$3\beta_1 + \sum_{n=2}^{+\infty} (n+2)\beta_n r^{n-1} < 1 \ , \qquad r \in (0,1),$$

Letting $r \to 1^-$ we get

$$\sum_{n=1}^{+\infty} (n+2)\beta_n \leq 1.$$

Therefore, by Theorem 2, we have

COROLLARY 5. *The function f_5 of the form (20) satisfies the condition (17) if and only if (16) holds for it.*

Let V_{H0} denote the subclass of V_H such that in (6) we have $\varphi = 0$, i.e. the class of functions of the form

$$f(z) = z - \sum_{n=2}^{+\infty} \alpha_n z^n + \sum_{n=1}^{+\infty} b_n \overline{z^n}, \quad z \in \Delta, \ b_1 \in \langle 0,1), \tag{22}$$

where $\alpha_n, b_n \geq 0$, $n = 2, 3, \ldots$.

Theorem 3. *If a function $f \in V_{H0}$ satisfies the condition (17), then the condition (3) is fulfilled.*

Proof. Let $f \in V_{H0}$ be of the form (22) and satisfy (17). Then we have (18) for it. Proceeding similar to that for the function f_5 gives

$$1 + b_1 + \sum_{n=2}^{+\infty} (b_n - \alpha_n) r^{n-1} > \left| -2b_1 - \sum_{n=2}^{+\infty} \left((n-1)\alpha_n + (n+1)b_n \right) r^{n-1} \right| ,$$

so

$$1 + b_1 + \sum_{n=2}^{+\infty} (b_n - \alpha_n) r^{n-1} > 2b_1 + \sum_{n=2}^{+\infty} \left((n-1)\alpha_n + (n+1)b_n \right) r^{n-1}$$

for $r \in (0,1)$. Therefore

$$\sum_{n=2}^{+\infty} n(\alpha_n + b_n) r^{n-1} < 1 - b_1 , \qquad r \in (0,1).$$

From this, letting $r \to 1^-$, we obtain (3), which completes the proof. □

Remark 1. *It is easily seen that for holomorphic functions h of the form (7) the conditions (3) and (16) reduce to (8) and the condition (17) coincides with (9). Of course, the function $z \mapsto \frac{h(z)}{z}$, $z \in \Delta$, has removeable singularity at the point 0, and so (9) may be considered in the whole disc Δ.*

4. In this section we make some remarks on convex harmonic functions. The following lemma is known.

Lemma A ([3], p. *108). If $f = h + \overline{g}$ of the form (1) is univalent, sense-preserving starlike in Δ, and if H and G are holomorphic functions defined by*

$$zH'(z) = h(z), \quad zG'(z) = -g(z), \quad H(0) = G(0) = 0, \tag{23}$$

then $F = H + \overline{G}$ is an univalent, sense-preserving function convex in Δ.

However, the converse to Lemma A is false ([3], p. 110). We know that the convexity of a harmonic function F follows from the condition (see

[5], [10], [12])

$$\sum_{n=2}^{+\infty} n^2 \left(|c_n| + |d_n|\right) \le 1 - |d_1|, \tag{24}$$

where

$$F = H + \overline{G}, \quad H(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad G(z) = \sum_{n=1}^{\infty} d_n z^n, \quad z \in \Delta,\ |d_1| < 1. \tag{25}$$

If $f = h + \overline{g}$ is of the form (1) and H, G are such that we have (23), then

$$nc_n = a_n, \qquad nd_n = -b_n, \qquad n = 1, 2, \ldots. \tag{26}$$

If F of the form (25) satisfies the condition (24), then by (26) we obtain the condition (3) for the function f, and conversely.

From the mentioned facts we obtain

THEOREM 4. *If a function F of the form (25) satisfies the condition (24), then*

$$\left|(zH'(z))'\right| > \left|(zG'(z))'\right|, \qquad z \in \Delta. \tag{27}$$

PROOF. Let $F = H + \overline{G}$ be of the form (25) and let it satisfy (24). Consider the function $f = h + \overline{g}$ such that the equalities (23) hold. Then by (26) f satisfies (3). From Theorems A, C we conclude that $J_f(z) > 0$, $z \in \Delta$, which gives

$$\left|(zH'(z))'\right| > \left|-(zG'(z))'\right|, \qquad z \in \Delta,$$

i.e. the condition (27). □

Moreover, we know the following theorem.

THEOREM G ([5], [10], [12]). *If a function F of the form (25) satisfies the condition (24), then $F(\Delta_r)$ is convex for each $r \in (0, 1)$.*

Consequently, according to Theorems C and G and Lemma A, for harmonic functions satisfying the condition (3) or (24), respectively, we have the "full"Alexander theorem.

Let us return to the class V_H.

Theorem 5. *If a function F of the form (25) belongs to the class V_H and satisfies the condition (27), then it satisfies (24).*

Proof. From (6) and (25) for a function $F \in V_H$ we have

$$(zH'(z))' = 1 - \sum_{n=2}^{+\infty} n^2 |c_n| \left(e^{-i\varphi} z\right)^{n-1}, \qquad z \in \Delta,$$

$$(zG'(z))' = |d_1| + \sum_{n=2}^{+\infty} n^2 |d_n| \left(e^{-i\varphi} z\right)^{n-1}, \qquad z \in \Delta,$$

so by (27) we get

$$\left| 1 - \sum_{n=2}^{+\infty} n^2 |c_n| r^{n-1} \right| > |d_1| + \sum_{n=2}^{+\infty} n^2 |d_n| r^{n-1}, \qquad r \in (0,1).$$

Hence we obtain (24). □

Consequently, by Theorems 4 and 5, we have

Corolary 6. *For a function $F \in V_H$ the conditions (24) and (27) are equivalent.*

Observe that functions f, F of the forms (1), (25), respectively, such that the equalities (26) hold, cannot belong to V_H simultaneously (see (6)), except for the holomorphic case (i.e. $b_n = d_n = 0$, $n = 1, 2, \ldots$).

Let $\mathcal{H}^c$ denotes the class of functions F of the form (25) univalent, sense-preserving in Δ and mapping Δ onto convex domains.

According to Theorem 5, Theorem G and Corollary 6, we obtain

Corolary 7. *If a function F of the form (25) belongs to the class V_H and satisfies the condition (27), then $F \in \mathcal{H}^c$.*

For a function F of the form (25) we denote

$$F^{\bullet\bullet}(z) := z\,(zH'(z))' + \overline{z\,(zG'(z))'}\,, \qquad z \in \Delta. \tag{28}$$

On account of (12) and (28), it is clear that $F^{\bullet\bullet} = (F^{\bullet})^{\bullet}$.

From Theorem 2 we obtain

THEOREM 6. *If a function F of the form (25) satisfies the condition*

$$\sum_{n=1}^{+\infty} n\left(n|c_n| + (n+2)|d_n|\right) \leq 2, \qquad c_1 = 1, \tag{29}$$

then

$$\left|\frac{F^{\bullet\bullet}(z)}{F^{\bullet}(z)} - 1\right| < 1\ , \qquad z \in \Delta \setminus \{0\}. \tag{30}$$

In consequance, we have

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right) \in (0,2), \quad r \in (0,1), \quad \theta \in \langle 0, 2\pi). \tag{31}$$

The last statement follows from the fact (see e.g. [11]) that

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right) = \mathrm{Re}\frac{F^{\bullet\bullet}(re^{i\theta})}{F^{\bullet}(re^{i\theta})}, \quad r \in (0,1), \quad \theta \in \langle 0, 2\pi).$$

Obviously, the condition (29) implies (16).

The function $f_3(z;b) = z + b\overline{z}^n$, $z \in \Delta$, $n \in \{2,3,\ldots\}$, for $|b| \leq \frac{1}{n(n+2)}$ satisfies the condition (29), and so (30) and (31), as well. If $b = \frac{1}{4}$, $n = 2$, then this function does not satisfy the condition (29).

Some other properties of the operators (12), (28) and their generalizations were considered e.g. in [21], [22].

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