Trudy Petrozavodskogo gosudarstvennogo universiteta
Seria “Matematika” Vypusk 13, 2006

UDK 517.54

ON COMPLEX HARMONIC TYPICALLY-REAL FUNCTIONS WITH A POLE AT THE POINT ZERO

Z. J. Jakubowski, A. Sibelska

Several mathematicians examined classes of meromorphic typically-real functions with a simple pole at the point zero. This article includes results concern class Q'_H of complex harmonic typically-real functions with a pole at the point zero. There are determined the relationships between this class and the class Q'_r of meromorphic typically-real funtions with a pole at the origin, which was investigated by S. A. Gelfer [4]. We present also coefficient estimates for functions of a subclass of the class Q'_H and properties of the Hadamard product with fuctions of the class Q'_H.

§ 1. Introduction

Meromorphic typically-real functions have been examined for a long time ([1], [2]). In several papers authors investigated functions with a pole at the infinity (e.g. [3]). Other matematicians (e.g. S. A. Gelfer [4], J. Zamorski [5], M. P. Remizowa [6], Z. J. Jakubowski, K. Skalska [7]) examined classes of functions, which are holomorphic typically-real in the ring $P := \{z \in \mathbf{C} : 0 < |z| < 1\}$, with a simple pole at the point $z = 0$. Our results extend these investigations to classes of complex harmonic typically-real functions in P with a simple pole at $z = 0$. Our study was inspired, among others, by [8], [9], [7] and [10].

[1]Key Words and Phrases: harmonic functions, typically-real functions, meromorphic functions, coefficient estimates, Hadamard product.
Mathematics Subject Classification: 30C45, 30C50, 31A05.

§ 2. On a class of typically-real functions with a pole at the zero

S. A. Gelfer in paper [4] considered, among others, the class Q'_r of functions H of the form

$$H(z) = -\frac{1}{z} + a_0 + a_1 z + \ldots, \; z \in P, \tag{1}$$

holomorphic typically-real in P and such that $H(z) \neq 0, \; z \in P$.

He showed also that the following properties hold:

(a) $H \in Q'_r \Rightarrow ImzImH(z) > 0, \, z \in P, z \neq \overline{z}$,

(b) $H \in Q'_r \Rightarrow a_n = \bar{a}_n, \, n = 0, 1, 2, \ldots,$

(c) $H \in Q'_r \Leftrightarrow \{Re\{\frac{z}{1-z^2} H(z)\} < 0 \wedge a_n = \bar{a}_n, \, n = 0, 1, 2, \ldots\}$.

Moreover, if Σ'_r denotes the class of functions of the form (1), with real coefficients, univalent in P and $H(z) \neq 0, \; z \in P$, then $\Sigma'_r \subset Q'_r$.

Definition 1. *Let Q'_H denote the class of complex functions f harmonic in P and such that*

(i) $f(z) = F(z) + \overline{G(z)}, \; F(z) = -\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, G(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in P$,

(ii) $f(z) \neq 0, \; z \in P$,

(iii) $ImzImf(z) > 0, \; z \in P, \; z \neq \overline{z}$.

Directly from the definition we obtain $\lim_{z \to 0} f(z) = \infty$ and each function $f \in Q'_H$ is locally univalent in some neighbourhood of the point $z = 0$. Moreover, if $f \in Q'_H$ and it is locally univalent function in $P \cap R$, then $f(x), \; x \in P \cap R$, is an increasing function on the intervals $(-1, 0)$ and $(0, 1)$. Besides, $\lim_{x \to -1^+} f(x) \geq 0$, $\lim_{x \to 0^-} f(x) = +\infty$, $\lim_{x \to 0^+} f(x) = -\infty, \lim_{x \to 1^-} f(x) \leq 0$.

If $H \in Q'_r$, then $H \in Q'_H$, of course. It is known that if a function $h(z) = z + a_2 z^2 + \ldots$ is holomorphic typically-real in the unit disc Δ, then $H = -\frac{1}{h} \in Q'_r$ ([11]). Consequently, this function H belongs to Q'_H.

It is know that $h'(x) > 0, \; x \in \Delta \cap R$. This property does not have to hold for harmonic typically-real functions.

Indeed, for the function $f_0(z) := z+a(z^3+\overline{z^3})$, $z \in \Delta$, $a \in \langle -\frac{1}{2}, -\frac{1}{6})$ we have $ImzImf(z) > 0$, $z \neq \overline{z} \in \Delta$ and $f_0'(x_a) = 0$, where $x_a = \sqrt{-\frac{1}{6a}} \in \langle -\frac{\sqrt{3}}{3}, 1)$.

Let Σ_H' denote the class of functions of the form (i), with real coefficients a_n, b_n, $n = 0, 1, 2, \ldots$, univalent and satisfaying the condition (ii). Then we have $\Sigma_H' \subset Q_H'$.

REMARK 1. *Let $f \in Q_H'$. From (iii) we conclude that $\overline{f(z)} = f(z)$ if and only if $z = \overline{z} \in P$. Hence $a_n - b_n = \overline{a_n - b_n}$, $n = 0, 1, 2, \ldots$.*

REMARK 2. *Let f be a function of the form (i) and $a_n = \overline{a_n}$, $b_n = \overline{b_n}$, $n = 0, 1, 2, \ldots$. If $z \in P$, then $\overline{f(\overline{z})} = f(z)$.*

Indeed, we have

$$\overline{f(\overline{z})} = \overline{F(\overline{z}) + \overline{G(\overline{z})}} = \overline{F(\overline{z})} + G(\overline{z}) = F(\overline{\overline{z}}) + \overline{G(z)} = f(z),\ z \in P.$$

The following two theorems determine relationship beetwen the classes Q_r' and Q_H'.

THEOREM 1. *If $f = F + \overline{G} \in Q_H'$, then $g = F - G$ is a typically-real function in P and exist $\lim_{x\to 1^-} g(x)$, $\lim_{x\to -1^+} g(x)$. If*

$$\lim_{x\to 1^-} g(x) \le 0, \quad \lim_{x\to -1^+} g(x) \ge 0, \tag{2}$$

then $g \in Q_r'$.

PROOF. Let $f = F + \overline{G} \in Q_H'$. Let us consider the function $g = F - G$. It is easy to observe that g is holomorphic in P and is of the form $g(z) = -\frac{1}{z} + (a_0 - b_0) + (a_1 - b_1)z + \ldots$, which is required in the class Q_r'. According to the remark 1, g is a real function in the set $P \cap R$. Moreover, $Imf = Im(F + \overline{G}) = Im(F - G) = Img$. Thus, on account of the condition (iii) from the definition of the class Q_H', we have $ImzImg(z) > 0$, $z \in P$, $z \neq \overline{z}$. Therefore g is a typically-real function.

Let us observe that $g(z) \neq 0$, $z \in P$.

If there existed a $z_0 \in P$, $z_0 \neq \overline{z_0}$, such that $g(z_0) = 0$, we would have $F(z_0) = G(z_0)$, thus $ImF(z_0) = ImG(z_0)$. Hence $Imf(z_0) = 0$, which contradicts (iii).

Since g is holomorphic typically-real in P, we have $g'(x) \neq 0$, $x \in (-1, 0) \cup (0, 1)$. Indeed, if there existed a $x_0 \in P \cap R$, such $g'(x) = 0$, the function g would be the double function in some neighbourhood of the point x_0. This contradicts the typically-reality of g.

The continuity of g' on the intervals $(-1,0),(0,1)$ and the fact that $\lim_{x\to 0^+} g'(x) = +\infty$, $\lim_{x\to 0^-} g'(x) = +\infty$ give $g'(x) > 0$, $x \in P \cap R$. Furthermore, $\lim_{x\to 0^+} g(x) = -\infty$, $\lim_{x\to 0^-} g(x) = +\infty$. Hence, from (2) and since g is increasing in $(-1,0)$ and $(0,1)$, it follows that does not exist $x_0 \in P \cap R$, such that $g(x_0) = 0$. □

The following example shows that the assumption (2) is needed.

EXAMPLE 1. *Let f_1 be the function of the form*

$$f_1(z) = F_1(z) + \overline{G_1(z)},$$

$$F_1(z) = -\frac{1}{z} + z,\ G_1(z) = b_1 z,\ b_1 < -1,\ z \in P. \tag{3}$$

Obviously, f_1 is of the form (i). For $z \in P \cap R$, we have $\overline{f_1(z)} = f_1(z)$. Moreover, $f_1(z) \neq 0$, $z \in P$ and $Imf_1(z) = Imz(\frac{1}{|z|^2} + 1 - b_1)$, thus f_1 belongs to Q'_H.

It is easy to check that the condition (2) does not hold.

Let us consider the function $g_1(z) = F_1(z) - G_1(z)$, $z \in P$. It is a holomorphic typically-real function. But g_1 takes the value zero at some points of the ring P, therefore $g_1 \notin Q'_r$.

Furthermore, we have

THEOREM 2. *If F and G are functions holomorphic in P of the form*

$$F(z) = -\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,\ G(z) = \sum_{n=0}^{\infty} b_n z^z,\ z \in P,$$

respectively, such that $g = F - G \in Q'_r$ and

$$Re\{F(x) + G(x)\} \neq 0,\ \ x \in P \cap R, \tag{4}$$

then $f = F + \overline{G} \in Q'_H$.

PROOF. Obviously, the function $f(z) = -\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^z}$ is of the form (i) from the definition of the class Q'_H and it is complex harmonic in P. Since $f(z) = g(z) + 2ReG(z)$, $z \in P$, we have $f(x) = \overline{f(x)}$ for $x \in P \cap R$. Moreover, $Imf(z) = Img(z), z \in P$, thus (iii) holds.

Let us observe that $f(z) \neq 0$, $z \in P$. Since $Imf(z) = Img(z) \neq 0$ for $z \in P$, $z \neq \overline{z}$, it suffices to show that $f(z) \neq 0$ for $z \in P \cap R$.

If there existed an $x_0 \in P \cap R$, such that $f(x_0) = 0$, we would have $F(x_0) + \overline{G(x_0)} = 0$, i.e. $Re\{F(x_0) + G(x_0)\} = 0$, $Im\{F(x_0) - G(x_0)\} = 0$. Hence and from (4) $f(x_0) \neq 0$, $x_0 \in P \cap R$. $\square$

Example 2. *Let* $F_2(z) = -\frac{1}{z} + \frac{4}{3} + i$, $G_2(z) = \frac{2}{3} + i$. *Then* $g_2(z) = F_2(z) - G_2(z)$ *is a function of the class* Q'_r.

Let $f_2(z) = F_2(z) + \overline{G_2(z)}, z \in P$. *We have* $f_2(z) = -\frac{1}{z} + 2, z \in P$ *and* $f_2(\frac{1}{2}) = 0$, *therefore* $f_2 \notin Q'_H$. *Clearly, the condition (4) does not hold.*

§ 3. Coefficient estimates

Applying the known result of S. A. Gelfer

([4], th.1), related to coefficient estimates for functions of the class Q'_r, in view of remark 1, for functions $f \in Q'_H$ satisfying the condition (2), we obtain the following estimates:

$$-2 \leq a_0 - b_0 \leq 2,$$
$$-1 \leq a_1 - b_1 \leq 3,$$
$$-4 \leq 4 \min_{0 \leq \theta \leq \pi} (sinn\theta sin\theta) \leq a_n - b_n \leq 4 \max_{0 \leq \theta \leq \pi} (sinn\theta sin\theta) \leq 4,\ n \geq 2.$$

Moreover, we have

Theorem 3. *If* $f = F + \overline{G} \in Q'_H$, *the condition (2) holds and*

$$|G'(z)| < |F'(z)|,\ \ z \in P, \tag{5}$$

then

$$|b_1| \leq 1,\ \ |b_2| \leq \frac{1}{2}, \tag{6}$$

$$|b_n| \leq \frac{2(n-1)(n-2)}{n},\ \ n = 3, 4, \ldots, \tag{7}$$

$$|a_1| \leq 4,\ \ |a_2| \leq \frac{9}{2}, \tag{8}$$

$$|a_n| \leq \frac{2(n^2 - n + 2)}{n},\ \ n = 3, 4, \ldots. \tag{9}$$

The estimates (6) are sharp. In case the first estimate, extremal functions are e.g. $f_1^*(z) = -\frac{1}{z} + \overline{z}$, $f_2^*(z) = -\frac{1}{z} - \overline{z}$. *In the second estimate, the equality sign occurs e.g. for the functions* $f_3^*(z) = -\frac{1}{z} + \frac{1}{2}\overline{z^2}$, $f_4^*(z) = -\frac{1}{z} - \frac{1}{2}\overline{z^2}, z \in P$.

PROOF. Let $f = F + \overline{G} \in Q'_H$ satisfy (2) and (5). Let us set $\omega(z) := \frac{G'(z)}{F'(z)}$, $z \in P$. Since $z = 0$ is at least double zero of the function ω, $|\omega(z)| < 1$, $z \in \Delta := \{z \in C : |z| < 1\}$, by Schwarz lemma we have $|\omega(z)| \le |z|^2 < |z|$, $z \in \Delta$. From theorem 1, in view of the condition (2), we obtain $g := F - G \in Q'_r$. Moreover, we have $G'(z) = \omega(z)F'(z)$. Thus $G'(z) = \omega(z)(g'(z) + G'(z))$ and therefore

$$z^2 G'(z) = \frac{\omega(z)}{1-\omega(z)} z^2 g'(z), \quad z \in \Delta. \tag{10}$$

It is know that an analytic function h is said to be subordinate to an analytic function l (written $h \prec l$) if $h(z) = l(\omega(z)), |z| < 1$ for some analytic function ω with $|\omega(z)| \le |z|$ ([12], p.190). Moreover, if l and L are given by the power series $\sum_{n=0}^{\infty} d_n z^n$, $\sum_{n=0}^{\infty} D_n z^n$, convergent in some disk $|z| < R$, $R > 0$, then we say that l is dominated by L and write $l(z) \ll L(z)$, if for each integer $n \ge 0$, $|d_n| \le D_n$ ([13], p.82).

We have thus $\frac{\omega(z)}{1-\omega(z)} \prec \frac{z}{1-z}$, $z \in \Delta$ and since $\varphi(z) := \frac{z}{1-z}$ is a convex function in Δ, writing $\frac{\omega(z)}{1-\omega(z)} = \sum_{k=2}^{\infty} c_k z^k$, we have $|c_k| \le 1$, $k = 2, 3, \ldots$ ([13]), II, p. 182). Thus

$$\frac{\omega(z)}{1-\omega(z)} \ll \frac{z^2}{1-z}. \tag{11}$$

Obviously, $z^2 g'(z) = 1 + \sum_{n=1}^{\infty} n(a_n - b_n) z^{n+1}$, $z \in \Delta$, and $z^2 G'(z) = \sum_{n=1}^{\infty} n b_n z^{n+1}$, $z \in \Delta$.

We mentioned that $g \in Q'_r$, therefore by the Gelfer's theorem ([4], th.1), we have $|a_1 - b_1| \le 3$ and $|a_n - b_n| \le 4$, $n = 2, 3, \ldots$. Let us consider the function $\psi(z) := 1 + 3z^2 + 2 \cdot 4z^3 + 3 \cdot 4z^4 + \cdots = 1 + 3z^2 + 4\sum_{n=2}^{\infty} n z^{n+1}$, $z \in \Delta$. From the definition of the coefficient domination $\ll$ ([13], I, p. 82) we conclude that

$$z^2 g'(z) \ll \psi(z), \quad z \in \Delta. \tag{12}$$

From (10), (11) and (12) we obtain $\sum_{n=1}^{\infty} n b_n z^{n+1} \ll \frac{z^2}{1-z}\psi(z)$. Consequently, the estimates (6) and (7) hold. Hence and from the above inequalities for coefficients of functions of the class Q'_r we have (8) and (9).

It is easy to check that the functions $f_1^*, f_2^*, f_3^*, f_4^*$ satisfy the assumptions of theorem 3. Thus estimates (6) are sharp. $\square$

Remark 3. *The class Q'_H includes functions, for which the above estimates do not hold, e.g. functions of the form (3). They have the coefficient b_1, which does not satisfy the first estimate (6). It is known that in this case the condition (2) does not hold.*

Remark 4. *Let us observe that substituting (5) for the stronger condition*

$$|G'(z)| < |zF'(z)|, \;\; z \in P \tag{13}$$

and applying the same method (the subordination $\prec$ and the coefficient domination $\ll$), we do not obtain estimates better then (7)-(9).

Let us note that the functions of the form (3) from example 1 do not satisfy the condition (5), and consequently, (13) does not hold for them. For the functions f_1^*, f_2^* from theorem 3 the inequality (5) holds, but the conditon $|G'(z)| < |z^2F'(z)|$, $z \in \Delta$, is false. However, the last inequality holds for the functions $f_3^*, f_4^* \in Q'_H$ and, in consequence, (5) and (13) are satisfied. The question, how much the mentioned inequalities restrict the class Q'_H is open.

The condition (5) is equivalent to locall univalence of the mapping f, of course ([14]; [15], p. 20).

Remark 5. *The inequalities (6) we can obtain immediately from the definition of the function ω, considering functions F, G of the form (i), such that the condition (5) holds in P, not necessarily typically-real, and comparing coefficients of the appopriate series.*

Indeed, taking $\omega(z) = \sum_{n=2}^{\infty} q_n z^n$ we get

$$\left(1 + \sum_{n=1}^{\infty} na_n z^{n+1}\right)\left(\sum_{n=2}^{\infty} q_n z^n\right) = \sum_{n=1}^{\infty} nb_n z^{n+1}, \; z \in \Delta.$$

Therefore, comparing the corresponding coefficients, we have

$$q_2 = 1 \cdot b_1, \quad q_3 = 2 \cdot b_2, \tag{14}$$

$$nb_n = q_{n+1} + q_{n-1} \cdot 1 \cdot a_1 + \ldots + q_2 \cdot (n-2) \cdot a_{n-2}, \; n = 3, 4, \ldots. \tag{15}$$

The function ω is a Schwarz function, thus $|q_n| \leq 1$, $n = 2, 3, \ldots$, ([13], I, p.87), and finally, in view of (14), we obtain the estimates (6).

It appears that if the assumptions of theorem 3 hold, then applaying the formulas (15) and the mentioned coefficient estimates for functions of the class Q'_r, we get results worse, then (7)-(9).

§ 4. The Hadamard product

Let α denote a fixed real number, $\alpha \neq -1, -\frac{1}{2}, -\frac{1}{3}, \ldots$. Let us consider the functions

$$k_\alpha(z) := z + \frac{1}{1+\alpha} z^2 + \ldots + \frac{1}{1+(n-1)\alpha} z^n + \ldots,\ z \in \Delta, \tag{16}$$

and

$$h_\alpha(z) := \frac{1}{z} + \frac{1}{1+\alpha} + \frac{1}{1+2\alpha} z + \ldots + \frac{1}{1+(n+1)\alpha} z^n + \ldots,\ z \in P. \tag{17}$$

It is known ([16]) that the function (16) is the solution of the equation

$$\alpha z k'_\alpha(z) + (1-\alpha) k_\alpha(z) = \frac{z}{1-z},\ z \in \Delta, \tag{18}$$

satisfying the conditions $k_\alpha(0) = k'_\alpha(0) - 1 = 0$ and for $\alpha > 0$ it can be expressed in the form

$$k_\alpha(z) = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-2} \frac{zt}{1-zt} dt,\ z \in \Delta \tag{19}$$

Moreover, for $\alpha \geq 0$ it is a typically-real function in Δ.

The mapping (17) (see [7]) is a solution of the equation

$$\alpha z h'_\alpha(z) + (1+\alpha) h_\alpha(z) = \frac{1}{z(1-z)},\ z \in P, \tag{20}$$

and for $\alpha > 0$ it is given by the formula

$$h_\alpha(z) = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}} \left(\frac{1}{zt} + \frac{1}{1-zt} \right) dt,\ z \in P. \tag{21}$$

Definition 2. *The Hadamard product $(f^1 * f^2)$, of two harmonic functions of the form*

$$f^1(z) = \frac{a_{-1}^{(1)}}{z} + \sum_{n=0}^{\infty} a_n^{(1)} z^n + \overline{\sum_{n=0}^{\infty} b_n^{(1)} z^n},\ f^2(z) = \frac{a_{-1}^{(2)}}{z} + \sum_{n=0}^{\infty} a_n^{(2)} z^n + \overline{\sum_{n=0}^{\infty} b_n^{(2)} z^n}$$

is called the function

$$(f^1 * f^2)(z) = \frac{a_{-1}^{(1)} a_{-1}^{(2)}}{z} + \sum_{n=0}^{\infty} a_n^{(1)} a_n^{(2)} z^n + \overline{\sum_{n=0}^{\infty} b_n^{(1)} b_n^{(2)} z^n}, \quad z \in P$$

Is worth mentioning that the above definition came into being on the basis of a classical idea of Jacques Salomon Hadamard (1865-1963), concerning the convolution of power series ([17]). Unfortunely, so few articles include a citation of the original Hadamard's paper on the convolutions.

In [18] (p. 248) it is given that J. S. Hadamard was in contact with Polish mathematicians (among others with Wacław Sierpiński) and was a member of the Polish Academy of Sciences. Professor Zygmunt Charzyński many times expressed hopes that we would wait with Hadamard to celebrate the centenary of his birthday.

We hope that this historical note compensates a bit for "our faults".

Definition 3. *Let* $Q^0_H := \{f \in Q'_H : b_0 = 0\}$.

Definition 4. $Q^0_H(\alpha) := \{u : u = f * (h_\alpha + \overline{k_\alpha}) : f \in Q^0_H\}$, $\alpha \in R$, $\alpha \neq -1, -\frac{1}{2}, -\frac{1}{3}, \ldots$, *where* h_α, k_α, *are expressed by (16) and (17), respectively.*

According to the forms of the functions k_α and h_α we assume in the definition 3 that $b_0 = 0$ and we leave a coefficient a_0, as in the form (i) of the function f.

Remark 6. *We have* $k_0(z) = \frac{z}{1-z}$, $z \in \Delta$, $h_0(z) = \frac{1}{z} + \frac{1}{1-z}$, $z \in P$, *thus* $f * (h_0 + \overline{k_0}) = f$, *hence* $Q^0_H(0) = Q^0_H$. *The function* k_0 *is univalent in* Δ. *However,* h_0 *is neither typically-real nor univalent in* P.

Let us observe that for any admissible α, the functions f_1^*, f_2^* (from th.3) belong to the class $Q^0_H(\alpha)$. Hence $Q^0_H \neq \emptyset$ and $Q^0_H \cap Q^0_H(\alpha) \neq \emptyset$. Moreover, in view of (16) and (17), assuming that $k_{+\infty}(z) = k_{-\infty}(z) = z$ $h_{+\infty}(z) = h_{-\infty}(z) = \frac{1}{z}$ we have $Q^0_H(+\infty) = Q^0_H(-\infty) = \{-\frac{1}{z} + \overline{b_1 z}\}$, where the coresponding function $f \in Q^0_H$ is of the form $f(z) = F(z) + \overline{G(z)}, F(z) = -\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$, $G(z) = \sum_{n=1}^{\infty} b_n z^n$, $z \in P$. If $b_1 \neq \overline{b_1}$, then $u = f * (h_\alpha + \overline{k_\alpha}), \alpha = -\infty, +\infty$ belong to $Q^0_H(+\infty) = Q^0_H(-\infty)$ and there are not typically-real.

Directly from definition 4 and theorem 3 we obtain

Corolary 1. *Let* $u \in Q^0_H(\alpha)$, $\alpha > 0$, *and let the corresponding function* $f \in Q^0_H$ *satisfy the assumption of theorem 3. If* u *is of the form* $u(z) = -\frac{1}{z} + \sum_{n=0}^{\infty} \hat{a}_n z^n + \overline{\sum_{n=1}^{\infty} \hat{b}_n z^n}$, *then the estimates*

$$|\hat{b}_1| \leq 1, \ |\hat{b}_2| \leq \frac{1}{2(1+\alpha)},$$

$$|\hat{b}_n| \leq \frac{2(n-1)(n-2)}{n(1+(n-1)\alpha)},\ n \geq 3,$$
$$|\hat{a}_1| \leq \frac{4}{1+2\alpha},\ |\hat{a}_2| \leq \frac{9}{2(1+3\alpha)},$$
$$|\hat{a}_n| \leq \frac{2(n^2-n+2)}{n(1+(n-1)\alpha)},\ n \geq 3$$

hold. Extremal functions for $|\hat{b}_1|$ *are e.g.* f_1^*, f_2^*. *The equality sign for* $|\hat{b}_2|$ *occurs e.g. for* $f_3^* * (h_\alpha + \overline{k_\alpha}) = -\frac{1}{z} + \frac{1}{2(1+\alpha)}\overline{z^2}$,
$f_4^* * (h_\alpha + \overline{k_\alpha}) = -\frac{1}{z} - \frac{1}{2(1+\alpha)}\overline{z^2}, \alpha > 0$.

Moreover, we have

THEOREM 1. *If* $u \in Q_H^0(\alpha)$, $\alpha \in R$, $\alpha \neq -1, -\frac{1}{2}, -\frac{1}{3}, \ldots$, $u = s + \overline{r}$, *then there exists* $f = F + \overline{G} \in Q_H^0$, *such that the system*

$$\begin{cases} \alpha z s'(z) + (1+\alpha)s(z) = F(z); \\ \alpha z r'(z) + (1-\alpha)r(z) = G(z); \quad z \in P. \end{cases} \tag{22}$$

holds. Conversely, for any $f = F + \overline{G} \in Q_H^0$, *the solution* $u = s + \overline{r}$, *where* $s(z) = F(z) * h_\alpha(z)$, $r(z) = G(z) * k_\alpha(z)$, $z \in P$, *of the system (22) belongs to the class* $Q_H^0(\alpha)$.

The proof follows by definitions 2, 4, the formulas (18),(20) and remark 4.

From theorem 1 and in view of (19), (21) it follows

COROLARY 2. *If* $u \in Q_H^0(\alpha)$, $\alpha > 0$, *then there exists* $f = F + \overline{G} \in Q_H^0$, *such that*

$$u(z) = \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}}\left(F(zt) + t^{-2}\overline{G(zt)}\right)dt,\ z \in P. \tag{23}$$

Conversely, if $f \in Q_H^0$, $f = F + \overline{G}$, *then* u *of the form (23) belongs to the class* $Q_H^0(\alpha)$.

We do not know, if or when for finite α the functions of the form (23) are typically-real in P.

The main results of this paper were presented during the XI-th Environmental Mathematically-Informatical Conference in Chełm (30.06.2005-03.07.2005)([19]).

References

[1] Wigner E. P. *On a class of analytic functions from the quantum theory of collisions* / E. P. Wigner // Ann. of Math. V. 53(1) (1951). P. 36–67.

[2] Goodman A. W. *Functions typically-real and meromorphic in the unit circle* / A. W. Goodman // Trans. Amer, Math. Soc. V. 81(1). (1956). P. 92–105.

[3] Walczak S. *Typical real functions in the exterior of unit circle* / S. Walczak // Comment. Math. V. 17. (1974). P. 513–531.

[4] Gelfer S. A. *On coefficients of typically-real functions* / S. A. Gelfer // DAN SSSR. V. 94(3). (1954). P. 373–376 (in Russian).

[5] Zamorski J. *On typically-real, meromorphic functions* / J. Zamorski // Annuals of PTM, Ser. I: Mathematical Papers. V. 6. (1961). P. 41–49 (in Polish).

[6] Remizowa M. P. *Extremal problems in the class of typically-real functions* / M. P. Remizowa // Izv. Vyss. Uc. Zaved. Mat. V. 32(1) (1963). P. 135–144 (in Russian).

[7] Jakubowski Z. J. *On typically-real functions meromorphic in the unit disc* / Z. J. Jakubowski, K. Skalska // Bull. Soc. Sci. Letters de Łódź. V. 40. 1. (71) (1990). P. 5–22.

[8] Clunie J. *Harmonic univalent mappings* / J. Clunie, T. Sheil-Small // Ann. Acad. Sci. Fenn., Ser. A. I. Math. V. 9. (1984). P. 3–25.

[9] Hengartner W. *Univalent harmonic functions* / W. Hengartner, G. Schober // Trans. Amer.Math. Soc. V. 299. (1987). P. 1–31.

[10] Jakubowski Z. J. *On complex harmonic typically-real functions* / Z. J. Jakubowski, W. Majchrzak, K. Skalska // J. Ramanujan Math. Soc. V. 9(1). (1994). P. 35–48.

[11] Rogosiński W. W. *Über positive harmonische Entwicklungen und typischreele Potenzreihen* / W. W. Rogosiński // Mat. Zeit. V. 35. (1932). P. 93–121.

[12] Duren P. *Univalent functions* / P. Duren. New York; Berlin; Heidelberg; Tokyo: Springer-Verlag, 1983.

[13] Goodman A. W. *Univalent functions* / A. W. Goodman. Mariner Publishing Company. Inc., 1983.

[14] Lewy H. *On the non-vashing of Jacobian in certain one-to-one mappings* / H. Lewy // Bull. Amer. Math. Soc. V. 42. (1936). P. 689–692.

[15] Duren P. *Harmonic mappings in the plane* / P. Duren // Cambridge Tracts in Math. 2004. V. 156.

[16] Jakubowski Z. J. *On some classes of analytic functions* / Z. J. Jakubowski // Complex Anal. and Appl. Sofia. (1989). P. 241–249.

[17] Hadamard J. S. *Theoreme sur les series entieres* / J. S. Hadamard // Acta Mathematica. V. 22. (1898). P. 55–65.

[18] *The Affiliate of Great Mathematicians.* Warsaw, 1989 (in Polish).

[19] Jakubowski Z. J. *On complex harmonic typically-real functions with a pole at the point zero* / Z. J. Jakubowski, A. Sibelska // Conference papers of the XI-th Environmental Mathematically-Informatical Conference. Chełm, 2005. P. 32–33 (in Polish).

Chair of Nonlinear Analysis
Faculty of Mathematics
University of Łódź
ul. S. Banacha 22, 90-238 Łódź, Poland
E-mail: zjakub@math.uni.lodz.pl
koszeka@math.uni.lodz.pl