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ON THE STRUCTURE OF D_W-APPELL DUNKL CLASSICAL ORTHOGONAL POLYNOMIALS

Abstract. In this paper, we investigate a class of classical orthogonal polynomials associated with the D_w-Dunkl operator. This class is defined via a lowering operator that combines the discrete translation operator with a reflection-invariant component. We characterize these polynomials showing that the only possible $T_{\theta,w}$-Appell classical sequences are of class one. The study concludes by explicitly determining the recurrence relations and structure formulas satisfied by such polynomials.

Key words: *orthogonal polynomials, Appell orthogonal polynomials, D_w-Dunkl operator*

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1. Introduction. Orthogonal polynomials play a fundamental role in modern analysis, approximation theory, and mathematical physics. Classical families such as Hermite, Laguerre, and Jacobi polynomials arise naturally in a wide range of contexts, including spectral methods for differential equations, probabilistic models, and quantum mechanics. The notion of *classicality*, originally introduced through the stability of orthogonality under differentiation, provides a unifying framework for these families. Extending this concept beyond the differential setting to discrete, q-analogue, and Dunkl-type operators considerably broadens its scope and raises new theoretical challenges.

An orthogonal polynomial sequence (OPS) $\{A_n\}_{n\geqslant 0}$ is called *D-classical* if the derived sequence $\{A'_n\}_{n\geqslant 0}$ remains orthogonal (see [9], Hahn, 1935). This characterization laid the foundation for subsequent generalizations involving alternative lowering operators.

In this direction, Hahn established parallel results in [10] for OPSs whose images under the forward difference operator ΔA_n or the q-difference operator $H_q A_n$ (for $n \geqslant 1$) also form orthogonal sequences. The operator H_q is defined by

$$H_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}, \qquad q \neq 1.$$

These extensions illustrate the flexibility of the concept of orthogonality: while differential operators model continuous phenomena, difference and q-difference operators naturally describe discrete and quantum dynamics. This viewpoint creates a bridge between orthogonal polynomials, combinatorics, and the theory of q-special functions, and becomes essential in contexts where classical differential models are no longer adequate.

More generally, let $\mathcal{O}$ be a lowering operator acting on $\mathcal{P}$, the space of polynomials in one variable. An OPS $\{A_n\}_{n\geqslant 0}$ is said to be $\mathcal{O}$-*classical* if there exists an OPS $\{Q_n\}_{n\geqslant 0}$, such that

$$\mathcal{O} A_n = Q_{n-1}, \qquad n \geqslant 1.$$

Formulating the theory in terms of a general operator $\mathcal{O}$ leads to a unified framework encompassing classical, semiclassical, q-analogue, and Dunkl-type orthogonal polynomials. In this setting, the structural properties of an OPS are determined directly by the algebraic and analytic features of the associated operator.

The theory of $\mathcal{O}$-classical OPSs has been extensively studied; see, for instance, [1], [4], [11], [12]. In particular, [4] introduced the notion of Dunkl-classical OPSs via the operator

$$T_\mu = D + 2\mu H_{-1}, \qquad \mu > -1/2,$$

and showed that the corresponding solutions are precisely the class-one D-classical OPSs [2], namely the generalized Hermite and generalized Gegenbauer polynomials. By combining differential operators with reflection symmetries, the Dunkl framework reveals how operator-induced symmetries govern both the algebraic structure and analytical behavior of orthogonal polynomials, thus providing a natural link between harmonic analysis and polynomial theory.

More recently, [5] proposed the q-Dunkl operator

$$T_{\mu,q} = H_q + \mu H_{-1},$$

and proved that, up to a dilation, the q^2-generalized Hermite polynomials are the only symmetric $T_{\mu,q}$-Appell classical OPSs. This result advances the broader problem of characterizing D_w-Appell Dunkl-classical (or, equivalently, Appell $T_{\theta,w}$-classical) orthogonal polynomial sequences. The operator $T_{\theta,w}$ is defined as:

$$T_{\theta,w}f(x) = D_\omega f(x) + \theta H_{-1}f(x), \ \ f \in \mathcal{P},$$

with

$$D_\omega f(x) = \frac{f(x+\omega) - f(x)}{\omega}.$$

Overall, this line of research reflects a natural progression in the theory of orthogonal polynomials: from differentiation-based classicality to discrete and q-difference analogues, and finally to Dunkl-type operators. Each extension not only deepens the theoretical understanding but also enriches the range of applications, particularly in q-deformed quantum mechanics, special function theory, and combinatorial analysis.

In this work we address the problem for the mixed Dunkl–difference operator acting on $\mathcal{P}$. We provide a complete classification of $T_{\theta,w}$-Appell classical monic orthogonal polynomial sequences, showing that every such sequence is necessarily D_{-w}-semiclassical of class one and that no $T_{\theta,w}$-Appell classical sequences of higher class can occur. We further establish that the associated canonical linear functional u_0 satisfies a Pearson-type functional equation of the form

$$D_{-w}\big((t+\tfrac{w}{2})\,u_0\big) + (a_2t^2 + a_1t + a_0)\,u_0 = 0, \qquad a_2 \neq 0,$$

and we express the coefficients a_0, a_1, a_2 explicitly through the recurrence parameters (β_n) and (γ_n). In addition, we derive explicit initial conditions and recursive relations for the sequences (β_n) and (γ_n), together with structural identities linking the operator $T_{\theta,w}$, the three-term recurrence, and the dual sequence $(u_n)_{n\geqslant 0}$, thereby clarifying the precise influence of the Dunkl parameter θ and the step w on the orthogonal properties. These findings extend and unify earlier characterizations of Dunkl-classical and q-Dunkl-classical orthogonal polynomials, positioning the D_w-Appell Dunkl framework naturally within the broader semiclassical theory.

Our paper is structured as follows: important preliminaries are given in Section 2, the proof of the main result and complete classification of $T_{\theta,w}$-Appell classical OPS are given in Section 3. Lastly, all important results and main contributions are summarized in the conclusion.

2. Preliminaries. Let $\mathcal{P}$ denote the vector space consisting of all polynomials in one variable with complex coefficients. Its algebraic dual space, containing all linear forms on $\mathcal{P}$, is denoted by $\mathcal{P}'$. For any linear functional $u \in \mathcal{P}'$ and polynomial $f \in \mathcal{P}$, the duality pairing is expressed as $\langle u, f\rangle$. The moments of the functional u are defined through evaluation on monomials, yielding the sequence:

$$(u)_n := \langle u, t^n\rangle, \quad n \in \mathbb{N}_0.$$

A linear form u is called *symmetric* if all its odd moments vanish, i.e., $(u)_{2n+1} = 0$ for all $n \geqslant 0$.

The motivation for considering symmetric functionals is strongly related to the structural simplifications they induce in the analysis of orthogonal polynomial systems. Indeed, when odd moments vanish, the recurrence coefficients and the related algebraic relations often assume simplified closed forms, making them especially suitable for studying classical and semiclassical families.

We now define several fundamental operations on $\mathcal{P}'$. For any form u, polynomial ψ, complex numbers $a \neq 0$ and b, and parameters $q \neq 1$, $w \neq 0$, the following forms are defined through duality: the derivative $Du = u'$, the product ψu, the scaling $h_a u$, the q-difference $H_q u$, and the weighted derivative $D_w u$ (see, for example, [1], [11], [13]),

$$\begin{gathered}\langle u', \phi\rangle := -\langle u, \phi'\rangle, \quad \langle \psi u, \phi\rangle := \langle u, \psi\phi\rangle, \ \phi \in \mathcal{P}, \\ \langle h_a u, \phi\rangle := \langle u, h_a\phi\rangle = \langle u, \phi(at)\rangle, \quad \langle \tau_b u, \phi\rangle := \langle u, \tau_{-b}\phi\rangle = \langle u, \phi(t-b)\rangle, \\ \langle H_q u, \phi\rangle := -\langle u, H_q\phi\rangle, \quad \langle D_w u, \phi\rangle := -\langle u, D_{-w}\phi\rangle,\end{gathered}$$

where

$$(H_q\phi)(t) = \frac{\phi(qt) - \phi(t)}{(q-1)t} \quad \text{and} \quad (D_w\phi)(t) = \frac{\phi(t+w) - \phi(t)}{w}.$$

Observe that these operators represent a variety of analytic transformations: the H_q operator generalizes differentiation to the q-calculus framework, while the D_w operator serves as a finite-difference analogue of the derivative. Their dual forms allow us to manipulate polynomial functionals consistently, which is fundamental to defining semiclassical orthogonal polynomials.

It is obvious that [1], [11]

$$H_q(\phi u) = (h_{q^{-1}}\phi)H_q u + q^{-1}(H_{q^{-1}}\phi)u, \ \phi \in \mathcal{P}, \ u \in \mathcal{P}'. \tag{1}$$

$$D_{-w}(\phi u) = (\tau_w \phi)D_{-w}u + (D_{-w}\phi)u, \ \phi \in \mathcal{P}, \ u \in \mathcal{P}'. \tag{2}$$

A linear functional u is *normalized* when its zeroth moment satisfies $(u)_0 = 1$.

The following lemma establishes an equivalence between two properties of a linear functional u acting on polynomials and its relationship with a polynomial sequence $\{A_n\}$.

Lemma 1. [13], [14] *Consider an arbitrary linear functional u belonging to the dual space $\mathcal{P}'$ of polynomials, and let m be any fixed positive integer; the following statements are equivalent:*

1) ***Orthogonality condition**: The functional u pairs non-zero with A_{m-1} ($\langle u, A_{m-1}\rangle \neq 0$) but vanishes on all higher-degree polynomials in the sequence ($\langle u, A_n\rangle = 0$ for all $n \geqslant m$).*
2) ***Basis representation**: The functional u admits a representation as a finite linear combination of the functionals $\{u_\nu\}$, which constitute the dual sequence of $\{A_n\}_{n\geqslant 0}$. More precisely, it can be written in the form $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$, where each coefficient λ_ν is a uniquely determined complex number, and the last coefficient λ_{m-1} is non-zero, i. e.,*

$$\exists \lambda_\nu \in \mathbb{C}, \quad 0 \leqslant \nu \leqslant m-1, \quad \lambda_{m-1} \neq 0 \quad \textit{such that} \ \ u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$$

This equivalence means that the vanishing of u on higher-degree polynomials is equivalent to u being spanned by the first m dual functionals, with the representation necessarily including the $(m-1)$-th functional. The non-vanishing condition $\lambda_{m-1} \neq 0$ corresponds directly to $\langle u, A_{m-1}\rangle \neq 0$, ensuring the representation does not terminate earlier than $m-1$.

A form u is said to be *regular* when there exists an associated polynomial sequence $\{A_n\}_{n\geqslant 0}$ satisfying the orthogonality condition:

$$\langle u, A_n A_m\rangle = r_n \delta_{n,m}, \quad \text{for all } n, m \geqslant 0,$$

where each r_n is non-zero. Such a sequence $\{A_n\}$ is called a monic orthogonal polynomial sequence (MOPS) with respect to u.

Note that u admits a representation $u = (u)_0 u_0$ with $(u)_0 \neq 0$. An important property of regular forms is that if a polynomial F satisfies $Fu = 0$, then F must be identically zero [15].

The following proposition further investigates the relation between MOPS and TTRR.

Proposition 1. [13], [14]. *Consider a monic polynomial sequence $\{A_n\}_{n\geqslant 0}$ where each polynomial has degree exactly n, along with its dual sequence $\{u_n\}_{n\geqslant 0}$ of linear functionals. The following conditions are equivalent:*

(i) *The polynomial sequence $\{A_n\}$ is orthogonal relative to the initial functional u_0, i.e., $\langle u_0, A_m A_n\rangle = 0$ for $m \neq n$ with nonzero norms.*

(ii) *Each dual functional u_n is explicitly determined by the initial functional u_0 through the scaling relation:*

$$u_n = \langle u_0, A_n^2\rangle^{-1} A_n u_0. \tag{3}$$

(iii) *The sequence $\{A_n\}$ satisfies a three-term recurrence relation (TTRR) of the form:*

$$\begin{cases} A_0(t) = 1, \\ A_1(t) = t - \beta_0, \\ A_{n+2}(t) = (t - \beta_{n+1})A_{n+1}(t) - \gamma_{n+1}A_n(t), \quad n \geqslant 0, \end{cases} \tag{4}$$

with coefficients β_n and γ_{n+1} defined by the moment relations:

$$\beta_n = \langle u_0, tA_n^2\rangle\langle u_0, A_n^2\rangle^{-1}; \gamma_{n+1} = \langle u_0, A_{n+1}^2\rangle\langle u_0, A_n^2\rangle^{-1} \neq 0, n \in \mathbb{N}_0. \tag{5}$$

This proposition shows that if the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is orthogonal with respect to u_0, then the form u_n can be written, via the dual basis expansion, as a normalized rescaling of the composition of A_n and u_0, ensuring the duality condition holds. To derive orthogonality from TTRR, we apply mathematical induction on the recurrence to prove pairwise orthogonality, where the base case holds by β_0's definition and the inductive step uses the recurrence structure with $\gamma_{n+1} \neq 0$ ensuring non-degeneracy.

Consider a MOPS $\{A_n\}_{n\geqslant 0}$ that is orthogonal with respect to a regular functional u_0. When we apply the scaling transformation $\widetilde{A}_n(t) = a^{-n}A_n(at)$ for some nonzero constant a, the resulting sequence

$\{\widetilde{A}_n\}_{n\geqslant 0}$ also constitutes a MOPS. This scaled sequence is orthogonal relative to the modified functional $\widetilde{u}_0 = h_{a^{-1}} u_0$ and follows the recurrence relations [14]:

$$\widetilde{A}_0(t) = 1, \quad \widetilde{A}_1(t) = t - \widetilde{\beta}_0,$$

and for $n \geqslant 0$,

$$\widetilde{A}_{n+2}(t) = (t - \widetilde{\beta}_{n+1})\widetilde{A}_{n+1}(t) - \widetilde{\gamma}_{n+1}\widetilde{A}_n(t).$$

The new recurrence coefficients are scaled versions of the original parameters, specifically $\widetilde{\beta}_n = a^{-1}\beta_n$ and $\widetilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

Recall that a regular form u is said to be D_{-w}-semiclassical if there exist polynomials Φ (monic) and Ψ with $\deg \Phi = d_1 \geqslant 0$ and $\deg \Psi = d_2 \geqslant 1$, satisfying the Pearson-type distributional equation:

$$D_{-w}(\Phi u) + \Psi u = 0. \tag{6}$$

The orthogonal polynomial sequence $\{A_n\}_{n\geqslant 0}$ associated with such a form is called D_w-semiclassical [1].

Under dilation transformations, the D_{-w}-semiclassical property is preserved [1]. If the form u satisfies (6), then the transformed form $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ satisfies the modified Pearson equation:

$$D_{-wa^{-1}}\big(\tilde{\phi}\tilde{u}\big) + \tilde{\psi}\tilde{u} = 0,$$

with $\tilde{\phi}(t) = a^{-t}\phi(at+b)$ and $\tilde{\psi}(t) = a^{1-t}\psi(at+b)$.

The class s of a D_{-w}-semiclassical form u is defined as $s = \max(d_1 - 2, d_2 - 1) \geqslant 0$, where this classification holds if and only if [1]:

$$\prod_{c\in\mathcal{Z}_\Phi} \{|\Psi(c+w) + (\theta_c\Phi)(c+w)| + |\langle u, \theta_{c+w}(\psi + \theta_c\phi)\rangle|\} > 0, \tag{7}$$

with $\mathcal{Z}_\Phi$ denoting the set of zeros of Φ and θ_c is the divided–difference operator at the point c $\Big($i. e., $\theta_c f(x) = \dfrac{f(x) - f(c)}{x - c}$, $f \in \mathcal{P}\Big)$. Forms satisfying $s = 0$ are conventionally referred to as D_{-w}-classical [1]. The next lemma is essential for the arguments that follow [2].

Lemma 2. *Consider an arbitrary MOPS $\{A_n\}_{n\geqslant 0}$ and let $M(t,n)$, $N(t,n)$ be polynomials in t for each fixed $n \geqslant 0$, satisfying $M(t,n)A_{n+1}(t) = N(t,n)A_n(t)$.*

If $\deg N(t,n) \leqslant n$ *for some index* n*, then both polynomials necessarily vanish, i. e.,*

$$M(t,n) = 0 \quad \textit{and} \quad N(t,n) = 0.$$

Let $T_{\theta,w}$ be a linear operator defined on the space $\mathcal{P}$ that acts on any polynomial ϕ as follows:

$$T_{\theta,w}(\phi) = D_w(\phi) + \theta H_{-1}(\phi), \quad \theta \in \mathbb{C}.$$

When $\theta = 0$, this operator simplifies to D_w (additional details appear in [11]). As w approaches zero, we obtain the limit:

$$\lim_{w\to 0} T_{\theta,w}\phi(t) = \phi'(t) + \theta\frac{\phi(t)-\phi(-t)}{2t} = T_\theta\phi(t),$$

where T_θ denotes the Dunkl operator introduced in [8] (further discussed in [4]).

This limiting case highlights an integrating principle; the finite difference operator with reflection symmetry naturally converges to the Dunkl differential reflection operator. In this sense, the $T_{\theta,w}$ operator can be regarded as a discrete deformation of Dunkl operators, connecting continuous and discrete orthogonal polynomial structures.

For the monomial sequence $\{t^{n+1}\}_{n\geqslant 0}$, the operator acts as:

$$T_{\theta,w}(t^{n+1}) = \Big[n+1+\theta\Big(\frac{1-(-1)^{n+1}}{2}\Big)\Big]t^n + \frac{1}{w}\sum_{k=0}^{n-1}\binom{n+1}{k}t^k w^{n+1-k}, n \geqslant 1.$$

This expression clearly shows that the operator $T_{\theta,w}$ reduces polynomial degrees by one, that is, $T_{\theta,w}$ is a *lowering operator.*

The additional parity-dependent term induced by θ demonstrates how reflection symmetries interact with finite differences to yield non-trivial modifications of the standard Appell sequences.

The transpose of $T_{\theta,w}$ satisfies the adjoint relation:

$$(T_{\theta,w})^t = -T_{\theta,-w}. \tag{8}$$

This adjoint relation is critical because it allows us to transfer operator actions between the polynomial basis and its dual functional system. Consequently, orthogonality conditions and recurrence properties can be smoothly translated across primal and dual frameworks, ensuring the structural consistency of the theory.

For any MPS $\{A_n\}_{n\geqslant 0}$, there exists a monic polynomial $\widehat{A}_n$ of degree n and a nonzero scalar θ_{n+1}, such that

$$T_{\theta,w}A_{n+1}(t) = \big(D_w A_{n+1}\big)(t) + \theta\big(H_{-1}A_{n+1}\big)(t) = \theta_{n+1}\,\widehat{A}_n(t), \qquad n \geqslant 0. \tag{9}$$

The constructed sequence $\{\widehat{A}_n\}$ is called the sequence derived from $\{A_n\}$ through the operator $T_{\theta,w}$. The scaling coefficient θ_{n+1}, which acts as a normalization factor, is defined as follows:

$$\theta_{n+1} := n + 1 + \theta\left(\frac{1-(-1)^{n+1}}{2}\right), \quad n \geqslant 0.$$

The alternating term $\dfrac{1-(-1)^{n+1}}{2}$ effectively acts as a parity-dependent switch, taking value 1 (when $n+1$ is odd, i. e., n is even) and 0 (when $n+1$ is even, i. e., n is odd).

Definition 1. *A MOPS $\{A_n\}_{n\geqslant 0}$ associated with the linear functional u_0 is said to be $T_{\theta,w}$-*Appell classical *when the derived sequence $\{T_{\theta,w}A_{n+1}\}_{n\geqslant 0}$ maintains orthogonality, where $T_{\theta,w}A_{n+1} = \theta_{n+1}A_n$ for all $n \geqslant 0$. When this condition holds, we correspondingly describe u_0 as a $T_{\theta,w}$-*Appell classical linear functional.

Consider now the dual basis $\{u_n\}_{n\geqslant 0}$ in the dual space $\mathcal{P}'$ that corresponds to the primal sequence $\{A_n\}_{n\geqslant 0}$. By Lemma 1 and (8), we obtain the important relation:

$$T_{\theta,-w}(u_n) = -\theta_{n+1}u_{n+1}, \quad n \geqslant 0. \tag{10}$$

This adjoint action, which maps each dual functional to a scaled version of the next functional, will serve as our cornerstone result for subsequent developments.

The MOPS $\{A_n\}_{n\geqslant 0}$ satisfies the standard three-term recurrence relation:

$$\begin{cases} A_0(t) = 1, & A_1(t) = t - \beta_0, \\ A_{n+2}(t) = (t-\beta_{n+1})A_{n+1}(t) - \gamma_{n+1}A_n(t), & \gamma_{n+1} \neq 0, \quad n \geqslant 0, \end{cases} \tag{11}$$

where the transformed sequence $\{T_{\theta,w}A_n\}_{n\geqslant 0}$ defined in (9) maintains orthogonality.

Now, we establish an operator product rule for arbitrary polynomials $\phi, \psi \in \mathcal{P}$:

$$T_{\theta,w}(\phi\psi) = (h_{-1}\phi)\cdot(T_{\theta,w}\psi) + \psi\cdot(T_{\theta,w}\phi) + (\phi - (h_{-1}\phi) + w(D_w\phi))\cdot(D_w\psi). \tag{12}$$

This decomposition shows how the operator distributes across polynomial products, featuring distinct interactions between the finite difference component D_w and the reflection operator h_{-1}.

3. Complete Classification of $T_{\theta,w}$-Appell Classical Orthogonal Polynomials. This lemma introduces a fundamental structural identity that links the recurrence coefficients β_n and γ_n to the action of the underlying operator. The novelty lies in the operator-based approach, which allows these coefficients to be characterized through explicit algebraic relations rather than through ad hoc polynomial manipulations. The method relies on exploiting the compatibility between the recurrence relation and the operator action, leading to identities that hold for a family of orthogonal polynomials depending on free parameters. These parameters play a crucial role in controlling the deformation of the classical structure and enable the extension to broader classes of polynomial families.

Lemma 3. *For any $T_{\theta,w}$-Appell classical sequence $\{A_n\}_{n\geqslant 0}$, the following structural relations hold:*

$$\frac{2t+w}{2t}\theta(h_{-1}A_{n+1})(t) = \left[\theta_{n+2} - \frac{\gamma_{n+1}}{\gamma_n}\theta_n - 1 + \frac{w\theta}{2t}\right]A_{n+1}(t) + \left[\frac{\gamma_{n+1}}{\gamma_n}\theta_n(t-\beta_n) - (t-\beta_{n+1}+w)\theta_{n+1}\right]A_n(t),\ n \geqslant 1. \tag{13}$$

Additionally, the initial recurrence coefficients are constrained to satisfy:

$$\beta_0 = -\frac{w}{2},\quad \beta_1 = \frac{w}{2},\quad \beta_2 = \frac{3w}{2},\quad \text{and}\quad \gamma_2 = \frac{2}{1+\theta}\gamma_1. \tag{14}$$

Proof. The proof develops through several steps that use the recurrence structure and operator properties:

Starting from the three-term recurrence (11) satisfied by our monic orthogonal sequence:

$$A_{n+2}(t) = (t-\beta_{n+1})A_{n+1}(t) - \gamma_{n+1}A_n(t),\quad \gamma_{n+1} \neq 0,\quad n \geqslant 0. \tag{15}$$

Applying the $T_{\theta,w}$ operator to both sides of (15), using the product rule (12), and utilizing the Appell classical property, we get:

$$\theta_{n+2}A_{n+1}(t) = -\theta_{n+1}(t+\beta_{n+1})A_n(t) + (1+\theta)A_{n+1}(t)$$

$$+(2t+w)(D_wA_{n+1})(t) - \gamma_{n+1}\theta_n A_{n-1}(t). \qquad (16)$$

This equation clarifies that the factor $(2t+w)$ reflects the scaling induced by the finite difference operator D_w, while the coefficients θ_{n+1},θ_{n+2} track the Appell-type shift. The presence of $-\gamma_{n+1}\theta_n A_{n-1}(t)$ underlines the coupling between forward and backward indices, which is the defining property of classical orthogonal systems.

Substituting the definition $T_{\theta,w} = D_w + \theta H_{-1}$ into (16) and eliminating A_{n-1} using the recurrence relation produces

$$\begin{aligned}\theta_{n+2}A_{n+1}(t) \quad &= \theta_{n+1}(t-\beta_{n+1}+w)A_n(t) + (1+\theta)A_{n+1}(t)\\ &+\frac{2t+w}{2t}\theta(h_{-1}A_{n+1})(t) - \frac{2t+w}{2t}\theta A_{n+1}(t)\\ &-\frac{\gamma_{n+1}}{\gamma_n}\theta_n\big((t-\beta_n)A_n(t) - A_{n+1}(t)\big),\end{aligned}$$

which proves the identity (13).

In this framework, the operator H_{-1}, by construction, functions to transform A_{n+1} by coupling it with polynomials of a lower degree. The multiplicative factor $\dfrac{2t+w}{2t}$ is crucial for ensuring that the calculus structural properties of the operator remain sound and that no singularities arise in the recurrence relations.

Letting $n=1$ in (13), with explicit expressions:

$$\begin{aligned}A_1(t) &= t-\beta_0,\\ A_2(t) &= t^2 - (\beta_0+\beta_1)t + (\beta_0\beta_1 - \gamma_1),\end{aligned}$$

generates two compatibility equations through coefficient matching:

$$2(\theta+1)(\beta_0+\beta_1) = 2(\beta_0+\beta_2-w), \qquad (17)$$

$$2(\beta_0\beta_1-\gamma_1) + \gamma_2(1+\theta) + 2(w-\beta_2)\beta_0 = 0 \qquad (18)$$

which consolidates to

$$\theta\beta_0(\beta_0+\beta_1) = -\gamma_1 + \gamma_2\frac{1+\theta}{2}. \qquad (19)$$

This intermediate identity highlights the precise balance among β_0, β_1, γ_1, γ_2. Such balances reveal underlying symmetries in the polynomial system and explain why only specific parameter values are admissible.

Now, evaluating the defining relation (9) at $n = 1$ gives:

$$\theta_2(t - \beta_0) = 2t + w - (\theta + 1)(\beta_0 + \beta_1),$$

and through degree comparison, we have:

$$\beta_0 + \beta_1 = \frac{w + 2\beta_0}{\theta + 1}. \tag{20}$$

Similarly, letting $n = 2$ in (9) produces:

$$\theta_3 A_2(t) = (3 + \theta)t^2 + (3w - 2(\beta_0 + \beta_1 + \beta_2))t + w^2 - w(\beta_0 + \beta_1 + \beta_2) + \beta_0\beta_1 - \gamma_1 - \gamma_2 + \beta_2(\beta_0 + \beta_1) + 2\theta(\beta_0\beta_1 - \gamma_1 - \gamma_2 + \beta_2(\beta_0 + \beta_1)).$$

Then, degree comparison combined with (20) gives

$$\beta_2 = 2w + \beta_0, \tag{21}$$

$$\beta_0(w + 2\beta_0) = \frac{1}{2}(w + 2\beta_0)^2. \tag{22}$$

Equation (22) gives two cases:
Case 1: If $w + 2\beta_0 \neq 0$, then (22) forces $w = 0$. However, this contradicts the parameter assumption that $w \neq 0$ (as required by the operator definition).
Case 2: If $w + 2\beta_0 = 0$, we directly obtain $\beta_0 = -w/2$. Substitution into (20) yields $\beta_1 = w/2$, then (21) gives $\beta_2 = 3w/2$. Finally, (19) produces $\gamma_2 = \dfrac{2}{1+\theta}\gamma_1$, which completes the proof. □

The following proposition establishes two coupled recurrence relations that govern the behavior of the sequences (β_n) and (γ_n). These relations provide structural constraints linking consecutive terms and highlight the dependence on the parameters θ_n and w. In particular, equation (23) describes the evolution of β_n, while equation (24) determines the propagation of γ_n for $n \geqslant 2$.

Proposition 2. *The sequences $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_{n+1}\}_{n\geqslant 1}$ satisfy the system:*

$$\theta_{n+2}\beta_{n+2} = \big(1 + (-1)^n\theta\big)\beta_{n+1} + \theta_{n+1}\beta_n + (2n + 3)w, \; n \geqslant 0, \tag{23}$$

$$\left(\frac{\theta_n}{\theta_{n-1}} - \frac{\theta_{n+2}}{\theta_n}\right)\gamma_n = -\frac{2}{\theta_{n-1}}\gamma_{n-1} - \beta_n(w+\beta_n) - \beta_{n+1}(w-\beta_{n+1}), \; n \geqslant 2. \tag{24}$$

Proof. Using the operator relation (13), we process equation (15) through two consecutive operations, application of the dilation operator h_{-1}, and multiplication by the rational coefficient

$$\theta\frac{2t+w}{2t}.$$

This particular choice of operator and multiplicative factor is not arbitrary. It is chosen to align the recurrence with the fundamental symmetry of the problem and to balance the coefficients that appear in later steps.

This approach yields the following sequence of equalities:

$$(h_{-1}A_{n+2})(t) = -(t+\beta_{n+1})(h_{-1}A_{n+1})(t) - \gamma_{n+1}(h_{-1}A_n)(t), \quad n \geqslant 0.$$

$$\theta\frac{2t+w}{2t}(h_{-1}A_{n+2})(t) = -\theta\frac{2t+w}{2t}(t+\beta_{n+1})(h_{-1}A_{n+1})(t) - \gamma_{n+1}\theta\frac{2t+w}{2t}(h_{-1}A_n)(t),\ n \geqslant 0,$$

$$\begin{aligned}
&\left\{\theta_{n+3} - \frac{\gamma_{n+2}}{\gamma_{n+1}}\theta_{n+1} - 1 + \frac{w\theta}{2t}\right\} A_{n+2}(t) \\
&\quad + \left\{\frac{\gamma_{n+2}}{\gamma_{n+1}}\theta_{n+1}(t-\beta_{n+1}) - \theta_{n+2}(t-\beta_{n+2}+w)\right\} A_{n+1}(t) \\
&\quad = -(t+\beta_{n+1})\Big\{\Big(\theta_{n+2} - \frac{\gamma_{n+1}}{\gamma_n}\theta_n - 1 + \frac{w\theta}{2t}\Big)A_{n+1}(t) \\
&\qquad + \Big(\frac{\gamma_{n+1}}{\gamma_n}\theta_n(t-\beta_n) - \theta_{n+1}(t-\beta_{n+1}+w)\Big)A_n(t)\Big\} \\
&\qquad - \gamma_{n+1}\Big\{\Big(\theta_{n+1} - \frac{\gamma_n}{\gamma_{n-1}}\theta_{n-1} - 1 + \frac{w\theta}{2t}\Big)A_n(t) \\
&\qquad + \Big(\frac{\gamma_n}{\gamma_{n-1}}\theta_{n-1}(t-\beta_{n-1}) - \theta_n(t-\beta_n+w)\Big)A_{n-1}(t)\Big\}.
\end{aligned}$$

The scaling by $\dfrac{2t+w}{2t}$ introduces a rational correction that is essential for producing well-structured coefficients for the comparison of terms.

A second application of the recurrence relation (15) gives the identity

$$M(t,n)A_{n+1}(t) = N(t,n)A_n(t), \quad n \geqslant 2, \tag{25}$$

where

$$M(t,n) = (\theta_{n+1} - \frac{\gamma_{n+1}}{\gamma_{n-1}}\theta_{n-1})t - \theta_{n+3}\beta_{n+1} - \frac{\gamma_{n+1}}{\gamma_n}\theta_n\beta_{n+1} + (\beta_{n+2} + \beta_{n+1} - w)\theta_{n+2} + \frac{\gamma_{n+1}}{\gamma_{n-1}}\theta_{n-1}\beta_{n-1} + w\theta + \frac{\gamma_{n+1}}{\gamma_n}\theta_n(w - \beta_n),$$

and

$$N(t,n) = (\theta_{n+1} - \frac{\gamma_{n+1}}{\gamma_{n-1}}\theta_{n-1})t^2 + \Big(\frac{\gamma_{n+1}}{\gamma_n}\theta_n(w - \beta_n - \beta_{n+1}) + w\theta_{n+1} + \frac{\gamma_{n+1}}{\gamma_{n-1}}\theta_{n-1}(\beta_n + \beta_{n-1})\Big)t + \gamma_{n+1}\Big(2 - \frac{\gamma_{n+2}}{\gamma_{n+1}}\theta_{n+1} + \frac{\gamma_n}{\gamma_{n-1}}\theta_n\Big) + \frac{\gamma_{n+1}}{\gamma_n}\theta_n\beta_n\beta_{n+1} + \theta_{n+1}\beta_{n+1}(w - \beta_{n+1}) - \frac{\gamma_{n+1}}{\gamma_{n-1}}\theta_{n-1}\beta_n\beta_{n-1} - \frac{\gamma_{n+1}}{\gamma_n}\theta_n\beta_n(w - \beta_n).$$

Since (25) must hold for all t, Lemma 2 implies $M(t,n) = N(t,n) = 0$ for all $n \geqslant 2$.

Thus from $N(t,n) = 0$ we obtain the recurrence

$$\gamma_{n+1} = \frac{\theta_{n+1}}{\theta_{n-1}}\gamma_{n-1}, \qquad n \geqslant 2. \tag{26}$$

This decouples the γ–sequence from the β–sequence.

Next, the remaining part of $N(t,n) = 0$ gives the constraint

$$\frac{\gamma_{n+1}}{\gamma_n}\theta_n(w - \beta_n - \beta_{n+1}) + w\theta_{n+1} + \frac{\gamma_{n+1}}{\gamma_{n-1}}\theta_{n-1}(\beta_n + \beta_{n-1}) = 0. \tag{27}$$

Substituting (26) into (27) yields

$$\frac{\gamma_{n+1}}{\gamma_n}\theta_n\big(w - \beta_n - \beta_{n+1}\big) = -\theta_{n+1}\big(w + \beta_n + \beta_{n-1}\big). \tag{28}$$

Similarly, the remaining conditions from $M(t,n) = 0$ are simplified, using (26) and (28), to

$$\theta_{n+2}\beta_{n+2} = (\theta_{n+3} - \theta_{n+2})\beta_{n+1} + \theta_{n+1}\beta_n + (\theta_{n+1} + \theta_{n+2})w - w\theta, \ n \geqslant 0. \tag{29}$$

Using the explicit form

$$\theta_k = k + \theta\frac{1 - (-1)^k}{2},$$

we verify that

$$\theta_{n+3} - \theta_{n+2} = 1 + (-1)^n\theta, \qquad (\theta_{n+1} + \theta_{n+2})w - w\theta = (2n+3)w,$$

which rewrites (29) exactly as (23).

Finally, eliminating γ_{n+1} and γ_{n+2} from the constant term of $N(t,n)=0$ using (26) yields (24). This completes the proof. □

The relations obtained in Lemma 3 and Proposition 2 show that the sequences $\{\beta_n\}$ and $\{\gamma_n\}$ are completely determined by a finite set of initial parameters. In fact, the values of β_0 and β_1 are fixed by Lemma 3, and the recurrence (23) determines all subsequent β_n. Hence, the sequence $\{\beta_n\}$ depends only on the parameters w and θ.

Similarly, Lemma 3 fixes γ_2 in terms of γ_1, and the recurrence (24) determines all remaining γ_n. Therefore, once $\gamma_1 \neq 0$ is chosen, the entire sequence $\{\gamma_n\}$ is uniquely determined. Consequently, for fixed parameters (w,θ), a $T_{\theta,w}$-Appell classical OPS depends on a single free parameter γ_1.

Theorem 1. *Let $\{A_n\}_{n\geqslant 0}$ be a $T_{\theta,w}$-Appell classical orthogonal polynomial sequence. Then, by Lemma 3 and Proposition 2, its recurrence coefficients $\{\beta_n\}$ and $\{\gamma_n\}$ are uniquely determined by the parameters (w,θ) and a single free parameter $\gamma_1 \neq 0$. Moreover, $\{A_n\}$ is a D_{-w}-semiclassical orthogonal polynomial sequence of class one.*

Proof. Using equations (10) and (3), we obtain

$$T_{\theta,-w}(A_n u_0) = -\chi_n A_{n+1} u_0, \;\; n \geqslant 0. \tag{30}$$

where

$$\chi_n := \theta_{n+1}\frac{\langle u_0, A_n^2\rangle}{\langle u_0, A_{n+1}^2\rangle} = \theta_{n+1}\gamma_{n+1}^{-1}, \;\; n \geqslant 0.$$

By applying the operator definitions from (1) and (2), we derive the following relationship for all non-negative integers $n \geqslant 0$:

$$\begin{aligned}(\tau_w A_n)D_{-w}(u_0) + (D_{-w}A_n)u_0 + \theta(h_{-1}A_n)(H_{-1}u_0) - \theta(H_{-1}A_n)u_0 \\ = -\chi_n A_{n+1}u_0.\end{aligned} \tag{31}$$

Now, for $n = 0$ in (30), we obtain

$$T_{\theta,-w}(u_0) = -(1+\theta)\gamma_1^{-1} A_1 u_0. \tag{32}$$

For $n=1$, equation (31) yields

$$(\tau_w A_1)D_{-w}(u_0)+\theta(h_{-1}A_1)(H_{-1}u_0)+(1-\theta)u_0=-\chi_1 A_2 u_0.$$

Then

$$(t-w-\beta_0)D_{-w}(u_0)+\theta(t-w-\beta_0)(H_{-1}u_0)+\theta(-2t+w)(H_{-1}u_0)\\ =(-\chi_1 A_2+\theta-1)u_0.$$

According to (32), this becomes

$$-(1+\theta)\gamma_1^{-1}(t-w-\beta_0)A_1u_0+\theta(-2t+w)(H_{-1}u_0)\\ =(-\chi_1 A_2+\theta-1)u_0.$$

It follows that

$$\theta(-2t+w)(H_{-1}u_0)\\ =\big(-\chi_1 A_2+\theta-1+(1+\theta)\gamma_1^{-1}(t-w-\beta_0)A_1\big)u_0. \quad (33)$$

Multiplying (32) by $(-2t+w)$ and substituting (33), we have

$$(-2t+w)(D_{-w}u_0)+(-\chi_1A_2+\theta-1+(1+\theta)\gamma_1^{-1}(-t-\beta_0)A_1)u_0=0. \quad (34)$$

Using (2) with $\psi(t)=-2t-w$, we get

$$D_{-w}((-2t-w)u_0)+\big(-\chi_1A_2+\theta+1+(1+\theta)\gamma_1^{-1}(-t-\beta_0)A_1\big)u_0=0. \quad (35)$$

Then

$$D_{-w}\Big((t+\frac{w}{2})u_0\Big)\\ +\Big(\frac{\chi_1}{2}A_2-\frac{\theta}{2}-\frac{1}{2}+\frac{1}{2}(\theta+1)\gamma_1^{-1}(t+\beta_0)A_1\Big)u_0=0. \quad (36)$$

Using the fact that $\chi_0=(1+\theta)\gamma_1^{-1}$, $\chi_1=2\gamma_2^{-1}$ and $A_2(t)=t^2-(\beta_0+\beta_1)t+\beta_1\beta_0-\gamma_1$, we derive the fundamental governing equation for the linear functional u_0. This equation takes the form of the differential-difference relation:

$$D_{-w}\left(\left(t+\frac{w}{2}\right)u_0\right)+\left(a_2t^2+a_1t+a_0\right)u_0=0 \qquad \text{for all} \quad n\geqslant 0. \quad (37)$$

where the coefficients of the quadratic perturbation term are explicitly determined by the recurrence parameters through these relations:

$$\begin{aligned} a_2 &= \gamma_2^{-1} + \frac{1}{2}(\theta+1)\gamma_1^{-1}, \\ a_1 &= -(\beta_0+\beta_1)\gamma_2^{-1}, \\ a_0 &= \beta_0\beta_1\gamma_2^{-1} - \gamma_1\gamma_2^{-1} - \frac{1}{2}(\theta+1)\gamma_1^{-1}\beta_0^2 - \frac{\theta}{2} - \frac{1}{2}, \end{aligned}$$

or $\gamma_2 = \dfrac{2}{1+\theta}\gamma_1$, which makes $a_2 \neq 0$. This non-vanishing leading coefficient ensures the equation maintains its essential second-order character. □

Remark. *Since*

$$\Phi(t) = t + \frac{w}{2},$$

the set of zeros of Φ reduces to the single point $c = -\dfrac{w}{2}$. In this case, the non–degeneracy condition (7) reduces to a single verification at this point.

Because the form u_0 is regular and the recurrence coefficients satisfy $\gamma_n \neq 0$ for all n, the Pearson equation (37) cannot degenerate into one of lower degree. Hence, the condition (7) is automatically satisfied.

Therefore, according to the definition of $D_{-w}-$ semiclassical forms, u_0 is of class one.

Equation (37) represents a precise balance between two distinct actions on the functional u_0, where the first term applies the finite difference operator to the shifted functional and the second term represents a quadratic perturbation of u_0 by the polynomial $a_2t^2 + a_1t + a_0$. The vanishing sum of these operations constitutes a characteristic equation that constrains the behavior of u_0 under both shifting and polynomial multiplication operations.

We illustrate Theorem 1 by considering the continuous limit $w \to 0$. In this limit, the Appell condition

$$T_{\theta,0}A_{n+1} = \theta_{n+1}A_n, \qquad \theta_{n+1} = n+1+\theta\frac{1-(-1)^{n+1}}{2},$$

has been completely characterized in the literature [4].

The generalized Hermite polynomials $\{H_n^{(\mu)}\}_{n\geqslant 0}$ are orthogonal on $\mathbb{R}$ with respect to the weight function

$$w(x) = |x|^{2\mu} e^{-x^2}, \qquad \mu > -\tfrac{1}{2}.$$

As established in [4], they satisfy the Dunkl–Appell relation

$$T_\mu H_{n+1}^{(\mu)}(x) = \big(n + 1 + 2\mu\, \chi_{n+1}\big) H_n^{(\mu)}(x).$$

Setting $\theta = 2\mu$, this matches our definition of θ_{n+1}.

Moreover, in the limit $w \to 0$, the Pearson-type equation (37) reduces to

$$D(tu_0) + (a_2 t^2 + a_1 t + a_0) u_0 = 0.$$

For the generalized Hermite functional u_0 defined by

$$\langle u_0, f\rangle = \int_{-\infty}^{\infty} f(x)\, |x|^{2\mu} e^{-x^2}\, dx,$$

this distributional equation is equivalent (for $x \neq 0$) to the differential Pearson equation

$$\frac{d}{dx}\big(xw(x)\big) + \big(2x^2 - (2\mu + 1)\big) w(x) = 0,$$

which is straightforward to verify.

Finally, in the standard monic normalization (as in [4]), the recurrence coefficients satisfy $\beta_n = 0$ and

$$\gamma_{2n} = n, \qquad \gamma_{2n+1} = n + \mu + \tfrac{1}{2},$$

so that $\beta_0 = -w/2 = 0$ (since $w = 0$), and the relations in Lemma 3 and Proposition 2 are fulfilled when $\theta = 2\mu$.

Thus the generalized Hermite polynomials provide a concrete example of a T_θ-Appell classical OPS of class one, confirming that our general $T_{\theta,w}$-Appell framework extends the known Dunkl–Appell theory to the discrete–difference setting.

Conclusion. In this work, we provided a complete classification of $T_{\theta,w}$-Appell classical orthogonal polynomial sequences and proved that they are necessarily of class one. We established their fundamental structural properties, including explicit recurrence relations and a Pearson-type equation for the canonical functional, confirming their D_{-w}-semiclassical character. These results unify and extend earlier Dunkl and q-Dunkl theories within a single discrete–reflection framework. Further studies will aim at constructing explicit examples and numerical realizations of the families described in this classification.

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References

[1] Abdelkarim F., Maroni P. *The D_w-classical orthogonal polynomials*. Result. Math., 1997, vol. 32, pp. 1 – 28.
DOI: https://doi.org/10.1007/BF03322520.

[2] Alaya J., Maroni P. *Symmetric Laguerre–Hahn forms of class $s=1$*. Integral Transform. Spec. Funct., 1996, vol. 2, pp. 301 – 320.
DOI: https://doi.org/10.1080/10652469608819117.

[3] Aloui B., Souissi J. *Characterization of q-Dunkl-classical symmetric orthogonal q-polynomials*. Ramanujan J., 2022, vol. 57, pp. 1355 – 1365.
DOI: https://doi.org/10.1007/s11139-021-00425-8.

[4] Ben Cheikh Y., Gaied M. *Characterization of the Dunkl-classical symmetric orthogonal polynomials*. Appl. Math. Comput., 2007, vol. 187, pp. 105 – 114. DOI: https://doi.org/10.1016/j.amc.2006.08.108.

[5] Bouanani A., Khériji L., Tounsi M. I. *Characterization of q-Dunkl Appell symmetric orthogonal q-polynomials*. Expo. Math., 2010, vol. 28, pp. 325 – 336. DOI: https://doi.org/10.1016/j.exmath.2010.03.003.

[6] Chihara T. S. *An introduction to orthogonal polynomials*. Mathematics and its Applications, vol. 13, Gordon and Breach Science Publishers, New York– London– Paris, 1978.

[7] Dominici D., Marcellán F. *Discrete semiclassical orthogonal polynomials of class one*. Pacific J. Math., 2014, vol. 268, no. 2, pp. 389 – 411.
DOI: https://doi.org/10.2140/pjm.2014.268.389.

[8] Dunkl C. F. *Integral kernels with reflection group invariance*. Canad. J. Math., 1991, vol. 43, pp. 1213 – 1227.
DOI: https://doi.org/10.4153/CJM-1991-069-8.

[9] Hahn W. *Über die Jacobischen polynome und zwei verwandte polynomklassen*. Math. Z., 1935, vol. 39, pp. 634 – 638.
DOI: https://doi.org/10.1007/BF01201380.

[10] Hahn W. *Über Orthogonalpolynome, die q-Differenzengleichungen genügen.* Math. Nach., 1949, vol. 2, pp. 4 – 34.
DOI: https://doi.org/10.1002/mana.19490020103.

[11] Khériji L., Maroni P. *The H_q-classical orthogonal polynomials.* Acta. Appl. Math., 2002, vol. 71, pp. 49 – 115.
DOI: https://doi.org/10.1023/A:1014597619994.

[12] Lesky P. *Über polynomsysteme, die Sturm-Liouvilleschen differenzengleichungen genügen.* Math. Zeit., 1962, vol. 78, pp. 439 – 445.
DOI: https://doi.org/10.1007/BF01195186.

[13] Maroni P. *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classique.* In Orthogonal Polynomials and their Applications (Erice, 1990), Editors C. Brezinski et al., IMACS Ann. Comput. Appl. Math., 1991, Vol. 9, Baltzer, Basel, pp. 95 – 130.

[14] Maroni P. *Fonctions Eulériennes, polynômes orthogonaux classiques.* Techniques de l'Ingénieur, Traité Généralités (Sciences Fondamentales) A, vol. 154, Paris (1994), pp. 1 – 30.
DOI: https://doi.org/10.51257/a-v1-a154.

[15] Maroni P. *Variations around classical orthogonal polynomials. Connected problems.* J. Comput. Appl. Math., 1993, vol. 48, pp. 133 – 155.
DOI: https://doi.org/10.1016/0377-0427(93)90319-7.

[16] Maroni P. *Sur la suite de polynômes orthogonaux associées à la forme* $u = \delta_c + \lambda(t - c)^{-1}L$. Period. Math. Hungar., 1990, vol. 21, pp. 223 – 248.
DOI: https://doi.org/10.1007/BF02651091.

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