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RATIONAL TYPE MULTI-VALUED MAPPINGS IN PARTIAL METRIC SPACE WITH APPLICATION TO NONLINEAR FRACTIONAL DIFFERENTIAL INCLUSION

Abstract. In this paper, we derive a new rational type common fixed point theorem in a complete partial metric space considering multi-valued mappings extending some existing concepts in literature. This work leads to a fixed point result which is utilized to study the solution to nonlinear fractional delay differential inclusion together with a periodic point result concerning the property P of the mappings. A rational type interpolative fixed point theorem is also included with application to fractional differential inclusion with integral boundary condition. Suitable examples are provided to demonstrate the results.

Key words: *fixed point, multi-valued mapping, partial metric space, fractional differential inclusion*

2020 Mathematical Subject Classification: *47H09, 47H10, 54C60*

1. Introduction. Rational type fixed point results have emerged as a key generalization of the usual contraction principle. Rational type fixed point theory is extensively used in solving differential equations, fractional differential equations ($\mathcal{FDE}$s), where solution operators show rational growth. In [9], Gupta et al. used interpolative Boyd-Wong type and Matkowski type contractions on partial b-metric space to establish some fixed point theorems ($\mathcal{FPT}$s). In 2023, Puvar et al. [14] discussed rational type cyclic contraction via C-class functions in G-metric spaces with common fixed point result. In nonlinear analysis, delay fractional differential inclusions ($\mathcal{FDI}s$) have emerged as a significant area because they offer an integrated framework for systems that feature memory, time delays, and nonlinear phenomena. Delay $\mathcal{FDE}$s are widely used in economic growth model, population dynamics, signal processing, control systems, etc. In

2025, Aldowah et al. [2] established rational type fixed point theorem to provide an application to economic growth model. Kebede and Lakoud [8] analyzed the existence and stability of solutions for a $\mathcal{FDE}$ model with Atangana-Baleanu Caputo derivatives and vaccination in 2024. Further, in 2025, El-Sayeed et al. [6] derived a unique solution to a constrained problem of $\mathcal{FDE}$ and Hyers-Ulam stability result. Some more literature can be found in [4], [7], [19], [20].

The use of multi-valued mappings in fixed point theory provides a new scope considering different aspects mainly in optimization problem with multiple solutions, control system with uncertain inputs, etc. The discussion on fuzzy multi-valued mappings using graph theoretical aspects in parametric metric space is done by Rafique et al. in [15]. Multi-valued Kikkawa-Suzuki type $\mathcal{FPT}$ with stability analysis is studied in [22]. For some detailed work in this direction, we refer to [13], [18].

In this paper, we focus on investigating the existence of solutions to the Caputo fractional delay differential inclusion via rational type multi-valued mapping with a variable coefficient given by

$$\begin{aligned}
&{}^{C}\mathfrak{D}_t^{\omega}\varrho(t) \in \mathcal{A}_0(t)\varrho(t) + \mathcal{A}_1(t)\varrho(t-\tau) + \mathfrak{F}(t, \varrho(t), \varrho(t-\tau)), \quad t \in [0, T] \\
&\text{with initial condition} \quad x(t) = \psi(t), \quad t \in [-\tau, 0],
\end{aligned}$$

where ${}^{c}\mathfrak{D}_t^{\omega}$ denotes the Caputo fractional derivative of order ω. Here, the Caputo fractional derivative describes memory effects or hereditary effects, whereas delay terms represent the impact of past states after a finite time delay.

In [21], Wang et al. discussed the existence and uniqueness of solution to the above problem by considering $\mathcal{A}_0, \mathcal{A}_1$ as constant system matrices of appropriate dimensions. As an extension of this work, in this paper we consider $\mathcal{A}_0(\cdot), \mathcal{A}_1(\cdot)$ as functions dependent on time. This will help us to deal with different time-dependent practical situations. For this, we develop a rational type common $\mathcal{FPT}$ in a complete partial metric space ($\mathcal{PMS}$), leading to a $\mathcal{FPT}$, which is utilized to study the existence of solution to different nonlinear problems. Although the nonlinear formulation creates considerable challenges, yet it connects many areas including fixed point theory, fractional differential equations, and multi-valued mappings. Following that, we also derive an interpolative hybrid $\mathcal{FPT}$, which is used to investigate the solution to nonlinear $\mathcal{FDI}$ with integral boundary condition. The advantage of our rational type contraction condition over classical contraction is that it is well-suited to the intrinsic structure of partial metric. Moreover, hybrid contractions frequently allow discontinuous mappings that possess fixed points. Therefore, the results of the paper are applicable to a larger family of operators dealing with real life situations.

The study of periodic points is essential in nonlinear analysis to describe the repeated and cyclic behaviour of mappings. Moreover, analysing periodic points gives a precise understanding of long-term dynamical system and stability properties of nonlinear operators. In the last section of the paper, we derive periodic point result considering the rational type multi-valued mapping via property P.

For the convenience of the readers, we present the list of abbreviations used throughout the discussion in the following manner:

$\mathcal{FDE}$: Fractional differential equation;

$\mathcal{FDI}$: Fractional differential inclusion;

$\mathcal{FPT}$: Fixed point theorem;

$\mathcal{PMS}$: Partial metric space.

2. Preliminaries. Before proceeding, let us revisit the key concepts of the $\mathcal{PMS}$.

For a non-empty set Ω, let $p : \Omega \times \Omega \to [0, \infty)$ be a mapping satisfying the following properties:

(i) $0 \leqslant p(\rho, \rho) \leqslant p(\rho, \sigma)$,

(ii) $p(\rho, \rho) = p(\rho, \sigma) = p(\sigma, \sigma)$ if and only if $\rho = \sigma$,

(iii) $p(\rho, \sigma) = p(\sigma, \rho)$,

(iv) $p(\rho, \eta) \leqslant p(\rho, \sigma) + p(\sigma, \eta) - p(\sigma, \sigma)$,

for all $\rho, \sigma, \eta \in \Omega$. Here p is termed as a partial metric on Ω and the pair (Ω, p) forms a $\mathcal{PMS}$ (refer to [12]). For example, on $\Omega = \mathbb{R}^+ \cup \{0\}$, $p_1(\varrho, \vartheta) = |\varrho - \vartheta| + |\varrho| + |\vartheta|$, $p_2(\varrho, \vartheta) = |\varrho - \vartheta| + \min\{\varrho, \vartheta\}$ and $p_3(\varrho, \vartheta) = |\varrho - \vartheta| + \max\{\varrho, \vartheta\}$ for all $\varrho, \vartheta \in \Omega$, are partial metrics.

In [12], Matthews defined the Cauchy sequence and convergence of sequences in $\mathcal{PMS}$. A sequence $\{\sigma_n\}$ in (Ω, p) is said to be Cauchy if and only if $\lim\limits_{n, m \to \infty} p(\sigma_n, \sigma_m)$ exists and is finite. A sequence $\{\sigma_n\}$ is said to be convergent to η in Ω if and only if $\lim\limits_{n \to \infty} p(\sigma_n, \eta) = \lim\limits_{n \to \infty} p(\sigma_n, \sigma_n) = p(\eta, \eta)$. The pair (Ω, p) is called complete if and only if every Cauchy sequence $\{\sigma_n\}$ in (Ω, p) converges to some point η in Ω, that is, $p(\eta, \eta) = \lim\limits_{n, m \to \infty} p(\sigma_n, \sigma_m)$.

Given a $\mathcal{PMS}$ (Ω, p), the induced metric $p^s : \Omega \times \Omega \to \mathbb{R}$ in Ω, given by $p^s(\rho,\sigma) = 2p(\rho,\sigma) - p(\rho,\rho) - p(\sigma,\sigma)$ for all $\rho,\sigma \in \Omega$, provides a metric on Ω (refer to [5]).

Lemma 1. [5] *Consider a* $\mathcal{PMS}$ (Ω, p)*;*

(i) A sequence $\{\sigma_n\}$ *is a Cauchy sequence in* (Ω, p) *if and only if* $\{\sigma_n\}$ *is a Cauchy sequence in the induced metric space* (Ω, p^s).

(ii) (Ω, p) *is complete if and only if the metric space* (Ω, p^s) *is complete. Moreover,*

$$\lim_{n\to\infty} p^s(\sigma_n, \eta) = 0 \quad \textit{iff} \quad p(\eta,\eta) = \lim_{n\to\infty} p(\sigma_n, \eta) = \lim_{m,n\to\infty} p(\sigma_n, \sigma_m).$$

For a $\mathcal{PMS}$ (Ω, p), let $\mathcal{P}(\Omega)$ denote the set of all subsets of Ω and $\mathcal{CB}^p(\Omega)$ denote the set containing all non-empty closed and bounded subsets of Ω with respect to p. For any $\mathcal{A}, \mathcal{B} \in \mathcal{CB}^p(\Omega)$, the partial Hausdorff metric $\mathcal{H}_p$ is defined by

$$\mathcal{H}_p(\mathcal{A}, \mathcal{B}) = \max\{\delta_p(\mathcal{A}, \mathcal{B}), \delta_p(\mathcal{B}, \mathcal{A})\},$$

where $p(a, \mathcal{A}) = \inf\{p(a,x) : x \in \mathcal{A}\}$, $\delta_p(A,B) = \sup\{p(a,B) : a \in A\}$, and $\delta_p(B,A) = \sup\{p(b,A) : b \in B\}$ (refer to [3]).

Lemma 2. [3] *Let* (Ω, p) *be a* $\mathcal{PMS}$, $\mathcal{A}, \mathcal{B} \in \mathcal{CB}^p(\Omega)$ *and* $h > 1$. *For any* $a \in \mathcal{A}$, *there exists* $b = b(a) \in \mathcal{B}$, *such that* $p(a,b) \leqslant h\mathcal{H}_p(\mathcal{A}, \mathcal{B})$.

Next, we recall the basic definitions of $\mathcal{FDE}$.

Definition 1. [10] *Given a function* $z \in L^1([0,1])$, *the Riemann-Liouville fractional integral of order* $\omega > 0$ *is defined by*

$$I_{0+}^{\omega} z(t) = \begin{cases} \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} z(s)\, ds\,, & \omega > 0 \\ z(t)\,, & \omega = 0\,. \end{cases}$$

Definition 2. [10] *For a function* $z \in AC^n([0{,}1])$, *the Caputo fractional derivative of order* $\omega > 0$ *is expressed as*

$$^{C}\mathfrak{D}_{0+}^{\omega} z(t) = \begin{cases} \frac{1}{\Gamma(n-\omega)} \int_0^t (t-s)^{n-\omega-1} z^{(n)}(s)\, ds\,, & \omega \notin \mathbb{N} \\ z^{(n)}(t)\,, & \omega \in \mathbb{N} \end{cases}$$

where $z^{(n)}(t) = \frac{d^n z(t)}{dt^n}$ with $n = \lfloor\omega\rfloor + 1$ and $\lfloor\omega\rfloor$ represents the integer part of real number ω.

3. Main Results. First, we derive a rational type common $\mathcal{FPT}$ in a complete $\mathcal{PMS}$.

Theorem 1. *Suppose $\mathfrak{S}, \mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ are two multi-valued mappings in a complete $\mathcal{PMS}$ (Ω, p). For some $\kappa \in [0, \frac{1}{2})$, let $\mathfrak{S}$ and $\mathfrak{T}$ satisfy the condition:*

$$\mathcal{H}_p(\mathfrak{S}\varrho, \mathfrak{T}\vartheta) \leqslant \kappa\Big(\max\{p(\varrho,\vartheta), p(\varrho, \mathfrak{S}\varrho), p(\vartheta, \mathfrak{T}\vartheta)\} + \frac{p(\varrho, \mathfrak{S}\varrho)p(\vartheta,\mathfrak{T}\vartheta)}{1 + p(\varrho, \vartheta)}\Big) \quad (1)$$

for all $\varrho, \vartheta \in \Omega$. Then $\mathcal{CF}(\mathfrak{S}, \mathfrak{T}) \neq \varnothing$, where $\mathcal{CF}(\mathfrak{S},\mathfrak{T})$ denotes the set of common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$.

Proof. Suppose $\varrho_0 \in \Omega$ and $h \in \big(1, \frac{1}{2\kappa}\big)$, $\kappa \neq 0$ (for $\kappa = 0$, we consider $h > 1$). By Lemma 2, for $\varrho_1 \in \mathfrak{S}\varrho_0$, there exists $\varrho_2 \in \mathfrak{T}\varrho_1$, such that

$$\begin{aligned}
p(\varrho_1, \varrho_2) &\leqslant h\mathcal{H}_p(\mathfrak{S}\varrho_0,\mathfrak{T}\varrho_1) \\
&\leqslant h\kappa\left(\max\{p(\varrho_0, \varrho_1), p(\varrho_0, \mathfrak{S}\varrho_0), p(\varrho_1, \mathfrak{T}\varrho_1)\} + \frac{p(\varrho_0, \mathfrak{S}\varrho_0)p(\varrho_1,\mathfrak{T}\varrho_1)}{1 + p(\varrho_0, \varrho_1)}\right) \\
&\leqslant h\kappa\left(\max\{p(\varrho_0,\varrho_1), p(\varrho_0, \varrho_1), p(\varrho_1, \varrho_2)\} + \frac{p(\varrho_0, \varrho_1)p(\varrho_1, \varrho_2)}{p(\varrho_0, \varrho_1)}\right) \\
&= h\kappa(\max\{p(\varrho_0, \varrho_1), p(\varrho_1, \varrho_2)\} + p(\varrho_1, \varrho_2)). \qquad (2)
\end{aligned}$$

If $p(\varrho_0, \varrho_1) \leqslant p(\varrho_1, \varrho_2)$, then from (2), $p(\varrho_1, \varrho_2) \leqslant 2h\kappa p(\varrho_1, \varrho_2)$, which gives $p(\varrho_1, \varrho_2) = 0$ and so, $\varrho_1 = \varrho_2$. Then $p(\varrho_0, \varrho_1) = 0$ and thus $\varrho_0 = \varrho_1 = \varrho_2 \in \mathcal{CF}(\mathfrak{S}, \mathfrak{T})$.

If $p(\varrho_1, \varrho_2) \leqslant p(\varrho_0, \varrho_1)$, then $p(\varrho_1, \varrho_2) \leqslant h\kappa\{p(\varrho_0, \varrho_1) + p(\varrho_1, \varrho_2)\}$, i. e., $p(\varrho_1, \varrho_2) \leqslant \Big(\dfrac{h\kappa}{1 - h\kappa}\Big)p(\varrho_0, \varrho_1) = \theta p(\varrho_0, \varrho_1)$, where $\theta = \dfrac{h\kappa}{1 - h\kappa} < 1$. By Lemma 2, for $\varrho_2 \in \mathfrak{T}\varrho_1$ there exists $\varrho_3 \in \mathfrak{S}\varrho_2$, such that

$$p(\varrho_2, \varrho_3) \leqslant h\mathcal{H}_p(\mathfrak{T}\varrho_1, \mathfrak{S}\varrho_2) \leqslant h\kappa(\max\{p(\varrho_1, \varrho_2), p(\varrho_2, \varrho_3)\} + p(\varrho_2, \varrho_3)). \quad (3)$$

If $p(\varrho_1, \varrho_2) \leqslant p(\varrho_2, \varrho_3)$, then from (3), $p(\varrho_2, \varrho_3) = 0$, and then $p(\varrho_1, \varrho_2) = 0$. Thus, $\varrho_1 = \varrho_2 = \varrho_3 \in \mathcal{CF}(\mathfrak{S}, \mathfrak{T})$.

Suppose $p(\varrho_2, \varrho_3) \leqslant p(\varrho_1, \varrho_2)$. From (3) we have $p(\varrho_2, \varrho_3) \leqslant h\kappa\{p(\varrho_1, \varrho_2) + p(\varrho_2, \varrho_3)\}$, i. e., $p(\varrho_2, \varrho_3) \leqslant \theta p(\varrho_1, \varrho_2) \leqslant \theta^2 p(\varrho_0, \varrho_1)$. Continuing in this way, we get a sequence $\{\varrho_n\}$ in Ω, such that $\varrho_{2n+1} \in \mathfrak{S}\varrho_{2n}$ and $\varrho_{2n+2} \in \mathfrak{T}\varrho_{2n+1}$ with

$$p(\varrho_{2n+1},\varrho_{2n+2}) \leqslant h\mathcal{H}_p(\mathfrak{S}\varrho_{2n}, \mathfrak{T}\varrho_{2n+1})$$

$$\leqslant h\kappa(\max\{p(\varrho_{2n},\varrho_{2n+1}),p(\varrho_{2n+1},\varrho_{2n+2})\}+p(\varrho_{2n+1},\varrho_{2n+2})).$$

If $p(\varrho_{2n},\varrho_{2n+1})\leqslant p(\varrho_{2n+1},\varrho_{2n+2})$, then $\varrho_{2n}=\varrho_{2n+1}=\varrho_{2n+2}\in\mathcal{CF}(\mathfrak{S},\mathfrak{T})$.

Suppose $p(\varrho_{2n+1},\varrho_{2n+2})\leqslant p(\varrho_{2n},\varrho_{2n+1})$. Then for $n\in\mathbb{N}\cup\{0\}$,

$$p(\varrho_{2n+1},\varrho_{2n+2})\leqslant\theta p(\varrho_{2n},\varrho_{2n+1})\leqslant\theta^{2n+1}p(\varrho_0,\varrho_1)$$

and $p(\varrho_{2n+2},\varrho_{2n+3})\leqslant\theta^{2n+2}p(\varrho_0,\varrho_1)$. Thus, for each $n\in\mathbb{N}\cup\{0\}$, we have $p(\varrho_n,\varrho_{n+1})\leqslant\theta^n p(\varrho_0,\varrho_1)$.

For all $m,n\in\mathbb{N}$, it is easy to see that $p^s(\varrho_n,\varrho_{n+m})\leqslant 2p(\varrho_n,\varrho_{n+m})\to 0$ as $n\to\infty$ and so, $\{\varrho_n\}$ is a Cauchy sequence in (Ω,p^s). Using Lemma 1, (Ω,p^s) is complete, since (Ω,p) is complete. So, there exists some $u\in\Omega$, such that $\lim_{n\to\infty}p^s(\varrho_n,u)=0$. By Lemma 1

$$p(u,u)=\lim_{n\to\infty}p(\varrho_n,u)=\lim_{n,\,m\to\infty}p(\varrho_n,\varrho_m)=0.$$

Now,

$$\begin{aligned}p(u,\mathfrak{S}u)&\leqslant p(u,\varrho_{2n+2})+\mathcal{H}_p(\mathfrak{S}u,\mathfrak{T}\varrho_{2n+1})\\&\leqslant p(u,\varrho_{2n+2})+\kappa\Big[\max\{p(u,\varrho_{2n+1}),p(u,\mathfrak{S}u),p(\varrho_{2n+1},\varrho_{2n+2})\}\\&\qquad\qquad+\frac{p(u,\mathfrak{S}u)p(\varrho_{2n+1},\varrho_{2n+2})}{1+p(u,\varrho_{2n+1})}\Big].\end{aligned}$$

Taking limit as $n\to\infty$, $p(u,\mathfrak{S}u)\leqslant\kappa p(u,\mathfrak{S}u)$, i. e., $p(u,\mathfrak{S}u)=0$ and so, $u\in\mathfrak{S}u$.

Similarly, $p(u,\mathfrak{T}u)=0$, i. e., $u\in\mathfrak{T}u$. Consequently, $u\in\mathcal{CF}(\mathfrak{S},\mathfrak{T})$. □

The following examples demonstrate the above theorem.

Example 1. Let $\Omega=[0,1]$ and $p(\varrho,\vartheta)=\max\{\varrho,\vartheta\}$. Then (Ω,p) is a complete $\mathcal{PMS}$. Let $\mathfrak{S},\mathfrak{T}:\Omega\to\mathcal{CB}^p(\Omega)$ be defined by

$$\mathfrak{S}\varrho=\Big[0,\frac{\varrho}{6}\Big]\text{ and }\mathfrak{T}\varrho=\Big[0,\frac{\varrho}{8}\Big],\ \varrho\in\Omega.$$

Now, for each $\varrho,\ \vartheta\in\Omega$,

$$\sup_{u\in[0,\frac{\varrho}{6}]}\ \inf_{v\in[0,\frac{\vartheta}{8}]}p(u,v)=\frac{\varrho}{6},\qquad\sup_{v\in[0,\frac{\vartheta}{8}]}\ \inf_{u\in[0,\frac{\varrho}{6}]}p(u,v)=\frac{\vartheta}{8},$$

and

$$\mathcal{H}_p(\mathfrak{S}\varrho,\mathfrak{T}\vartheta)=\max\Big\{\frac{\varrho}{6},\frac{\vartheta}{8}\Big\}\leqslant\frac{1}{6}\max\{\varrho,\vartheta\}=\frac{\nu}{6},\quad\text{where }\nu=\max\{\varrho,\vartheta\}.$$

Also, $p(\varrho,\mathfrak{S}\varrho)=\varrho$, $p(\vartheta,\mathfrak{T}\vartheta)=\vartheta$, $p(\varrho,\vartheta)=\nu$ and so,

$$\kappa\Big(\max\{p(\varrho,\vartheta),p(\varrho,\mathfrak{S}\varrho),p(\vartheta,\mathfrak{T}\vartheta)\}+\frac{p(\varrho,\mathfrak{S}\varrho)p(\vartheta,\mathfrak{T}\vartheta)}{1+p(\varrho,\vartheta)}\Big)=\kappa\Big(\nu+\frac{\varrho\vartheta}{1+\nu}\Big).$$

Thus, Theorem 1 is satisfied for $\frac{1}{6}\leqslant\kappa<\frac{1}{2}$. Clearly $0\in\mathcal{CF}(\mathfrak{S},\mathfrak{T})$ here.

Example 2. Let $\Omega=[0,1]$ and $p(\varrho,\vartheta)=|\varrho-\vartheta|+|\varrho|+|\vartheta|$, then (Ω,p) is a complete $\mathcal{PMS}$. Let two multi-valued mappings $\mathfrak{S},\mathfrak{T}:\Omega\to\mathcal{CB}^p(\Omega)$ be defined by $\mathfrak{S}\varrho=\Big\{\frac{\varrho}{11^n}:n\in\mathbb{N}\Big\}\cup\{0\}$ and $\mathfrak{T}\varrho=\Big\{\frac{\varrho}{13^n}:n\in\mathbb{N}\Big\}\cup\{0\}$, $\varrho\in\Omega$. For each $\varrho,\vartheta\in\Omega$, we have,

$$\mathcal{H}_p(\mathfrak{S}\varrho,\mathfrak{T}\vartheta)=\max\Big\{\frac{2\varrho}{11},\frac{2\vartheta}{13}\Big\}\leqslant\frac{2}{11}\max\{\varrho,\vartheta\}.$$

Also, $p(\varrho,\mathfrak{S}\varrho)=2\varrho$, $p(\vartheta,\mathfrak{T}\vartheta)=2\vartheta$. Now, RHS of (1) is

$$\kappa\Big(|\varrho-\vartheta|+|\varrho|+|\vartheta|+\frac{4\varrho\vartheta}{1+|\varrho-\vartheta|+|\varrho|+|\vartheta|}\Big)>\kappa(|\varrho-\vartheta|+|\varrho|+|\vartheta|).$$

So, for any value of $\kappa\in\Big[\frac{2}{11},\frac{1}{2}\Big)$, condition (1) is valid for all $\varrho,\vartheta\in\Omega$. Clearly, $0\in\mathcal{CF}(\mathfrak{S},\mathfrak{T})$.

Corollary 1. *Consider $\mathfrak{T}:\Omega\to\mathcal{CB}^p(\Omega)$ as a multi-valued mapping on a complete $\mathcal{PMS}$ (Ω,p). For $0\leqslant\kappa<\frac{1}{2}$, if $\mathfrak{T}$ satisfies the condition*

$$\mathcal{H}_p(\mathfrak{T}\varrho,\mathfrak{T}\vartheta)\leqslant\kappa\Big(\max\{p(\varrho,\vartheta),p(\varrho,\mathfrak{T}\varrho),p(\vartheta,\mathfrak{T}\vartheta)\}+\frac{p(\varrho,\mathfrak{T}\varrho)p(\vartheta,\mathfrak{T}\vartheta)}{1+p(\varrho,\vartheta)}\Big),\quad(4)$$

for all $\varrho,\vartheta\in\Omega$, then $\mathcal{F}(\mathfrak{T})\neq\varnothing$, where $\mathcal{F}(\mathfrak{T})$ denotes the set of all fixed points of $\mathfrak{T}$.

Applying the above result, we now examine the solvability of a delay $\mathcal{FDI}$.

Consider the following Caputo $\mathcal{FDI}$ with the variable coefficient:

$${}^{C}\mathfrak{D}_t^{\omega}\varrho(t)\in\mathcal{A}_0(t)\varrho(t)+\mathcal{A}_1(t)\varrho(t-\tau)+\mathfrak{F}(t,\varrho(t),\varrho(t-\tau)),\quad t\in[0,T]\quad(5)$$

with initial condition $\varrho(t)=\psi(t)$, $t\in[-\tau,0]$, where ${}^{C}\mathfrak{D}_t^{\omega}$ denotes the Caputo fractional derivative of order $0<\omega<1$, $\varrho:[-\tau,T]\to\mathbb{R}^n$ is the state vector, $\mathcal{A}_0,\mathcal{A}_1$ are continuous and time dependent bounded linear operators, $\mathfrak{F}:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{CB}^p(\mathbb{R}^n)$ is a multi-valued closed and bounded map and $\psi \in C([-\tau,0],\mathbb{R}^n)$ with $\psi(0) \neq 0 \in \mathbb{R}^n$ is the past state with the norm defined as $\|\psi\| = \sup_{t\in[-\tau,0]} \|\psi(t)\|$ (refer to [21]).

The next lemma is similar to Theorem 3.1 of [21].

Lemma 3. *If $\varrho : [-\tau, T] \to \mathbb{R}^n$ is a continuously differentiable function, such that*

$$\varrho(t) = \begin{cases} \psi(t), & t \in [-\tau, 0] \\ \psi(0) + \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} [A_0(s)\varrho(s) + A_1(s)\varrho(s-\tau) \\ \qquad\qquad + f(s, \varrho(s), \varrho(s-\tau))]\, ds, & t \in [0,T] \end{cases}$$

where $f(s,\varrho(s),\varrho(s-\tau)) \in \mathfrak{F}(s,\varrho(s),\varrho(s-\tau))$, then $\varrho(t)$ is a solution to the system (5).

Let $\Omega = C([-\tau,T],\mathbb{R}^n)$ with $\|\varrho\| = \sup_{t\in[-\tau,T]} \|\varrho(t)\|, \varrho \in \Omega$. Then Ω is a complete $\mathcal{PMS}$ with p defined by

$$p(\varrho,\vartheta) = \|\varrho - \vartheta\| + \|\varrho\| + \|\vartheta\| \quad \text{for all } \varrho,\vartheta \in \Omega. \tag{6}$$

Theorem 2. *For the $\mathcal{FDI}$ (5), assume that the following conditions hold:*

(i) for all $\varrho, \vartheta \in \Omega$ and $t \in [0,T]$, $f_\varrho(t) := f(t,\varrho(t),\varrho(t-\tau))$, $f_\vartheta(t) := f(t,\vartheta(t),\vartheta(t-\tau)) \in \mathfrak{F}(t,\varrho(t),\varrho(t-\tau))$ be Lipschitz continuous with positive constant $\mathcal{N}_0$, such that $f(t,0,0) = 0$ and for any $t \in [0,T]$,

$$\|f_\varrho(t) - f_\vartheta(t)\| \leqslant \mathcal{N}_0(\|\varrho(t) - \vartheta(t)\| + \|\varrho(t-\tau) - \vartheta(t-\tau)\|).$$

(ii) $\mathcal{A}_0, \mathcal{A}_1 : [0,T] \to \mathcal{L}(\mathbb{R}^n)$, (where $\mathcal{L}(\mathbb{R}^n)$ is the space of all bounded linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$) be continuous and bounded operators with $e_{\mathcal{A}_0} = \sup_{t\in[0,T]} \|\mathcal{A}_0(t)\|$, $e_{\mathcal{A}_1} = \sup_{t\in[0,T]} \|\mathcal{A}_1(t)\|$ satisfying

$$\frac{T^\omega}{\Gamma(\omega+1)}(e_{\mathcal{A}_0} + e_{\mathcal{A}_1} + 2\mathcal{N}_0) \geqslant 2\|\psi\|.$$

Then the system (5) has a solution.

Proof. Let $\mathcal{M} = \frac{1}{4}\Big(\frac{\lambda}{2|psi(0)\|} - 1\Big)$, where $\lambda = \frac{T^\omega}{\Gamma(\omega+1)}(e_{\mathcal{A}_0} + e_{\mathcal{A}_1} + 2\mathcal{N}_0)$. Consider Ω and p as defined in (6). We take the closed and bounded subset

$$\Omega_\mathcal{M} = \{\varrho \in \Omega : \|\varrho\| \leqslant \mathcal{M}\}.$$

Let $\mathfrak{T} : \Omega_{\mathcal{M}} \to \mathcal{CB}^p(\Omega_{\mathcal{M}})$ be defined by

$$\mathfrak{T}\varrho = \left\{ u \in \Omega : u(t) = \begin{cases} \psi(t), & t \in [-\tau, 0] \\ \psi(0) + \frac{1}{\Gamma(\omega)} \int\limits_0^t (t-s)^{\omega-1} \big(\mathcal{A}_0(s)\varrho(s) + \mathcal{A}_1(s)\varrho(s-\tau) \\ \qquad + f(s, \varrho(s), \varrho(s-\tau))\big)\, ds, & t \in [0, T] \end{cases} \right\}, \quad (7)$$

where $f(s, \varrho(s), \varrho(s-\tau)) \in \mathfrak{F}(s, \varrho(s), \varrho(s-\tau))$.

Let $\varrho, \vartheta \in \Omega_{\mathcal{M}}$ be arbitrary. For $u \in \mathfrak{T}\varrho$, $v \in \mathfrak{T}\vartheta$ and $t \in [0, T]$,

$$\begin{aligned} \|u(t) - v(t)\| &= \Bigg\| \frac{1}{\Gamma(\omega)} \int\limits_0^t (t-s)^{\omega-1} (\mathcal{A}_0(s)\varrho(s) + \mathcal{A}_1(s)\varrho(s-\tau) + f_\varrho(s))\, ds \\ &\quad - \frac{1}{\Gamma(\omega)} \int\limits_0^t (t-s)^{\omega-1} (\mathcal{A}_0(s)\vartheta(s) + \mathcal{A}_1(s)\vartheta(s-\tau) + f_\vartheta(s))\, ds \Bigg\| \\ &\leqslant \frac{1}{\Gamma(\omega)} \int\limits_0^t |t-s|^{\omega-1} \big(\|\mathcal{A}_0(s)\|\, \|\varrho(s) - \vartheta(s)\| \\ &\quad + \|\mathcal{A}_1(s)\|\, \|\varrho(s-\tau) - \vartheta(s-\tau)\| + \|f_\varrho(s) - f_\vartheta(s)\| \big)\, ds. \end{aligned}$$

Taking sup over $s \in [0, T]$, we get

$$\begin{aligned} \|u(t) - v(t)\| &\leqslant \frac{1}{\Gamma(\omega)} \int\limits_0^t |t-s|^{\omega-1} \big((e_{\mathcal{A}_0} + e_{\mathcal{A}_1}) \|\varrho - \vartheta\| \\ &\quad + \sup_{s \in [0,T]} \|\mathcal{N}_0 (\|\varrho(s) - \vartheta(s)\| + \|\varrho(s-\tau) - \vartheta(s-\tau)\|)\| \big)\, ds \\ &= \frac{1}{\Gamma(\omega)} \int\limits_0^t |t-s|^{\omega-1} (e_{\mathcal{A}_0} + e_{\mathcal{A}_1} + 2\mathcal{N}_0) \|\varrho - \vartheta\|\, ds. \quad (8) \end{aligned}$$

Again, taking sup over $t \in [0, T]$ and using $\lambda = \dfrac{T^\omega}{\Gamma(\omega+1)} (e_{\mathcal{A}_0} + e_{\mathcal{A}_1} + 2\mathcal{N}_0)$, we get,

$$\|u - v\| \leqslant \frac{T^\omega}{\Gamma(\omega+1)} (e_{\mathcal{A}_0} + e_{\mathcal{A}_1} + 2\mathcal{N}_0) \|\varrho - \vartheta\| = \lambda \|\varrho - \vartheta\|. \quad (9)$$

Also,

$$\begin{aligned}\|u(t)\| \leqslant \|\psi(0)\| + \frac{1}{\Gamma(\omega)} \int\limits_0^t |t-s|^{\omega-1} (\|\mathcal{A}_0(s)\| \, \|\varrho(s)\| \\ + \|\mathcal{A}_1(s)\| \, \|\varrho(s-\tau)\| + \|f_\varrho(s)\|) \, ds \\ \leqslant \|\psi(0)\| + \frac{1}{\Gamma(\omega)} \int\limits_0^t |t-s|^{\omega-1} (e_{\mathcal{A}_0} \|\varrho(s)\| + e_{\mathcal{A}_1} \|\varrho(s-\tau)\| \\ + \mathcal{N}_0(\|\varrho(s)\| + \|\varrho(s-\tau)\|)) \, ds.\end{aligned}$$

So, $\|u\| \leqslant \|\psi(0)\| + \lambda \|\varrho\|$.

Similarly, $\|v\| \leqslant \|\psi(0)\| + \lambda \|\vartheta\|$. Now,

$$\begin{aligned}\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) \leqslant \sup_{u \in \mathfrak{T}\varrho, v \in \mathfrak{T}\vartheta} p(u,v) = \sup_{u \in \mathfrak{T}\varrho, v \in \mathfrak{T}\vartheta} (\|u-v\| + \|u\| + \|v\|) \\ \leqslant \lambda p(\varrho, \vartheta) + 2\|\psi(0)\|. \qquad (10)\end{aligned}$$

Again, $p(\varrho, \vartheta) \leqslant 2(\|\varrho\| + \|\vartheta\|) \leqslant 4\mathcal{M} = \dfrac{\lambda}{2\|\psi(0)\|} - 1$, i. e., $1 + p(\varrho, \vartheta) \leqslant$ $\leqslant \dfrac{\lambda}{2\|\psi(0)\|}$, i. e., $2\|\psi(0)\| \leqslant \dfrac{\lambda}{1 + p(\varrho, \vartheta)}$. So, from (10):

$$\begin{aligned}\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) \leqslant \lambda p(\varrho, \vartheta) + \frac{\lambda}{1 + p(\varrho, \vartheta)} \\ \leqslant \lambda \Big(\max\{p(\varrho, \vartheta), p(\varrho, \mathfrak{T}\varrho), p(\vartheta, \mathfrak{T}\vartheta)\} + \frac{p(\varrho, \mathfrak{T}\varrho) p(\vartheta, \mathfrak{T}\vartheta)}{1 + p(\varrho, \vartheta)} \Big).\end{aligned}$$

Hence, condition (4) is satisfied for $\lambda = \kappa$. By Corollary 1, $\mathfrak{T}$ has a fixed point, i. e., system (5) has a solution. □

Now, we illustrate the above result with the help of a numerical example.

Example 3. Consider the following delay $\mathcal{FDE}$:

$$\begin{aligned}{}^C\mathfrak{D}_t^{\frac{1}{2}} \varrho(t) = \Big(\frac{3}{751} + \frac{4}{625} \sin(2\pi t)\Big) \varrho(t) + \frac{9}{975} \cos(\pi t) \varrho(t-1) \\ + \frac{11}{871} \arctan(\varrho(t)) + \frac{4}{992} \frac{\sin(\varrho(t)) \cos(\varrho(t-1))}{1 + t^2}, \quad t \in [0,1]\end{aligned}$$

with initial condition $\varrho(t) = \frac{e^t}{1000}$, $t \in [-1, 0]$.

Here $n = 1$, $T = 1$, $\tau = 1$, $\omega = \dfrac{1}{2}$,

$$\mathcal{A}_0(t) = \frac{3}{751} + \frac{4}{625}\sin(2\pi t), \quad \mathcal{A}_1(t) = \frac{9}{975}\cos(\pi t),$$

$$f_\varrho(t) = \frac{11}{871}\arctan(\varrho(t)) + \frac{4}{992}\frac{\sin(\varrho(t))\cos(\varrho(t-1))}{1+t^2} = f(t, \varrho(t), \varrho(t-1))$$

and $\psi(t) = \dfrac{e^t}{1000}$. We take $\Omega = C([-1,1], \mathbb{R})$ with the sup-norm and the complete partial metric on $\mathcal{X}$ as defined by (6).

For each $\varrho, \vartheta \in \Omega$ and $t \in [0, 1]$:

$$\begin{aligned}
\|f_\varrho(t) - f_\vartheta(t)\| &\leqslant \frac{11}{871}|\arctan\varrho(t) - \arctan\vartheta(t)| \\
&\quad + \frac{4}{992(1+t^2)}|\sin\varrho(t)\cos\varrho(t-1) - \sin\vartheta(t)\cos\vartheta(t-1)| \\
&\leqslant \frac{11}{871}|\varrho(t) - \vartheta(t)| + \frac{4}{992}\left(|\varrho(t) - \vartheta(t)| + |\varrho(t-1) - \vartheta(t-1)|\right) \\
&\leqslant 0.016|\varrho(t) - \vartheta(t)| + 0.004|\varrho(t-1) - \vartheta(t-1)| \\
&\leqslant 0.016(|\varrho(t) - \vartheta(t)| + |\varrho(t-1) - \vartheta(t-1)|).
\end{aligned}$$

So, for $\mathcal{N}_0 = 0.016$, condition (i) of Theorem 2 is satisfied. Clearly, $\mathcal{A}_0, \mathcal{A}_1 : [0,1] \to \mathcal{L}(\mathbb{R})$ are continuous and bounded linear operators with $e_{\mathcal{A}_0} = 0.01$, $e_{\mathcal{A}_1} = 0.009$, and $\|\psi\| = 0.001$.

Also, $\dfrac{T^\omega}{\Gamma(\omega+1)}(e_{\mathcal{A}_0} + e_{\mathcal{A}_1} + 2\mathcal{N}_0) = 0.57 \geqslant 2\|\psi(0)\|$, which shows that condition (ii) is satisfied. Hence, by Theorem 2 the delay $\mathcal{FDE}$ has a solution.

3.1. Rational type interpolative $\mathcal{FPT}$ and application to $\mathcal{FDI}$ with integral boundary condition.

Theorem 3. *Consider $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ as a multi-valued mapping on a complete $\mathcal{PMS}$ (Ω, p). For $\kappa \in [0, \frac{1}{2})$ with $0 < \gamma < 1$, if $\mathfrak{T}$ satisfies the following condition:*

$$\begin{aligned}
\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) \leqslant \kappa(\max\{p(\varrho,\vartheta), p(\varrho,\mathfrak{T}\varrho), p(\vartheta,\mathfrak{T}\vartheta)\})^\gamma& \\
\times\left(p(\varrho,\vartheta) + \frac{p(\varrho,\mathfrak{T}\varrho) + p(\vartheta,\mathfrak{T}\vartheta)}{2}\right)^{1-\gamma}& \qquad (11)
\end{aligned}$$

for all $\varrho, \vartheta \in \Omega$, then $\mathcal{F}(\mathcal{T}) \neq \varnothing$.

Proof. For $\varrho_0 \in \Omega$ and $h \in \left(1, \frac{1}{2\kappa}\right)$ with $k \neq 0$ (for $\kappa = 0$, we consider $h > 1$), using Lemma 2, we generate a sequence $\{\varrho_n\}$ in Ω with $\varrho_n \in \mathfrak{T}\varrho_{n-1}$, $n \in \mathbb{N}$, such that

$$p(\varrho_n, \varrho_{n+1}) \leqslant h\mathcal{H}_p(\mathfrak{T}\varrho_{n-1}, \mathfrak{T}\varrho_n).$$

By (11),

$$\begin{aligned} p(\varrho_n, \varrho_{n+1}) &\leqslant h\kappa(\max\{p(\varrho_{n-1}, \varrho_n), p(\varrho_n, \varrho_{n+1})\})^\gamma \\ &\quad \times \left(p(\varrho_{n-1}, \varrho_n) + \frac{p(\varrho_{n-1}, \varrho_n) + p(\varrho_n, \varrho_{n+1})}{2}\right)^{1-\gamma}. \end{aligned} \tag{12}$$

If $p(\varrho_{n-1}, \varrho_n) \leqslant p(\varrho_n, \varrho_{n+1})$, then $\varrho_{n-1} = \varrho_n = \varrho_{n+1}$ is a fixed point of $\mathfrak{T}$. Suppose $p(\varrho_n, \varrho_{n+1}) \leqslant p(\varrho_{n-1}, \varrho_n)$. From (12),

$$\begin{aligned} p(\varrho_n, \varrho_{n+1}) &\leqslant h\kappa(p(\varrho_{n-1}, \varrho_n))^\gamma(2p(\varrho_{n-1}, \varrho_n))^{1-\gamma} \\ &= k^n p(\varrho_0, \varrho_1), \quad \text{where } k = 2^{1-\gamma}h\kappa. \end{aligned}$$

Proceeding in a manner similar to Theorem 1, we can show that $\{\varrho_n\}$ is a Cauchy sequence in (Ω, p), and for some $u \in \Omega$

$$p(u,u) = \lim_{n\to\infty} p(\varrho_n, u) = \lim_{n,\, m\to\infty} p(\varrho_n, \varrho_m) = 0. \tag{13}$$

Now,

$$\begin{aligned} p(u, \mathfrak{T}u) &\leqslant \mathcal{H}_p(u, \varrho_{n+1}) + p(\mathfrak{T}\varrho_n, \mathfrak{T}u) \\ &\leqslant p(u, \varrho_{n+1}) + \kappa(\max\{p(\varrho_n, u), p(\varrho_n, \mathfrak{T}\varrho_n), p(u, \mathfrak{T}u)\})^\gamma \\ &\qquad \left(p(\varrho_n, u) + \frac{p(\varrho_n, \mathfrak{T}\varrho_n) + p(u, \mathfrak{T}u)}{2}\right)^{1-\gamma}. \end{aligned}$$

Taking limit as $n \to \infty$,

$$p(u, \mathfrak{T}u) \leqslant \kappa(p(u, \mathfrak{T}u))^\gamma\left(\frac{1}{2}p(u, \mathfrak{T}u)\right)^{1-\gamma} = \frac{\kappa}{2^{1-\gamma}}p(u, \mathfrak{T}u).$$

So, $p(u, \mathfrak{T}u) = 0$, and thus $u \in \mathfrak{T}u$. □

Example 4. Consider Ω, p and $\mathfrak{T}$ as in Example 1. Then $\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) = \frac{1}{8}p(\varrho, \vartheta)$ and (11) is satisfied for $\kappa \in \left[\frac{1}{8}, \frac{1}{2}\right)$. So, by Theorem 3, $\mathcal{F}(\mathfrak{T}) \neq \varnothing$. Clearly $0 \in \mathcal{F}(\mathfrak{T})$ here.

Example 5. Suppose $\Omega = \{2^{-n} : n = 0,1,2,\dots\} \cup \{0\}$ and $p(\varrho,\vartheta) = |\varrho-\vartheta|+|\varrho|+|\vartheta|$ is such that (Ω, p) is a complete $\mathcal{PMS}$. Consider $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ defined by

$$\mathfrak{T}\varrho = \begin{cases} \{0\}, & \varrho = 0 \\ \{2^{-(n+1)}, 0\}, & \varrho = 2^{-n}, \quad n = 0,1,2,\dots \end{cases}$$

For each $\varrho, \vartheta \in \Omega$, we consider the following cases:
Case 1. When $\varrho = \vartheta = 0$; then condition (11) is trivially true.
Case 2. When $\varrho = 0$, $\vartheta = 2^{-n}$,

$$\sup_{u\in\{0\}} \inf_{v\in\mathfrak{T}2^{-n}} p(u,v) = 0 \quad \text{and} \quad \sup_{v\in\mathfrak{T}2^{-n}} \inf_{u\in\{0\}} p(u,v) = \frac{1}{2^n}.$$

So, $\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) = \frac{1}{2^n}$. Again, $p(\varrho,\vartheta) = 2^{1-n}$, $p(\varrho, \mathfrak{T}\varrho) = 0$ and $p(\vartheta, \mathfrak{T}\vartheta) = 2^{1-n}$. Thus,

$$\begin{aligned} \kappa(\max\{p(\varrho,\vartheta), p(\varrho,\mathfrak{T}\varrho), p(\vartheta,\mathfrak{T}\vartheta)\})^\gamma \Big(p(\varrho,\vartheta) + \frac{p(\varrho,\mathfrak{T}\varrho)+p(\vartheta,\mathfrak{T}\vartheta)}{2}\Big)^{1-\gamma} \\ = \kappa(2^{1-n})^\gamma (2^{1-n}+2^{-n})^{1-\gamma} = \kappa 2^{1-n}\Big(\frac{3}{2}\Big)^{1-\gamma}. \quad (14) \end{aligned}$$

For $\kappa = \frac{9}{20}$, $\gamma = \frac{1}{2}$ and using (14), we see that (11) is satisfied.
Case 3. When $\varrho = \vartheta = 2^{-n}$,

$$\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) = \max\Big\{\frac{1}{2^n}, \frac{1}{2^n}\Big\} = \frac{1}{2^n}.$$

Also, $p(\varrho,\vartheta) = 2^{1-n}$, $p(\varrho,\mathfrak{T}\varrho) = p(\vartheta,\mathfrak{T}\vartheta) = 2^{1-n}$. Now,

$$\begin{aligned} \kappa(\max\{p(\varrho,\vartheta), p(\varrho,\mathfrak{T}\varrho), p(\vartheta,\mathfrak{T}\vartheta)\})^\gamma \Big(p(\varrho,\vartheta) + \frac{p(\varrho,\mathfrak{T}\varrho)+p(\vartheta,\mathfrak{T}\vartheta)}{2}\Big)^{1-\gamma} \\ = \kappa(2^{1-n})^\gamma (2^{2-n})^{1-\gamma} = \kappa 2^{1-n} 2^{1-\gamma}. \quad (15) \end{aligned}$$

Taking $\kappa = \frac{9}{20}$, $\gamma = \frac{1}{2}$ and using (15), condition (11) is satisfied. Thus, all the conditions of Theorem 3 are fulfilled. Clearly, $0 \in \mathcal{F}(\mathcal{T})$ here.

The following interpolative $\mathcal{FPT}$ of Hardy-Rogers type can also be derived.

Theorem 4. *Let $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ be a multi-valued mapping in a complete $\mathcal{PMS}$ (Ω, p). For $\kappa \in [0,1)$ and $\gamma \in (0,1)$, $\mathfrak{T}$ satisfies*

$$\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) \leqslant \kappa(\max\{p(\varrho,\vartheta), p(\varrho,\mathfrak{T}\varrho), p(\vartheta,\mathfrak{T}\vartheta)\})^{\gamma}\left(\frac{p(\varrho,\mathfrak{T}\vartheta)+p(\vartheta,\mathfrak{T}\varrho)}{1+2p(\varrho,\vartheta)}\right)^{1-\gamma}.$$

Then $\mathfrak{T}$ has a fixed point.

The proof of the above result is analogous to Theorem 3.

Now, we apply Theorem 3 to the following nonlinear $\mathcal{FDI}$ with integral boundary conditions.

$$\begin{aligned} &{}^{c}\mathfrak{D}_t^{\omega}\varrho(t) \in \mathfrak{F}(t,\varrho(t)) \quad (0 < t < 1, 1 < \omega \leqslant 2) \\ &\text{with} \quad \varrho(0) = 0, \quad \varrho(1) = \int_0^{\xi} x(s)\,ds \quad (0 < \xi < 1). \end{aligned} \tag{16}$$

Let $\Omega = C([0,1],\mathbb{R})$, which is a complete $\mathcal{PMS}$ with

$$p(\varrho,\vartheta) = \|\varrho - \vartheta\| + \|\varrho\| + \|\vartheta\|,$$

for each $\varrho, \vartheta \in \Omega$, where $\|\varrho\| = \sup_{s\in[0,1]} |\varrho(s)|$. Define $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ by

$$\begin{aligned} \mathfrak{T}\varrho = \bigg\{ u \in \Omega : u(t) &= \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} f(s,\varrho(s))\,ds \\ &\quad - \frac{2t}{(2-\xi^2)\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} f(s,\varrho(s))\,ds \\ &+ \frac{2t}{(2-\xi^2)\Gamma(\omega)} \int_0^{\xi} \left(\int_0^s (s-m)^{\omega-1} f(m,\varrho(m))\,dm \right) ds, \quad t \in [0,1] \bigg\}, \end{aligned} \tag{17}$$

where $f(t,\varrho(t)) \in \mathfrak{F}(t,\varrho(t))$.

Theorem 5. *Consider the $\mathcal{FDI}$ (16). Assume that the following condition holds: for each $g \in \mathfrak{F}(t,\varrho(t))$ and $\varrho, \vartheta \in \Omega$ with $1 < \omega \leqslant 2$, such that $g(m,0) = 0$ and*

$$|g(m,\varrho(m)) - g(m,\vartheta(m))| \leqslant \frac{\Gamma(\omega+1)}{5} |\varrho(m) - \vartheta(m)| \quad \textit{for all } m \in [0,1].$$

Then the system (16) has a solution.

Proof. Let $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ be as given by (17) and $\varrho, \vartheta \in \Omega$. For $u \in \mathfrak{T}\varrho$, $v \in \mathfrak{T}\vartheta$, we have

$$
\begin{aligned}
|u(t) - v(t)| \leqslant \frac{1}{\Gamma(\omega)} \Bigg(\int_0^t |(t-s)^{\omega-1}| \, |f(s, \varrho(s)) - f(s, \vartheta(s))| \, ds \\
+ \frac{2t}{(2-\xi^2)} \int_0^1 (1-s)^{\omega-1} |f(s, \varrho(s)) - f(s, \vartheta(s))| \, ds \\
+ \frac{2t}{(2-\xi^2)} \int_0^\xi \left(\int_0^s |s-m|^{\omega-1} |f(m, \varrho(m)) - f(m, \vartheta(m))| \, dm \right) ds \Bigg) \\
\leqslant \frac{\Gamma(\omega+1)}{5\Gamma(\omega)} \Bigg(\int_0^t |(t-s)^{\omega-1}| \, |\varrho(s) - \vartheta(s)| \, ds + \frac{2t}{2-\xi^2} \int_0^1 (1-s)^{\omega-1} |\varrho(s)) - \vartheta(s)| \, ds \\
+ \frac{2t}{2-\xi^2} \int_0^\xi \left(\int_0^s |s-m|^{\omega-1} |\varrho(m) - \vartheta(m)| \, dm \right) ds \Bigg).
\end{aligned}
$$

Taking sup over $s \in [0,1]$,

$$
\begin{aligned}
|u(t) - v(t)| \leqslant \frac{\Gamma(\omega+1)}{5\Gamma(\omega)} \|\varrho - \vartheta\| \Bigg(\int_0^t |(t-s)^{\omega-1}| \, ds \\
+ \frac{2t}{(2-\xi^2)} \int_0^1 (1-s)^{\omega-1} \, ds + \frac{2t}{(2-\xi^2)} \int_0^\xi \left(\int_0^s |s-m|^{\omega-1} | \, dm \right) ds \Bigg).
\end{aligned}
$$

Again, taking sup over $t \in [0,1]$,

$$
\|u - v\| \leqslant \frac{1}{\Gamma(\omega)} \frac{\Gamma(\omega+1)}{5} \left(\frac{1}{\omega} + \frac{2}{(2-\xi^2)} \frac{1}{\omega} + \frac{2\xi^{\omega+1}}{(2-\xi^2)\omega(\omega+1)} \right) \|\varrho - \vartheta\|,
$$

i. e., $\|u - v\| \leqslant \frac{1}{2} \|\varrho - \vartheta\|$. Also,

$$
\begin{aligned}
|u(t)| \leqslant \frac{1}{\Gamma(\omega)} \Bigg(\int_0^t |(t-s)^{\omega-1}| |f(s,\varrho(s))|\,ds + \frac{2t}{2-\xi^2} \int_0^1 (1-s)^{\omega-1} |f(s,\varrho(s))|\,ds \\
+ \frac{2t}{2-\xi^2} \int_0^\xi \Bigg(\int_0^s |s-m|^{\omega-1} |f(m,\varrho(m))|\,dm \Bigg) ds \Bigg) \\
\leqslant \frac{1}{\Gamma(\omega)} \frac{\Gamma(\omega+1)}{5} \Bigg(\int_0^t |(t-s)^{\omega-1}| |\varrho(s)|\,ds + \frac{2t}{(2-\xi^2)} \int_0^1 (1-s)^{\omega-1} |\varrho(s)|\,ds \\
+ \frac{2t}{2-\xi^2} \int_0^\xi \Bigg(\int_0^s |s-m|^{\omega-1} |\varrho(m)|\,dm \Bigg) ds \Bigg) \\
\leqslant \frac{1}{\Gamma(\omega)} \frac{\Gamma(\omega+1)}{5} \|x\| \Bigg(\int_0^t |(t-s)^{\omega-1}|\,ds + \frac{2t}{(2-\xi^2)} \int_0^1 (1-s)^{\omega-1}\,ds \\
+ \frac{2t}{2-\xi^2} \int_0^\xi \Bigg(\int_0^s |s-m|^{\omega-1}|\,dm \Bigg) ds \Bigg).
\end{aligned}
$$

Taking sup over $s \in [0,1]$ and then over $t \in [0,1]$, we get

$$
\|u\| \leqslant \frac{1}{\Gamma(\omega)} \frac{\Gamma(\omega+1)}{5} \|\varrho\| \Big(\frac{1}{\omega} + \frac{2}{(2-\xi^2)} \frac{1}{\omega} + \frac{2\xi^{\omega+1}}{(2-\xi^2)\omega(\omega+1)} \Big),
$$

i. e., $\|u\| \leqslant \frac{1}{2}\|\varrho\|$. Similarly, $\|v\| \leqslant \frac{1}{2}\|\vartheta\|$. Now,

$$
\begin{aligned}
\mathcal{H}_p(\mathfrak{T}\varrho, \mathfrak{T}\vartheta) \leqslant \sup_{u\in\mathfrak{T}\varrho, v\in\mathfrak{T}\vartheta} p(u,v) \leqslant \frac{1}{2} p(\varrho,\vartheta) \\
\leqslant \frac{1}{2} (\max\{p(\varrho,\vartheta), p(\varrho,\mathfrak{T}\varrho), p(\vartheta,\mathfrak{T}\vartheta)\})^\gamma \Big(p(\varrho,\vartheta) + \frac{p(\varrho,\mathfrak{T}\varrho) + p(\vartheta,\mathfrak{T}\vartheta)}{2} \Big)^{1-\gamma}.
\end{aligned}
$$

Thus, all the conditions of Theorem 3 are satisfied; so, the system (16) has a solution. □

3.2. Rational type periodic point result. This section presents a periodic point result for rational type multi-valued mapping. Here, we require the following definition.

Definition 3. *Let Ω be a non-empty set and $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ be a multi-valued mapping. For $n \in \mathbb{N}$ with $n > 1$, the n-th iterate $\mathfrak{T}^n$ is defined by*

$\mathfrak{T}^n u = \bigcup\limits_{v \in \mathfrak{T}^{n-1}u} \mathfrak{T}v$. *If* $\mathcal{F}(\mathfrak{T}) = \mathcal{F}(\mathfrak{T}^n)$ *for all* $n \in \mathbb{N}$, *then* $\mathfrak{T}$ *is said to have the property* P *(refer to [1]).*

The property P is essential in fixed point theory and nonlinear analysis, as it establishes a strong connection between fixed points and periodic points of a mapping. The merit of property P lies in its ability to detect all periodic points and to establish that there are no nontrivial cycles (refer to [11], [16], [17] for more details).

Theorem 6. *On a complete* $\mathcal{PMS}$ (Ω, p), *let a mapping* $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ *satisfies the condition (4). Then* $\mathfrak{T}$ *has the property* P.

Proof. Suppose $u \in \mathcal{F}(\mathfrak{T}^n)$ for some $n > 1$. We prove that $u \in \mathfrak{T}u$. Now,

$$\begin{aligned} p(\mathfrak{T}u, u) &\leqslant \mathcal{H}_p(\mathfrak{T}\mathfrak{T}^n u, \mathfrak{T}\mathfrak{T}^{n-1} u) \\ &\leqslant \kappa\Big(\max\{\mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n-1} u), \mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n+1} u), \mathcal{H}_p(\mathfrak{T}^{n-1} u, \mathfrak{T}^n u)\} \\ &\qquad\qquad + \frac{\mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n+1} u)\mathcal{H}_p(\mathfrak{T}^{n-1} u, \mathfrak{T}^n u)}{1 + \mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n-1} u)}\Big) \\ &\leqslant \kappa\big(\max\{\mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n+1} u), \mathcal{H}_p(\mathfrak{T}^{n-1} u, \mathfrak{T}^n u)\} + \mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n+1} u)\big). \end{aligned}$$

If $\mathcal{H}_p(\mathfrak{T}^{n-1} u, \mathfrak{T}^n u) \leqslant \mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n+1} u)$, then

$$\mathcal{H}_p(\mathfrak{T}^{n+1} u, \mathfrak{T}^n u) \leqslant 2\kappa \mathcal{H}_p(\mathfrak{T}^{n+1} u, \mathfrak{T}^n u) < \mathcal{H}_p(\mathfrak{T}^{n+1} u, \mathfrak{T}^n u),$$

which is a contradiction. Thus, $\mathcal{H}_p(\mathfrak{T}^{n+1} u, \mathfrak{T}^n u) \leqslant \mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n-1} u)$ and so,

$$\begin{aligned} \mathcal{H}_p(\mathfrak{T}^{n+1} u, \mathfrak{T}^n u) &\leqslant \kappa(\mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n-1} u) + \mathcal{H}_p(\mathfrak{T}^{n+1} u, \mathfrak{T}^n u)) \\ &\leqslant \frac{\kappa}{1-\kappa}\mathcal{H}_p(\mathfrak{T}^n u, \mathfrak{T}^{n-1} u) \\ &\leqslant \Big(\frac{\kappa}{1-\kappa}\Big)^n \mathcal{H}_p(\mathfrak{T}u, u). \end{aligned}$$

Therefore,

$$p(\mathfrak{T}u, u) \leqslant \mathcal{H}_p(\mathfrak{T}^{n+1} u, \mathfrak{T}^n u) \leqslant \Big(\frac{\kappa}{1-\kappa}\Big)^n \mathcal{H}_p(\mathfrak{T}u, u).$$

Taking limit as $n \to \infty$, we get $p(\mathfrak{T}u, u) = 0$, i. e., $u \in \mathfrak{T}u$. $\square$

Example 6. Let $\Omega = [0,1]$ and $p(\varrho,\vartheta) = |\varrho-\vartheta| + |\varrho| + |\vartheta|$, $\varrho,\vartheta \in \mathcal{X}$. Let $\mathfrak{T} : \Omega \to \mathcal{CB}^p(\Omega)$ be defined by

$$\mathfrak{T}\varrho = \begin{cases} \{[0,\frac{\varrho}{5}]\}, & \varrho \in [0,1) \\ \{1\}, & \varrho = 1. \end{cases}$$

For $u \in [0,1)$, $\mathfrak{T}u = \big[0,\dfrac{u}{5}\big]$ and so, $\mathfrak{T}^2 u = \big[0,\dfrac{u}{25}\big]$. Continuing in a similar manner, we get $\mathfrak{T}^n u = \big[0,\dfrac{u}{5^n}\big]$ for all $n \in \mathbb{N}$. Again,

$$\mathcal{H}_p(\mathfrak{T}\varrho,\mathfrak{T}\vartheta) = \frac{1}{5}\max\{\varrho,\vartheta\} = \frac{\beta}{5}, \quad \text{where } \beta = \max\{\varrho,\vartheta\},$$

$p(\varrho,\mathfrak{T}\varrho) = 2\varrho$, $p(\vartheta,\mathfrak{T}\vartheta) = 2\vartheta$ and $p(\varrho,\vartheta) = 2\max\{\varrho,\vartheta\} = 2\beta$. Now, R.H.S. of (4) is

$$\kappa\Big(\max\{2\beta,2\varrho,2\vartheta\} + \frac{4\varrho\vartheta}{1+2\beta}\Big) = 2\kappa\Big(\beta + \frac{2\varrho\vartheta}{1+2\beta}\Big) \geqslant 2\kappa\beta.$$

Hence, Theorem 6 is satisfied for $\kappa \in [\frac{1}{10},\frac{1}{2})$ and so, $\mathfrak{T}$ has property P. Clearly, for all $n \in \mathbb{N}$, $\mathcal{F}(\mathfrak{T}) = \mathcal{F}(\mathfrak{T}^n) = \{0,1\}$ here.

4. Future Problems. In the course of future work, the Ulam-Hyers stability of delay $\mathcal{FDI}$ can be studied with the help of Theorem 3. In 2025, Aldowah et al. [2] established the rational type $\mathcal{FPT}$ to provide an application to the economic growth model. Therefore, to investigate the stability of the economic growth model with delayed investment response or delay in policy implementation via equation (5) is another scope for future study. Furthermore, considering the broad research in fuzzy metric spaces (refer to [15], [16]), the application of rational type contraction conditions in fuzzy multi-valued mappings is a natural extension.

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References

[1] Abbas M., Parvaneh V., Razani A. *Periodic points of T-Ciric generalized contraction mappings in ordered metric spaces.* Georgian Math. J., 2012, vol. 19, no. 4, pp. 597–610.
DOI: https://doi.org/10.1515/gmj-2012-0036

[2] Aldwoah K., Shah S. K., Mustafa A., Almalahi M. A., Hassan M., Alsulami A. *Rational-type fixed-point theorems in b-metric spaces and their application to economic growth and market equilibrium.* Bound. Value Probl., 2025, vol. 2025, no. 1, pp. 11.
DOI: https://doi.org/10.1186/s13661-025-01996-x

[3] Aydi H., Abbas M., Vetro C. *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces.* Topology Appl., 2012, vol. 159, no. 14, pp. 3234 – 3242.
DOI: https://doi.org/10.1016/j.topol.2012.06.012

[4] Bentahar S., Azennar R., Mentagui D. *Application to fractional differential equations via the α-fixed point theorem.* Adv. Fixed Point Theory, 2024, vol. 14, Article ID 47. DOI: https://doi.org/10.28919/afpt/7909

[5] Bukatin M., Kopperman R., Matthews S. G., Pajoohesh H. *Partial metric spaces.* Amer. Math. Monthly, 2009, vol. 116, no. 8, pp. 708 – 718.
DOI: https://doi.org/10.4169/193009709X460831

[6] El-Sayed A., El-Sayed W., Msaik K. M., Ebead H. R. *Fractional Order Delay Differential Equation Constrained by Nonlocal and Weighted Delay Integral Equations.* Int. J. Anal. Appl., 2025, vol. 23, pp. 57 – 57.
DOI: https://doi.org/10.28924/2291-8639-23-2025-57

[7] Johansyah M. D., Supriatna A. K., Rusyaman E., Saputra J. *Application of fractional differential equation in economic growth model: A systematic review approach.* AIMS Math., 2021, vol. 6, no. 9, pp. 10266 – 10280.
DOI: https://doi.org/10.3934/math.2021594

[8] Kebede S. G., Lakoud A. G. *Existence and stability of solution for time-delayed nonlinear fractional differential equations.* Appl. Math. Sci. Eng., 2024, vol. 32, no. 1, pp. 2314649.
DOI: https://doi.org/10.1080/27690911.2024.2314649

[9] Gupta A., Mansotra R. *Fixed-point theorems using interpolative Boyd-Wong-type contractions and interpolative matkowski-type contractions on partial b-metric spaces.* Probl. Anal. Issues Anal., 2025, vol. 14(32), no. 1, pp. 107 – 118. DOI: https://doi.org/10.15393/j3.art.2025.16990

[10] Kilbas A. A., Srivastava H. M., Trujillo J. J. *Theory and applications of fractional differential equations.* North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006, vol. 204.
DOI: https://doi.org/10.1016/S0304-0208(06)80001-0

[11] Kumam P., Rahimi H., Rad G. S. *The existence of fixed and periodic point theorems in cone metric type spaces.* J. Nonlinear Sci. Appl., 2014, vol. 7, no. 4, pp. 255 – 263.
DOI: https://dx.doi.org/10.22436/jnsa.007.04.03

[12] Matthews S. G. *Partial metric topology.* Ann. N. Y. Acad. Sci., 1994, vol. 728, no. 1, pp. 183 – 197.
DOI: https://doi.org/10.1111/j.1749-6632.1994.tb44144.x

[13] Nadler Jr S. B. *Multi-valued contraction mappings.* Pac. J. Math., 1969, vol. 30, no. 2, pp. 475 – 488.
DOI: https://doi.org/10.2140/pjm.1969.30.475

[14] Puvar S. V., Vyas R. G. *Rational type cyclic contraction in G-metric spaces.* Probl. Anal. Issues Anal., 2023, vol. 12(30), no. 3, pp. 119 – 131.
DOI: https://doi.org/10.15393/j3.art.2023.13110

[15] Rafique M., Nazir T., Kalita H. *Fixed points of fuzzy multivalued F-contractive mappings with a directed graph in parametric metric spaces.* Probl. Anal. Issues Anal., 2024, vol. 13(31), no. 3, pp. 79 – 100.
DOI: https://doi.org/10.15393/j3.art.2024.15870

[16] Razani A. *A contraction theorem in fuzzy metric spaces.* Fixed Point Theory Appl., 2005, vol. 2005, no. 3, pp.257 – 265.
DOI: https://doi.org/10.1155/FPTA.2005.257

[17] Roshan J. R., Parvaneh V., Kadelburg Z. *Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces.* J. Nonlinear Sci. Appl., 2014, vol. 7, no. 4, pp. 229 – 245.
DOI: https://dx.doi.org/10.22436/jnsa.007.04.01

[18] Sagheer D., Batul S., Urooj I., Aydi H., Kumar S. *New multivalued F-contraction mappings involving α-admissibility with an application.* J. Inequal. Appl., 2023, vol. 2023, no. 109.
DOI: https://doi.org/10.1186/s13660-023-03016-x

[19] Sagheer D., Qawaqneh H., Batul S., Urooj I., Rahman Z., Aydi H. *Solving Fractional Differential Equations in b-Metric Spaces Tailored with a Directed Graph.* Eur. J. Pure Appl. Math., 2025, vol. 18, no. 4, pp. 6811 – 6811. DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6811

[20] Singh N. K., Ghosh S. *Common fixed point theorems on extended b-metric space using rational-type contraction.* Fixed Point Theory Algorithms Sci. Eng., 2025, vol. 2025, no. 1, pp. 22.
DOI: https://doi.org/10.1186/s13663-025-00804-6

[21] Wang F. F., Chen D. Y., Zhang X. G., Wu Y. *The existence and uniqueness theorem of the solution to a class of nonlinear fractional order system with time delay.* Appl. Math. Lett., 2016, vol. 53, pp. 45 – 51.
DOI: https://doi.org/10.1016/j.aml.2015.10.001

[22] Zaman M. S., Goswami N. *A new variant of Kikkawa-Suzuki type fixed point theorem for multi-valued mappings with stability analysis and application*

to Volterra integral inclusion. TWMS J. Appl. Eng. Math., 2025, vol. 15, no. 2, pp. 486 – 500.

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