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APPROXIMATION RESULTS CONNECTED WITH THE SECOND HANKEL-CLIFFORD TRANSFORM AND IT APPLICATIONS

Abstract. We give direct and inverse approximation theorems for functions defined on the semiaxis and the Hankel-Clifford translation in appropriate L^2 space with weight. There are two forms of inverse theorem in a sharpened Timan-Zygmund form, in L^2 space and in the corresponding Sobolev space. As a corollary, we obtain several equivalence statements generalizing the Titchmarsh equivalence theorem. Some conditions of integrability with a power weight for Hankel-Clifford transform are discussed and their sharpness is proved under some restrictions.

Key words: *second Hankel-Clifford transform, Hankel-Clifford translation, direct and inverse approximation theorems, integrability*

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1. Introduction. Let $f\colon \mathbb{R} \to \mathbb{C}$ be a Lebesgue integrable function on $\mathbb{R}$, i.e., $f \in L^1(\mathbb{R})$. Then the Fourier transform of f is defined by

$$\widehat{f}(x) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}} f(t) e^{-itx}\, dt, \quad x \in \mathbb{R}.$$

It is known that $\widehat{f}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to\pm\infty} \widehat{f}(x) = 0$, i.e., $\widehat{f} \in C_0(\mathbb{R})$.

In the case $1 < p \leqslant 2$, the Fourier transform of a function $f \in L^p(\mathbb{R})$ is defined as a limit of integral $(2\pi)^{-1/2} \int\limits_{-a}^{b} f(x) e^{-itx}\, dx$ in $L^q(\mathbb{R})$, where $1/p + 1/q = 1$ and $a, b \to +\infty$. In particular, $\widehat{f} \in L^q(\mathbb{R})$ and a Hausdorff-Young type inequality

$$\|\widehat{f}\|_q \leqslant C\|f\|_p := C\Big(\int\limits_{\mathbb{R}} |f(t)|^p\, dt \Big)^{1/p}, \quad f \in L^p(\mathbb{R}), \quad 1 < p \leqslant 2, \qquad (1)$$

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holds. For $p = 2$, the Plancherel equality $\|\widehat{f}\|_2 = \|f\|_2$ is valid instead of (1). To find more about these definitions and statements, see [18, Ch. III and IV] or [3, Ch. 5].

If, in addition, $\widehat{f} \in L^1(\mathbb{R})$ and $f \in C(\mathbb{R})$, i.e., f is continuous on $\mathbb{R}$, then the inversion formula

$$f(t) = (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{f}(x) e^{itx}\, dx$$

takes place for all $t \in \mathbb{R}$ (see [3, Ch. 5, p. 192]). In this case, we have $\lim\limits_{x\to\infty} f(x) = 0$, that is $f \in C_0(\mathbb{R})$.

For $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, let us consider the modulus of smoothness of order $k \in \mathbb{N} = \{1,2,\dots\}$

$$\omega_k(f,\delta)_p = \sup_{0\leqslant h\leqslant\delta} \|\mathring{\Delta}_h^k f\|_p, \quad \mathring{\Delta}_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + (k-2j)h/2).$$

If $\omega_1(f,\delta)_p = O(\delta^\alpha)$, $0 < \alpha \leqslant 1$, then we write $f \in Lip(\alpha, p)$. If $f \in C_0(\mathbb{R})$ or f is uniformly continuous and bounded on $\mathbb{R}$, i.e., $f \in BUC(\mathbb{R})$, then by definition

$$\omega_k(f,\delta)_\infty = \sup_{0\leqslant h\leqslant\delta} \|\mathring{\Delta}_h^k f\|_\infty := \sup_{0\leqslant h\leqslant\delta} \sup_{x\in\mathbb{R}} |\mathring{\Delta}_h^k f(x)|.$$

Denote by Φ the set of continuous and increasing on $\mathbb{R}_+ = [0,\infty)$ functions ω, such that $\omega(0) = 0$. If $\omega \in \Phi$ and $\int\limits_0^\delta t^{-1}\omega(t)\, dt = O(\omega(\delta))$, $\delta \geqslant 0$, then ω belongs to the Bary class B; if $\omega \in \Phi$ and $\delta^m \int\limits_\delta^\infty t^{-m-1}\omega(t)\, dt = O(\omega(\delta))$ for some $m > 0$ and all $\delta > 0$, then ω belongs to the Bary-Stechkin class B_m (see [1]). We say that $\omega \in \Phi$ satisfies the Δ_2-condition if $\omega(2x) \leqslant C\omega(x)$, $x \in \mathbb{R}_+$.

If $\sigma \geqslant 0$, $f(z)$ is an entire function, and for any $\varepsilon > 0$ there exists $A(\varepsilon)$, such that $|f(z)| \leqslant Ae^{(\sigma+\varepsilon)|z|}$ for all $z \in \mathbb{C}$, then $f(z)$ belongs to the class E_σ. We introduce the best approximation of order $\sigma > 0$ in $L^p(\mathbb{R})$ by formula $A_\sigma(f)_p = \inf\{\|f - g\|_p \colon g \in L^p(\mathbb{R}) \cap E_\sigma\}$. The next result was obtained by Bernstein in the case $p = \infty$, $m = 1$, and by Akhiezer in the case $1 \leqslant p < \infty$, $m = 1, 2$. The following result see in the monograph of Timan [17, Ch. 5, § 5.1].

Theorem 1. *Let $1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$, $f \in L^p(\mathbb{R})$. Then*

$$A_\sigma(f)_p \leqslant C\omega_m(f, 1/\sigma)_p, \quad \sigma > 0.$$

The inverse approximation theorems also is connected with names of Bernstein and Timan, see [17, Ch. 6, section 6.1.6].

Theorem 2. *Let $1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$, $f \in L^p(\mathbb{R})$. Then*

$$\omega_m(f, 1/n)_p \leqslant Cn^{-m} \sum_{i=0}^{n} (i+1)^{k-1} A_i(f)_p, \quad n \in \mathbb{N}.$$

The following integrability result of Titchmarsh is well known, see [18, Ch. 4, Theorem 84]. It is an analog of Szász conditions for generalized absolute convergence of Fourier series, see [15].

Theorem 3. *Let $1 < p \leqslant 2$, $0 < \alpha \leqslant 1$, $f \in Lip(\alpha, p)$. Then $\widehat{f}(t) \in L^\beta(\mathbb{R})$ for all β satisfying the inequality*

$$\frac{p}{p + \alpha p - 1} < \beta \leqslant q = \frac{p}{p-1}.$$

More recent results on weighted integrability of Fourier transform see in [9] and [7].

The aim of the present paper is to obtain analogues and generalizations of Theorems 1 and 2 for the second Hankel-Clifford transform in weighted L^2 space. Also, the equivalence of modulus of smoothness and K-functional is obtained. These results are applied to problems of weighted integrability of cited above transform with power weight. The corresponding sufficient conditions were obtained earlier in [22] and [23]. Here we give them in another form and study their sharpness.

There are known many works with direct and inverse approximation theorems connected with famous integral transforms, see, e.g., the papers of Tyr, Daher et al [19], [20], [21]. In the last paper, Steklov means are used for construction of a modulus of smoothness and this is a special case due to the nature of Mellin transform. In [20], the direct approximation theorem is given together with the equivalence of modulus of smoothness and appropriate K-functional. In [19], the direct and inverse theorems are obtained for the modulus of smoothness constructed by q-Dunkl transform. The main differences of the present paper from the cited above ones are: (a) we consider a modulus of smoothness of order $m > 0$ instead of natural

m; (b) inverse theorems are given for an integrable with weight function and for function from a Sobolev space in sharpened Timan-Zygmund form; (c) a general form of Titchmarsh equivalence theorems, see Corollary 2 and Corollary 5, are obtained as consequence of approximation results.

2. Definitions. Let $1 \leqslant p < \infty$, $\mu \geqslant 0$, $\mathbb{R}_+ = [0, +\infty)$, and $L^p_\mu(\mathbb{R}_+)$ be the space of all real-valued measurable functions, such that $\|f\|_{L^p_\mu} = \Big(\int\limits_0^\infty |f(x)|^p x^\mu \, dx \Big)^{1/p} < \infty$. If χ_E is the indicator of a set $E \subset \mathbb{R}_+$ and $f\chi_E \in L^p_\mu(\mathbb{R}_+)$, then $f \in L^p_\mu(E)$. The space $L^\infty_\mu(\mathbb{R}_+)$ coincides with the space of bounded measurable functions with the norm $\|f\|_{L^\infty_\mu} = \|f\|_\infty = \sup_{x\in\mathbb{R}_+} |f(x)|$.

The normalized Bessel function $j_\nu(x)$ of the first kind and order $\nu > -1/2$ given by

$$j_\nu(x) = \Gamma(\nu+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} (x/2)^{2n},$$

where $\Gamma(\alpha)$, $\alpha > 0$, is the Euler gamma function. Then

$$c_\mu(x) = \Gamma^{-1}(\mu+1) j_\mu(2\sqrt{x}) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu+k+1)}, \quad x \geqslant 0, \tag{2}$$

is called the Bessel-Clifford function of the first kind of order $\mu \geqslant 0$, see, e.g., [5].

It is easy to check that $c_\mu(x)$ is a solution of the differential equation $xy'' + (\mu+1)y' + y = 0$.

Hayek [6] introduced the second Hankel-Clifford transform of $f \in L^1_\mu(\mathbb{R}_+)$ by

$$h_{2,\mu}(f)(y) = \int\limits_0^{+\infty} c_\mu(yx) f(x) x^\mu \, dx.$$

Using (2) and Lemma 2, we have $|c_\mu(x)| \leqslant \Gamma^{-1}(\mu+1)$, $x \in \mathbb{R}_+$, and

$$\|h_{2,\mu}(f)\|_{L^\infty_\mu} \leqslant \Gamma^{-1}(\mu+1) \|f\|_{L^1_\mu}, \quad f \in L^1_\mu(\mathbb{R}_+). \tag{3}$$

For $\mu \geqslant 0$, the transform $h_{2,\mu}$ can be extended from $L^1_\mu(\mathbb{R}_+) \cap L^2_\mu(\mathbb{R}_+)$ onto $L^2_\mu(\mathbb{R}_+)$ and the Plancherel type equality

$$\|h_{2,\mu}(f)\|_{L^2_\mu} = \|f\|_{L^2_\mu}, \quad f \in L^2_\mu(\mathbb{R}_+), \tag{4}$$

holds; see [6] or [8]. Moreover, for $f \in L^2_\mu(\mathbb{R}_+)$, we have $h_{2,\mu}(h_{2,\mu}(f)) = f$. Using Riesz-Thorin interpolation theorem, see [2, Ch. 1, Theorem 1.1.1]), we obtain a Hausdorff-Young type inequality for $f \in L^p_\mu(\mathbb{R}_+)$, $1 < p \leqslant 2$,

$$\|h_{2,\mu}(f)\|_{L^q_\mu} \leqslant C\|f\|_{L^p_\mu}, \quad q = p/(p-1). \tag{5}$$

Let $\Delta(x,y,z)$ be the area of triangle with sides x, y, z, if the triangle exists, and zero in other cases. For $\mu \geqslant 0$ we set

$$D_\mu(x,y,z) = \frac{\Delta^{2\mu+1}(x,y,z)}{2^{2\mu}(xyz)^\mu \Gamma(\mu+\frac{1}{2})\sqrt{\pi}},$$

when $\Delta(x,y,z) > 0$ and $D_\mu(x,y,z) = 0$ otherwise. Then $D_\mu(x,y,z)$ is non-negative and symmetric in x, y, z. Prasad, Singh, and Dixit [12] defined the generalized Hankel-Clifford translation of $f \in L^1_\mu(\mathbb{R}_+)$ by

$$T_x(f)(y) = \int_0^{+\infty} f(z) D_\mu(x,y,z) z^\mu dz, \quad 0 < x, y < \infty.$$

We set also $T_0 f(y) = f(y)$. Using Lemma 1.3 from [12], we have

$$h_{2,\mu}(T_x(f))(y) = c_\mu(xy) h_{2,\mu}(f)(y), \quad f \in L^1_\mu(\mathbb{R}_+), \quad y \geqslant 0. \tag{6}$$

By Lemma 2.3 in [22], this result is also valid for $f \in L^p_\mu(\mathbb{R}_+)$, $1 < p \leqslant 2$, and a.e. $y \in \mathbb{R}_+$. Now we can introduce the difference of order $m > 0$ with step $t > 0$ by

$$\Delta^m_{t,\mu,hc} f(x) = (I - \Gamma(\mu+1)T_t)^m f(x) = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} \Gamma^i(\mu+1) T_t^i f(x),$$

where I is the identical operator, $\binom{m}{i} = m(m-1)\dots(m-i+1)/i!$ for $i \in \mathbb{N}$ and $\binom{m}{0} = 1$, and the modulus of smoothness of order m in $L^p_\mu(\mathbb{R}_+)$, $1 \leqslant p < \infty$, by

$$\omega_m(f,\delta)_{p,\mu,hc} = \sup_{0 \leqslant t \leqslant \delta} \|\Delta^m_{t,\mu,hc} f\|_{L^p_\mu}.$$

Since the series $\sum\limits_{i=0}^{\infty} |\binom{m}{i}|$ converges, by Lemma 1, for $1 \leqslant p < \infty$, $\mu \geqslant 0$, and $\delta \geqslant 0$, we have $\omega_m(f,\delta)_{p,\mu,hc} \leqslant C\|f\|_{L^p_\mu}$.

Let $S(0,+\infty)$ be the set of all infinitely differentiable functions $\psi(x)$ on $(0,+\infty)$, such that

$$\rho_{m,k}(\psi) = \sup_{0<x<\infty} |x^m \psi^{(k)}(x)| < \infty$$

for all $m,k \in \mathbb{Z}_+ = \{0,1,\dots\}$. In [8], the authors proved that $h_{2,\mu}$ is an automorphism of $S(0,+\infty)$. Let $B_\mu(\psi) = x\psi'' + (\mu+1)\psi$ and $\psi \in S(0,+\infty)$. Then

$$h_{2,\mu}(B_\mu^j(\psi))(y) = (-y)^i h_{2,\mu}(\psi)(y), \quad y>0, \quad j \in \mathbb{N}, \tag{7}$$

see [8, Proposition 6] or (1.14) in [12].

Now we can consider the dual space $S'(0,v+\infty)$ consisting of linear continuous functionals on $S(0,v+\infty)$. By definition,

$$\langle B_\mu^j f, \psi\rangle = \langle f, B_\mu^j \psi\rangle, \quad f \in S'(0,v+\infty), \quad \psi \in S(0,v+\infty).$$

For $\mu \geqslant 0$, $1 \leqslant p < \infty$, and $m \in \mathbb{N}$, we define the Sobolev space $W_{p,\mu}^m(\mathbb{R}_+)$ that consists of all $f \in L_\mu^p(\mathbb{R}_+) \subset S'(0,v+\infty)$, such that $B_\mu^j f \in L_\mu^p(\mathbb{R}_+)$ for $j = 1,2,\dots,m$.

Also, we consider the space $S_e(\mathbb{R}_+)$ as the space of $\varphi X_{[0,+\infty)}$, where φ are even Schwartz functions. Then $S_e(\mathbb{R}_+) \subset S(0,\infty)$ and $S_e(\mathbb{R}_+)$ is dense in all $L_\mu^p(\mathbb{R}_+)$, $1 \leqslant p < \infty$. Using the density arguments, we extend (7) to the case of $\psi \in W_{p,\mu}^m(\mathbb{R}_+)$, $m \in \mathbb{N}$, $1 \leqslant p < \infty$, $i = 1,2,\dots,m$.

By the analogy with the minimal property of Fourier series with respect to an orthonormal system, we define the best approximation in $L_\mu^2(\mathbb{R}_+)$ by

$$E_\sigma(f)_{2,\mu,hc} = \Big(\int_\sigma^\infty |h_{2,\mu}(f)(y)|^2 y^\mu \, dy \Big)^{1/2}.$$

Let us consider also $P_\sigma(f) = h_{2,\mu}(h_{2,\mu}(f)\chi_{[0,\sigma]})$, where $\sigma > 0$ and χ_E is the indicator of a set E. Since the function $j_\mu(\sqrt{x}\cdot)$ can be presented as power series, it is infinitely differentiable. Therefore, one can see that $P_\sigma(f)(x)$ is also infinitely an differentiable function. As it is noted above, $h_{2,\mu}^{-1}(f) = h_{2,\mu}(f)$ for $f \in L_\mu^2(\mathbb{R}_+)$. Therefore, $h_{2,\mu}(P_\sigma(f)) = h_{2,\mu}(f)X_{[0,\sigma]}$ for $f \in L_\mu^2(\mathbb{R}_+)$.

We define a K-functional of order $m \in \mathbb{N}$ for $f \in L_\mu^2(\mathbb{R}_+)$ as follows:

$$K_m(f,t)_{2,\mu,hc} = \inf\{\|f-g\|_{L_\mu^2} + t\|B_\mu^m(g)\|_{L_\mu^2} : g \in W_{2,\mu}^m(\mathbb{R}_+)\}.$$

3. Auxiliary propositions. Lemma 1 was proved by Prasad and Singh [13, Lemma 1.1].

Lemma 1. *Let $1 \leqslant p < \infty$, $\mu \geqslant 0$, $f \in L^p_\mu(\mathbb{R}_+)$. Then*

$$\|\Gamma(\mu+1)T_t(f)\|_{L^p_\mu} \leqslant \|f\|_{L^p_\mu}.$$

Lemma 2 was proved by Platonov, see [11] and [10], the left-hand inequality (iii) see in [24].

Lemma 2. *Let $\mu \geqslant 0$. Then*
(i) $|j_\mu(x)| \leqslant 1$ for $x \geqslant 0$ and $j_\mu(x) < 1$ for $x > 0$;
(ii) $1 - j_\mu(x) \geqslant C > 0$ for $x \geqslant 1$;
(iii) for some $C_2 > C_1 > 0$ and all $x \in [0,1]$ the inequality

$$C_1 x^2 \leqslant 1 - j_\mu(x) \leqslant C_2 x^2$$

holds.

Lemma 3 follows from (6) and (7), see also [23, Lemma 3].

Lemma 3. *Let $1 \leqslant p \leqslant 2$, $\mu \geqslant 0$, $f \in L^p_\mu(\mathbb{R}_+)$, $m > 0$, $t \geqslant 0$. Then*

$$h_{2,\mu}(\Delta^m_{t,\mu,hc} f)(y) = (1 - j_\mu(2\sqrt{yt}))^m h_{2,\mu}(f)(y) \quad \text{for a.e.} \quad y \in \mathbb{R}_+.$$

For $k \in \mathbb{N}$ and $f \in W^k_{p,\mu}(\mathbb{R}_+)$, we have

$$h_{2,\mu}(\Delta^m_{t,\mu,hc} B^k_\mu(f))(y) = (1 - j_\mu(2\sqrt{yt}))^m (-y)^k h_{2,\mu}(f)(y)$$

for a.e. $y \in \mathbb{R}_+$.

Lemma 4. *(i) Let $r \in \mathbb{N}$, $f \in W^r_{2,\mu}(\mathbb{R}_+)$. Then*

$$\omega_r(f,\delta)_{2,\mu,hc} \leqslant C\delta^r \|B^r_\mu(f)\|_{L^2_\mu}, \quad \delta \geqslant 0.$$

(ii) If $m, r \in \mathbb{N}$, $m > r$, $f \in W^r_{2,\mu}(\mathbb{R}_+)$, then

$$\omega_m(f,\delta)_{2,\mu,hc} \leqslant C\delta^r \omega_{m-r}(B^r_\mu(f),\delta)_{2,\mu,hc}, \quad \delta \geqslant 0.$$

Proof. (i) By Lemmas 3, 2, (7) and (4), we have

$$\|\Delta^r_{t,\mu,hc} f\|^2_{L^2_\mu} = \int_0^\infty (1 - j_\mu(2\sqrt{yt}))^{2r} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy$$
$$\leqslant \int_0^{1/(4t)} C_1 (yt)^{2r} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy + \int_{1/(4t)}^\infty 2^{2r} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy$$

$$\leqslant C_2 \int_0^\infty (yt)^{2r} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy$$

$$= C_2 t^{2r} \int_0^\infty |h_{2,\mu}(B_\mu^r(f))(y)|^2 y^\mu \, dy = C_2 t^{2r} \|B_\mu^r(f)\|_{L_\mu^2}^2. \quad (8)$$

From (8) we deduce the result of (i).

(ii) Using the statement of (i), we obtain

$$\|\Delta_{t,\mu,hc}^m f\|_{L_\mu^2}^2 = \|\Delta_{t,\mu,hc}^r(\Delta_{t,\mu,hc}^{m-r} f)\|_{L_\mu^2}^2 \leqslant C_2 T^{2r} \|B_\mu^r(\Delta_{t,\mu,hc}^{m-r} f)\|_{L_\mu^2}^2$$

$$= C_2 t^{2r} \|\Delta_{t,\mu,hc}^{m-r}(B_\mu^r(f))\|_{L_\mu^2}^2 \leqslant C_2 t^{2r} \omega_{m-r}(B_\mu^r(f), t)_{2,\mu,hc}.$$

From the last inequality, we deduce the result of (ii). □

Lemma 5 may be found in [22].

Lemma 5. *Let $\mu \geqslant 0$, $m > 0$, and $G(t)$ be a non-negative measurable function on $\mathbb{R}_+$, such that*

$$\int_y^\infty G(t) t^\mu \, dt = O(\omega(1/y)), \quad y > 0. \quad (9)$$

(i) If $\omega \in B_m$, then $t^{m+\mu} G(t) \in L_{loc}^1(\mathbb{R}_+)$ and

$$\int_0^y t^m G(t) t^\mu \, dt = O(y^m \omega(1/y)), \quad y > 0. \quad (10)$$

(ii) If $\omega \in B \cap \Delta_2$, $t^{m+\mu} G(t) \in L_{loc}^1(\mathbb{R}_+)$ and (10) holds, then (9) is valid.

Lemma 6 can be derived from Lemma 3 in [1].

Lemma 6. *Let $\omega \in \Phi$ and $m > 0$. If $\omega \in B_m$, then*

$$\sum_{k=1}^n k^{m-1} \omega(k^{-1}) = O(n^m \omega(n^{-1})), \quad n \in \mathbb{N}. \quad (11)$$

Conversely, if (11) holds and ω is bounded on $\mathbb{R}_+$, then $\omega \in B_m$. Also, if $\omega \in B_m$, then ω satisfies the Δ_2-condition.

4. Approximation results in $L^2_\mu(\mathbb{R}_+)$.

Theorem 4. *Let $\mu \geqslant 0$, $f \in L^2_\mu(\mathbb{R}_+)$, $m > 0$. Then the Jackson inequality*

$$E_\sigma(f)_{2,\mu,hc} \leqslant C\omega_m(f,\sigma^{-1})_{2,\mu,hc}, \quad \sigma > 0,$$

holds.

Proof. By the definition of best approximation, equality (4), and Lemmas 2(ii), 3, we have, for $\sigma > 0$:

$$E_\sigma(f)_{2,\mu,hc} = \Big(\int_\sigma^\infty |h_{2,\nu}(f)(y)|^2 y^\mu\, dy\Big)^{1/2}$$

$$\leqslant C_1\Big(\int_\sigma^\infty (1-2\sqrt{y/\sigma})^{2m}|h_{2,\nu}(f)(y)|^2 y^\mu\, dy\Big)^{1/2}$$

$$\leqslant C_1\Big(\int_0^\infty \sigma(1-2\sqrt{y/\sigma})^{2m}|h_{2,\nu}(f)(y)|^2 y^\mu\, dy\Big)^{1/2}$$

$$= C_1\|h_{2,\mu}(\Delta^m_{\sigma^{-1},\mu,hc}f)\|_{L^2_\mu} = C_1\|\Delta^m_{\sigma^{-1},\mu,hc}f\|_{L^2_\mu} \leqslant C_1\omega_m(f,\sigma^{-1})_{2,\mu,hc}.$$

□

Corollary 1. *Let $\mu \geqslant 0$, $m, r \in \mathbb{N}$, $f \in W^r_{2,\mu}(\mathbb{R}_+)$. Then*

$$E_\sigma(f)_{2,\mu,hc} \leqslant \frac{C}{\sigma^r}\omega_m(B^r_\mu f,\sigma^{-1})_{2,\mu,hc}, \quad \sigma > 0.$$

Proof. By Theorem 4 and Lemma 4, we have

$$E_\sigma(f)_{2,\mu,hc} \leqslant C_1\omega_{m+r}(f,\sigma^{-1})_{2,\mu,hc} \leqslant \frac{C_2}{\sigma^r}\omega_m(B^r_\mu f,\sigma^{-1})_{2,\mu,hc}, \quad \sigma > 0.$$

□

Theorem 5 is a sharpened version of the inverse approximation theorem in $L^2_\mu(\mathbb{R}_+)$ of Timan-Zygmund type, see [4, Ch. 7, § 3, Theorem 3.4].

Theorem 5. *Let $\mu \geqslant 0$, $m > 0$, $f \in L^2_\mu(\mathbb{R}_+)$. Then*

$$\omega_m(f,n^{-1})_{2,\mu,hc} \leqslant Cn^{-m}\Big(\sum_{j=1}^n j^{2m-1}E^2_{j-1}(f)_{2,\mu,hc}\Big)^{1/2}, \quad n \in \mathbb{N}.$$

Proof. By (4) and Lemma 3, we have

$$\begin{aligned}\|\Delta^m_{t,\mu,hc}f\|^2_{L^2_\mu} &= \|h_{2,\mu}(\Delta^m_{t,\mu,hc}f)\|^2_{L^2_\mu}\\ &= \int_0^\infty (1-j_\mu(2\sqrt{yt}))^{2m}|h_{2,\mu}(f)(y)|^2 y^\mu\,dy\\ &= \Big(\int_0^n + \int_n^\infty\Big)(1-j_\mu(2\sqrt{yt}))^{2m}|h_{2,\mu}(f)(y)|^2 y^\mu\,dy = I_1+I_2. \qquad (12)\end{aligned}$$

Using Lemma 2 (i), we obtain

$$I_2 \leqslant \sum_n^\infty 2^{2m}|h_{2,\mu}(f)(y)|^2 y^\mu\,dy = 2^{2m}E_n^2(f)_{2,\mu,hc}. \qquad (13)$$

If $0 \leqslant t \leqslant n^{-1}$, $y \in [0,n]$, then $2\sqrt{yt} \in [0,2]$. Since the function $(1-j_\mu(x))/x^2$ is continuous and bounded from above on $[1,2]$, the right-hand inequality of Lemma 2 (iii) is valid also for $x \in [0,2]$, and

$$\begin{aligned}I_1 &\leqslant C_1\sum_{k=1}^n \int_{k-1}^k (yt)^{2m}|h_{2,\mu}(f)(y)|^2 y^\mu\,dy\\ &\leqslant \frac{C_1}{n^{2m}}\sum_{k=1}^n k^{2m}\Big(\int_{k-1}^\infty - \int_k^\infty\Big)|h_{2,\mu}(f)(y)|^2 y^\mu\,dy\\ &= \frac{C_1}{n^{2m}}\sum_{k=1}^n k^{2m}(E_{k-1}(f)_{2,\mu,hc}-E_k(f)_{2,\mu,hc})\\ &= \frac{C_1}{n^{2m}}\Big(E_0^2(f)_{2,\mu,hc}+\sum_{k=1}^{n-1}((k+1)^{2m}-k^{2m})E_k^2(f)_{2,\mu,hc}-\\ &\quad -n^{2m}E_n^2(f)_{2,\mu,hc}\Big) \leqslant \frac{C_2}{n^{2m}}\sum_{k=1}^n k^{2m-1}E_{k-1}^2(f)_{2,\mu,hc}. \qquad (14)\end{aligned}$$

Since $\sum_{k=1}^n k^{2m-1} \leqslant C_3 n^{2m}$, $n \in \mathbb{N}$, we find that

$$I_2 \leqslant C_4 n^{-2m} \times \sum_{k=1}^n k^{2m-1}E_{k-1}^2(f)_{2,\mu,hc}$$

and (12), (13), and (14) imply the inequality of Theorem 5. □

From Theorems 4 and 5 we deduce a sharpened version of a Titchmarsh equivalence-type result from [23, Theorem 4].

Corollary 2. *Let $\mu \geqslant 0$, $m > 0$, $f \in L^2_\mu(\mathbb{R}_+)$ and $\omega \in B_m$. Then the conditions*

(i) $\omega_m(f,\delta)_{2,\mu,hc} = O(\omega(\delta))$, $\delta \geqslant 0$;

and

(ii) $E_N(f)_{2,\mu,hc} = O(\omega(N^{-1}))$, $N > 0$,

are equivalent.

If $\omega \in B \cap B_m$, then the conditions (i), (ii),

(iii) $\int\limits_0^N y^{2m}|h_{2,\mu}(f)(y)|^2 y^\mu\,dy = O(N^{2m}\omega^2(N^{-1}))$, $N > 0$,

and

(iv) $\int\limits_N^{2N} |h_{2,\mu}(f)(y)|^2 y^\mu\,dy = O(\omega^2(N^{-1}))$, $N > 0$,

are also equivalent.

Proof. The implication (i)⇒ (ii) follows from Theorem 4. By Theorem 5 and the condition $\omega^2 \in B_{2m}$ equivalent to $\omega \in B_m$, see [16, Corollary 4.10], from (ii) we deduce $\omega_m(f,\delta)_{2,\mu,hc} \leqslant C_1\omega(\delta)$ for $\delta = n^{-1}$; $n \in \mathbb{N}$. But $\omega \in B_m$ satisfies Δ_2-condition and, for $t \in [(n+1)^{-1}, n^{-1}]$, $n \in \mathbb{N}$, one has

$$\omega_m(f,t)_{2,\mu,hc} \leqslant \omega_m(f,n^{-1})_{2,\mu,hc} \leqslant C_1\omega(n^{-1}) \leqslant C_1\omega(2/(n+1))$$
$$\leqslant C_2\omega((n+1)^{-1}) \leqslant C_2\omega(t).$$

The last inequality holds for all $t \in [0,1]$. If $\omega_m(f,1)_{2,\mu,hc} = 0$, then (i) is obvious; elsewhere, by Lemma 1,

$$\omega_m(f,t)_{2,\mu,hc} \leqslant \frac{\sum\limits_{i=0}^{\infty} |\binom{m}{i}|\|f\|_{L^2_\mu}}{\omega_m(f,1)_{2,\mu,hc}}\omega_m(f,1)_{2,\mu,hc} \leqslant C_3\omega_m(f,t)_{2,\mu,hc}$$

for $t \geqslant 1$ and (i) follows from (ii).

Since (ii) can be written as

$$\int\limits_N^\infty |h_{2,\mu}(f)(y)|^2 y^\mu\,dy = O(\omega^2(N^{-1})), \quad N > 0, \tag{15}$$

and, by [16, Corollary 4.10], the conditions $\omega \in B \cap B_m$ and $\omega^2 \in B \cap B_{2m}$ are equivalent, we conclude that (15)⇔(iii) due to Lemma 5.

Finally, (15)⇒(iv) and the inverse implication is stated as in the proof of [23, Theorem 4] with help of the condition $\omega^2 \in B$. □

Theorem 6 is an analog of Nikol'skii-Stechkin inequality for trigonometric polynomials and entire functions of exponential type, see [17, section 4.8.61].

Theorem 6. *Let $\mu \geqslant 0$, $m \in \mathbb{N}$, $f \in L^2_\mu(\mathbb{R}_+), \sigma > 0$. Then*

$$\|B^m_\mu(P_\sigma(f))\|_{L^2_\mu} \leqslant C\sigma^m \omega_m(f, \sigma^{-1})_{2,\mu,hc}.$$

Proof. From definition of P_σ and its properties, see section 2, it is easy to see that $B^m_\mu(P_\sigma(f))$ exists in the usual sense and belongs to $L^2_\mu(\mathbb{R}_+)$. Then by (7) we have:

$$\|B^m_\mu(P_\sigma(f))\|_{L^2_\mu} = \|h_{2,\mu}(B^m_\mu(P_\sigma(f)))\|_{L^2_\mu} = \|(\cdot)^m X_{[0,\sigma]} h_{2,\mu}(f)\|_{L^2_\mu}$$

$$= \left\| \frac{(\cdot)^m X_{[0,\sigma]}}{(1 - j_\mu(2\sqrt{\cdot/\sigma}))^m} (1 - j_\mu(2\sqrt{\cdot/\sigma}))^m h_{2,\mu}(f) \right\|_{L^2_\mu}. \tag{16}$$

For $x \in [0, \sigma]$, we have $0 \leqslant 2\sqrt{x/\sigma} \leqslant 2$. By the part (iii) of Lemma 2, we have $(1 - j_\nu(y))/y^2 \geqslant C_1 > 0$ for $y \in (0, 1]$. Then the continuous function $(1 - j_\nu(y))/y^2$ has the minimal value on $[1, 2]$, and, by Lemma 2 (i), this value is positive. Therefore, the left-hand inequality of Lemma 2 (iii) is valid for $x \in [0, 2]$ and

$$\frac{x^m}{(1 - j_\mu(2\sqrt{x/\sigma}))^m} \leqslant \frac{x^m}{C_2(x/\sigma)^m} = C_3\sigma^m, \quad x \in (0, \sigma].$$

From (16) and Lemma 3, we deduce that

$$\|B^m_\mu(P_\sigma(f))\|_{L^2_\mu} \leqslant C_3\sigma^m \|(1 - j_\mu(2\sqrt{\cdot/\sigma}))^m h_{2,\mu}(f)\|_{L^2_\mu}$$

$$= C_3\sigma^m \|h_{2,\mu}(\Delta^m_{\sigma^{-1},\mu,hc} f)\|_{L^2_\mu} = C_3\sigma^m \|\Delta^m_{\sigma^{-1},\mu,hc} f\|_{L^2_\mu}$$

$$\leqslant C_3\sigma^m \omega_m(f, \sigma^{-1})_{2,\mu,hc}.$$

□

Corollary 3. *Under conditions of Theorem 6, we have*

$$\|B^m_\mu(P_\sigma(f))\|_{L^2_\mu} \leqslant C\sigma^m \|f\|_{L^2_\mu}$$

From Theorems 4 and 6, we derive the equivalence of modulus of smoothness and K-functional of order $m \in \mathbb{N}$.

Theorem 7. *Let $\mu \geqslant 0$, $m \in \mathbb{N}$, $f \in L^2_\mu(\mathbb{R}_+)$. Then there exist $C_2 > C_1 > 0$, such that*

$$C_1\omega_m(f,t)_{2,\mu,hc} \leqslant K_m(f,t^m)_{2,\mu,hc} \leqslant C_2\omega_m(f,t)_{2,\mu,hc}, \quad t \geqslant 0. \tag{17}$$

Proof. In the proof of Theorem 6, we state that $P_\sigma(f) \in W^m_{2,\mu}(\mathbb{R}_+)$ for all $\sigma > 0$ and $m \in \mathbb{N}$. Also, it is easy to see that

$$E^2_\sigma(f)_{2,\mu,hc} = \int_\sigma^\infty |h_{2,\mu}(f)(y)|^2 y^\mu \, dy \tag{18}$$

$$= \|h_{2,\mu}(f - P_\sigma(f))\|_{L^2_\mu} = \|f - P_\sigma(f)\|_{L^2_\mu}. \tag{19}$$

Therefore, by Theorems 4 and 6 we have:

$$K_m(f,t^m)_{2,\mu,hc} \leqslant \|f - P_{1/t}(f)\|_{L^2_\mu} + t^m\|B^m_\mu(P_{1/t}(f))\|_{L^2_\mu}$$

$$\leqslant C_1(\omega(f,t)_{2,\mu,hc} + t^m(1/t)^m\omega(f,t)_{2,\mu,hc}) = 2C_1\omega(f,t)_{2,\mu,hc}$$

and the right-hand inequality (17) is proved.

The inverse estimate is obtained in a standard way. For $g \in W^m_{2,\mu}(\mathbb{R}_+)$, by Lemma 4,

$$\omega_m(f,t)_{2,\mu,hc} \leqslant \omega_m(f-g,t)_{2,\mu,hc} + \omega_m(g,t)_{2,\mu,hc}$$

$$\leqslant 2^m\|f-g\|_{L^2_\mu} + t^m\|B^m_\mu(g)\|_{L^2_\mu}.$$

Taking the infimum over $g \in W^m_{2,\mu}(\mathbb{R}_+)$, we derive the left-hand inequality (17). □

Corollary 4. *Let $\mu \geqslant 0$, $m \in \mathbb{N}$, $f \in L^2_\mu(\mathbb{R}_+)$. Then, for $\lambda \geqslant 1$ and $t \geqslant 0$, we have $\omega_m(f,\lambda t)_{2,\mu,hc} \leqslant C\lambda^m\omega_m(f,t)_{2,\mu,hc}$.*

Theorem 8 shows the important property of approximating aggregate and is an analog of the famous Stechkin result about trigonometrical polynomial of the best approximation, see [14].

Theorem 8. *Let $\mu \geqslant 0$, $m \in \mathbb{N}$, $f \in L^2_\mu(\mathbb{R}_+)$. Then, for any $\sigma > 0$, the inequality*

$$\omega(P_\sigma(f),t)_{2,\mu,hc} \leqslant C\omega(f,t)_{2,\mu,hc}, \quad t \geqslant 0. \tag{20}$$

Proof. By the definition of modulus of smoothness, (19), and Theorem 4, we have:

$$\omega(P_\sigma(f),t)_{2,\mu,hc} \leqslant \omega(f-P_\sigma(f),t)_{2,\mu,hc} + \omega(f,t)_{2,\mu,hc} \leqslant 2^m E_\sigma(f)_{2,\mu,hc}$$

$$+\omega(f,t)_{2,\mu,hc} \leqslant C_1\omega(f,\sigma^{01})_{2,\mu,hc} + \omega(f,t)_{2,\mu,hc}.$$

By the last inequality, we obtain (20) for $t \geqslant \sigma^{-1}$. If $t < \sigma^{-1}$, then, by Lemma 4, Theorem 6 and Corollary 4, we find that

$$\omega(P_\sigma(f),t)_{2,\mu,hc} \leqslant C_2\|B_\mu^m(P_\sigma(f))\|_{2,\mu,hc}t^m \leqslant C_3(\sigma t)^m\omega_m(f,\sigma^{-1})_{2,\mu,hc}$$

$$\leqslant C_4(\sigma t)^m(\sigma^{-1}/t)\omega_m(f,t)_{2,\mu,hc} = C_4\omega_m(f,t)_{2,\mu,hc}.$$

Theorem is proved. □

Now we prove a sharpened version on the inverse approximation theorem for functions from the Sobolev space.

Theorem 9. *Let $\mu \geqslant 0$, $r \in \mathbb{N}$, $f \in W_{2,\mu}^r(\mathbb{R}_+)$. Then*

$$E_\sigma^2(B_\mu^r(f))_{2,\mu,hc} \leqslant C\Big(\sigma^{2r}E_\sigma^2(f)_{2,\mu,hc} + \sigma\sum_{j=2}^{\infty}(j\sigma)^{2r-1}E_{j\sigma}^2(f)_{2,\mu,hc}\Big).$$

Proof. By the definition of best approximation in $L_\mu^2(\mathbb{R}_+)$ and (7), we have

$$E_\sigma^2(B_\mu^r(f))_{2,\mu,hc} = \int_\sigma^\infty |h_{2,\mu}(B_\mu^r(f))(y)|^2 y^\mu\,dy$$

$$= \int_\sigma^\infty y^{2r}|h_{2,\mu}(f)(y)|^2y^\mu\,dy = \sum_{j=1}^\infty \int_{j\sigma}^{(j+1)\sigma} y^{2r}|h_{2,\mu}(f)(y)|^2y^\mu\,dy$$

$$\leqslant \sum_{j=1}^\infty((j+1)\sigma)^{2r}(E_{j\sigma}^2(f)_{2,\mu,hc} - E_{(j+1)\sigma}^2(f)_{2,\mu,hc})$$

$$= 2^{2r}\sigma^{2r}E_\sigma^2(f)_{2,\mu,hc} + \sum_{j=2}^\infty((j+1)^{2r} - j^{2r})\sigma^{2r}E_{j\sigma}^2(f)_{2,\mu,hc}$$

$$\leqslant C_1\Big(\sigma^{2r}E_\sigma^2(f)_{2,\mu,hc} + \sum_{j=2}^\infty j^{2r-1}\sigma^{2r}E_{j\sigma}^2(f)_{2,\mu,hc}\Big). \tag{21}$$

To justify the summation by parts in (21), we set $A_j = E^2_{j\sigma}(f)_{2,\mu,hc}$ and write

$$\sum_{j=1}^{N}(j+1)^{2r}(A_j - A_{j+1}) = 2^{2r}A_1 + \sum_{j=2}^{N}((j+1)^{2r} - j^{2r})A_j - (N+1)^{2r}A_{N+1}. \tag{22}$$

But we can estimate as in (21):

$$\infty > E^2_{\sigma}(B^r_{\mu}(f))_{2,\mu,hc} \geqslant \sum_{j=1}^{\infty}(j\sigma)^{2r}(A_j - A_{j+1}).$$

Thus, the left-hand side of (22) converges as $N \to +\infty$. This implies that

$$(N+1)^{2r}A_{N+1} = \sum_{j=N+1}^{\infty}(N+1)^{2r}(A_j - A_{j+1}) \leqslant \sum_{j=N+1}^{\infty}(j+1)^{2r}(A_j - A_{j+1})$$

and the right-hand side tends to zero as $N \to +\infty$. Therefore, from (22) we deduce that

$$\sum_{j=1}^{\infty}(j+1)^{2r}(A_j - A_{j+1}) = 2^{2r}A_1 + \sum_{j=2}^{\infty}((j+1)^{2r} - j^{2r})A_j$$

and (21) holds. □

Corollary 5. *Let $\mu \geqslant 0$, $m,r \in \mathbb{N}$, $\omega \in B_m \cap B$ and $f \in W^r_{2,\mu}(\mathbb{R}_+)$. Then the conditions*

$$\omega_m(B^r_{\mu}(f),t)_{2,\mu,hc} = O(\omega(t)), \quad t \geqslant 0; \tag{23}$$

$$E_{\sigma}(f)_{2,\mu,hc} = O(\sigma^{-r}\omega(\sigma^{-1})), \quad \sigma > 0; \tag{24}$$

$$\int_t^{\infty} y^{2r}|h_{2,\mu}(y)|^2 y^{\mu}\,dy = O(\omega^2(t^{-1})), \quad t > 0; \tag{25}$$

$$\int_0^{t} y^{2m+2r}|h_{2,\mu}(y)|^2 y^{\mu}\,dy = O(t^{2m}\omega^2(t^{-1})), \quad t > 0; \tag{26}$$

$$\omega_{m+r}(f,t)_{2,\mu,hc} = O(t^r\omega(t), \quad t \geqslant 0; \tag{27}$$

are equivalent.

Proof. By Corollary 2 and (7), the conditions (23), (25), and (26) are equivalent. By Corollary 1, we have (23)⇒(24), while, by Lemma 4, (23)⇒(27). By Theorem 5, Lemma 6, and the condition $\omega^2 \in B_{2m}$, from (24) we deduce that

$$\omega_{m+r}(f, n^{-1})_{2,\mu,hc} \leqslant \frac{C_1}{n^{m+r}} \Big(\sum_{j=1}^{n} j^{2(m+r)-1} j^{-2r} \omega^2(j^{-1}) + \|f\|_{L^2_\mu} \Big)^{1/2}$$
$$\leqslant \frac{C_2}{n^{m+r}} (n^{2m} \omega^2(n^{-1}))^{1/2} = C_2 n^{-r} \omega(n^{-1}).$$

As in Corollary 2, we obtain that (27) holds and (24)⇒(27). But, by Theorem 4, (27)⇒(24). Finally, it is easy to see that $\omega \in B_m$ implies $t^r \omega(t) \in B_{m+r}$, i.e., $\varphi(t) = t^{2r}\omega^2(t)$ belongs to $B_{2(m+r)}$ and, by Corollary 2, (27)⇒(26)⇒(23). We conclude that all five conditions are equivalent. □

5. Some remarks on weighted integrability of second Hankel-Clifford transform. In [22, Corollary 4.2], the following results was obtained.

Theorem 10. *Let $1 < p \leqslant 2$, $1/p + 1/q = 1$, $\mu \geqslant 0$, $f \in L^p_\mu(\mathbb{R}_+)$, $r \in (0, q)$, $m > 0$. If $\alpha > (r/q - 1)(\mu + 1)$ and the integral*

$$\int_1^\infty y^{\alpha - r(\mu+1)/q} \omega_m^r(f, y^{-1})_{p,\mu,hc} y^\mu \, dy$$

converges, then $y^\alpha |h_{2,\mu}(f)(y)|^r \in L^1_\mu(\mathbb{R}_+)$.

In the case $p = 2$, we can obtain similar result for best approximations.

Theorem 11. *Let $\mu \geqslant 0$, $f \in L^2_\mu(\mathbb{R}_+)$, $r \in (0, 2)$, $\alpha > (r/2 - 1)(\mu + 1)$ and the integral*

$$\int_1^\infty y^{\alpha - r(\mu+1)/2} E_y^r(f)_{2,\mu,hc} y^\mu \, dy$$

is finite. Then $y^\alpha |h_{2,\mu}(f)(y)|^r \in L^1_\mu(\mathbb{R}_+)$.

Proof. By definition of the best approximations, we have:

$$\int_{2^j}^{2^{j+1}} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy \leqslant \int_{2^j}^{\infty} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy = E^2_{2^j}(f)_{2,\mu,hc}.$$

Using the above inequality and the Hölder inequality with exponents $2/r$ and $2/(2-r)$, we obtain

$$\int_{2^j}^{2^{j+1}} y^\alpha |h_{2,\mu}(f)(y)|^r y^\mu\,dy \leqslant \Big(\int_{2^j}^{2^{j+1}} |h_{2,\mu}(f)(y)|^2 y^\mu\,dy\Big)^{r/2} \times$$
$$\times \Big(\int_{2^j}^{2^{j+1}} y^{2\alpha/(2-r)} y^\mu\,dy\Big)^{1-r/2} \leqslant C_1 2^{j(\alpha+(\mu+1)(1-r/2)))} E^r_{2^j}(f)_{2,\mu,hc}$$
$$\leqslant C_2 \int_{2^{j-1}}^{2^j} y^{\alpha-r(\mu+1)/2} E^r_y(f)_{2,\mu,hc} y^\mu\,dy. \tag{28}$$

Summing up (28) over $j = 1, 2, \ldots$, we obtain

$$\int_2^\infty y^\alpha |h_{2,\mu}(f)(y)|^r y^\mu\,dy \leqslant C_2 \int_1^\infty y^{\alpha-r(\mu+1)/2} E^r_y(f)_{2,\mu,hc} y^\mu\,dy. \tag{29}$$

On the other hand, we have

$$\int_0^2 y^\alpha |h_{2,\mu}(f)(y)|^r y^\mu\,dy \leqslant \Big(\int_0^2 |h_{2,\mu}(f)(y)|^2 y^\mu\,dy\Big)^{r/2} \times$$
$$\times \Big(\int_0^2 y^{2\alpha/(2-r)} y^\mu\,dy\Big)^{1-r/2} \leqslant C_3 \|h_{2,\mu}(f)\|_{L^2_\mu} < \infty, \tag{30}$$

if $2\alpha + (2-r)(\mu+1) > 0$, i.e., for $\alpha > (r/2-1))\mu + 1$. From (28), (29), and (30), we deduce the result of theorem. □

Theorem 12 shows that Theorem 11 is sharp.

Theorem 12. *Let $\mu \geqslant 0$, $r \in (0,2)$, $\alpha > (r/2-1)(\mu+1)$ and $\omega \in B \cap \Delta_2$ be such that*

$$\int_1^\infty y^{\alpha-r(\mu+1)/2} \omega^r(y^{-1}) y^\mu\,dy = \infty.$$

Then there exists $f_0 \in L^2_\mu(\mathbb{R}_+)$, such that $E_y(f_0)_{2,\mu,hc} \leqslant C\omega(y^{-1})$, $y \geqslant 1$, but $y^\alpha |h_{2,\mu}(f_0)(y)|^r \notin L^1_\mu(\mathbb{R}_+)$.

Proof. Let us consider the Dirichlet-Hankel-Clifford kernels

$$D_{y.hc}(x) = \int_0^y c_\mu(xt) f(t) t^\mu \, dt = h_{2,\mu}(X_{[0,y]})(x).$$

By the Plancherel equality (4), we have

$$\|D_{y,hc}\|^2_{L^2_\mu} = \|X_{[0,y]}\|^2_{L^2_\mu} = \int_0^y t^\mu \, dt = (\mu+1)^{-1} y^{\mu+1}.$$

Let us define

$$f_0 = \sum_{j=1}^{\infty} \omega(2^{-j})(D_{2^j,hc} - D_{2^{j-1},hc}) 2^{-j(\mu+1)/2}.$$

Then $h_{2,\mu}(f_0) = \sum\limits_{j=1}^{\infty} \omega(2^{-j}) 2^{-j(\mu+1)/2} X_{[2^{j-1},2^j]}$ in $L^2_\mu(\mathbb{R}_+)$ and

$$E_{2^n}(f_0)_{2,c} = \Big\| \sum_{j=n+1}^{\infty} \omega(2^{-j}) 2^{-j(\mu+1)/2} X_{[2^{j-1},2^j]} \Big\|_{L^2_\mu} \leqslant$$

$$\leqslant C_1 \sum_{j=n+1}^{\infty} \omega(2^{-j}) 2^{-j(\mu+1)/2} 2^{j(\mu+1)/2} \leqslant C_2 \omega(2^{-n-1})$$

due to the condition $\omega \in B$, see [16, Lemma 4.5].

For $y \in [2^n, 2^{n+1}]$, $n \in \{0, 1, \dots\}$, we find that

$$E_y(f_0)_{2,\mu,hc} \leqslant E_{2^n}(f_0)_{2,\mu,hc} \leqslant C_2 \omega(2^{-n-1}) \leqslant C_2 \omega(y^{-1}).$$

On the other hand,

$$\int_1^\infty y^\alpha |h_{2,\mu}(f_0)(y)|^r y^\mu \, dy = \sum_{j=1}^{\infty} \int_{2^{j-1}}^{2^j} y^\alpha \omega^r(2^{-j}) 2^{-jr(\mu+1)/2} y^\mu \, dy$$

$$\geqslant \sum_{j=1}^{\infty} \int_{2^{j-1}}^{2^j} y^\alpha \omega^r(y^{-1}) y^{-r(\mu+1)/2} y^\mu \, dy = \int_1^\infty y^{\alpha - r(\mu+1)/q} \omega^r(y^{-1}) y^\mu \, dy$$

due to the condition $\omega \in \Delta_2$. Since the last integral is equal to infinity, Theorem 12 is proved. □

Also, we can show the sharpness of Theorem 10.

Corollary 6. *Let $\mu \geqslant 0$, $r \in (0,2)$, $m > 0$, $\alpha > (r/2-1)(\mu+1)$ and $\omega \in B \cap B_m$ be such that*

$$\int_1^\infty y^{\alpha - r(\mu+1)/2} \omega^r(y^{-1}) y^\mu \, dy = \infty.$$

Then there exists $f_0 \in L^2_\mu(\mathbb{R}_+)$, such that $\omega_m(f_0,\delta)_{2,\mu,hc} = O(\omega(\delta))$, $\delta \geqslant 0$, but $y^\alpha |h_{2,\mu}(f_0)(y)|^r \notin L^1_\mu(\mathbb{R}_+)$.

Proof. We consider the same function f_0 as in the proof of Theorem 12. Then $y^\alpha |h_{2,\mu}(f_0)(y)|^r \notin L^1_\mu(\mathbb{R}_+)$, $E_y(f_0)_{2,\mu,hc} \leqslant C_1 \omega(y^{-1})$, $y \geqslant 1$, and $E_y(f_0)_{2,\mu,hc} = E_1(f_0)_{2,\mu,hc} \leqslant C_1 \omega(y^{-1})$ for $0 < y < 1$. By Corollary 2, we conclude that $\omega_m(f_0,\delta)_{2,\mu,hc} = O(\omega(\delta))$, $\delta \geqslant 0$. □

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