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P. Parida, B. B. Jena, S. K. Paikray

DEGREE APPROXIMATION OF SIGNALS OF CERTAIN LIPSCHITZ CLASS BY THE PRODUCT OF NATARAJAN-EULER SUMMABILITY OF ITS FOURIER SERIES

Abstract. In this paper, we explore the Natarajan-Euler product summability method and its applications to Fourier series. We establish several novel results concerning the degree of approximation of Fourier series for functions belonging to the classes $Lip(\alpha, r)$ and $Lip(\zeta(t), r)$, where $r \geqslant 1$, and $0 < \alpha \leqslant 1$. The $Lip(\alpha, r)$-class consists of functions satisfying the Lipschitz condition characterized by α and r, while $Lip(\zeta(t), r)$ extends this to variable moduli of smoothness $\zeta(t)$. Using the Natarajan-Euler product means, we develop new bounds and approximation properties, demonstrating the effectiveness of this summability technique in refining Fourier approximations for these function classes. The results contribute to the deeper understanding of the interplay between summability methods and functional smoothness, offering insights into both theoretical and applied aspects of harmonic analysis and approximation theory.

Key words: *Natarajan-Euler summability method; Product summability; Lipschitz class; Fourier series*

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1. Introduction. The approximation of Fourier series plays a pivotal role in Mathematics and Engineering, offering significant utility in diverse fields, such as signal analysis, image processing, and system design in modern telecommunications. This classical area of harmonic analysis has seen substantial growth since the early works of Fourier, who introduced trigonometric series as a tool to solve heat conduction problems. Over time, researchers extended his ideas to study convergence and approximation properties for various types of functions. The introduction of summability methods, such as Cesàro, Abel, and Euler, means further enhanced the understanding of Fourier series convergence and divergence, providing more general and practical tools for approximating functions.

Modern developments have focused on the degree of approximation of Fourier series and conjugate Fourier series for 2π periodic functions in various functional spaces. These include Lipschitz, Hölder, Besov, and Zygmund spaces (see [2], [3], [4], [7], [9], [10], and [14]), which generalize smoothness and regularity conditions. Summability techniques have become indispensable in this context, offering refined convergence results and error bounds. Studies have demonstrated their applications in signal processing, where Fourier methods aid in filtering and reconstruction, and in image processing for feature extraction and noise reduction (see [5], [8], [11], [12], [13], [15]).

In line with these advancements, this paper investigates the degree of approximation of Fourier series for functions in the Lipschitz class $Lip(\alpha, r)$. Using the Natarajan-Euler product means, we establish new results that bridge summability techniques and functional smoothness. These findings contribute to the growing body of research that connects classical Fourier analysis with contemporary applications and theory.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. The transformation

$$t_n = \sum_{k=0}^{n} \lambda_{n-k} s_k = \sum_{k=0}^{n} \lambda_k s_{n-k}$$

defines the sequence $\{t_n\}$ of Natarajan means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{\lambda_n\}$, where λ_n is non-negative real or complex numbers, such that $\sum |\lambda_n| < \infty$. If $t_n \to s$, as $n \to \infty$, where s is a finite number, then we say that the series $\sum a_n$ is summable by the Natarajan method or (M, λ_n)-summable to the sum s. It is known that the necessary and sufficient condition for the regularity of Natarajan method (M, λ_n) is that $\sum \lambda_n = 1$.

Let

$$T_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k.$$

Then $\{T_n\}$ defines the (E, q), the Euler method of sequence $\{s_n\}$. If $T_n \to s$, as $n \to \infty$, where s is a finite number, then we say that the series $\sum a_n$ is summable by the Euler method (E, q) to s. Clearly (E, q) method is regular.

Now, we define the (M,λ_n) mean of the (E,q) transform or the product Natarajan-Euler method $(M,\lambda_n)(E,q)$ transform of the partial sum s_n of the series $\sum a_n$. If τ_n is the n-th $(M,\lambda_n)(E,q)$ transform of the series $\sum a_n$, then

$$\tau_n = \sum_{k=0}^{n} \lambda_k T_{n-k} = \sum_{k=0}^{n} \lambda_k \frac{1}{(1+q)^{n-k}} \sum_{\nu=0}^{n-k} \binom{n-k}{\nu} s_\nu.$$

If $\tau_n \to s$, as $n \to \infty$, where s is a finite number, then the series $\sum a_n$ is said to be $(M,\lambda_n)(E,q)$ product summable to the sum s.

Let $f(t)$ be the 2π-periodic function that is L-integrable over the interval $(0,2\pi)$. Then the Fourier series of $f(t)$ is given by

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \equiv \sum_n A_n(x). \tag{1}$$

The p-norm $\|.\|_p$ of a function of $f \in L^p[0,2\pi]$ is defined as

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx \right)^{\frac{1}{p}}, \quad p \geqslant 1.$$

The degree of approximation of a function $f \in L^p$, usually denoted by $E_n(f)$, is given by

$$E_n(f) = \min_{p_n} \|p_n - f\|_p,$$

where p_n is a trigonometric polynomial of degree n, called a trigonometric approximation of f.

For $0 < \alpha \leqslant 1$, if

$$|f(x+t) - f(x)| = \mathcal{O}(|t|^\alpha),\ t > 0,$$

then the function f is said to be of class $Lip\ \alpha$ or $Lip(\alpha)$.

If, for $0 < \alpha \leqslant 1, r \geqslant 1,\ t > 0$,

$$\int_0^{2\pi} (|f(x+t) - f(x)|^r dx)^{\frac{1}{r}} = \mathcal{O}(|t|^\alpha),$$

then the function f is said to be of class $Lip(\alpha, r)$.

We use the following notation throughout this paper:

$$\phi(x,t) = f(x,t) + f(x-t) - 2f(x)$$

and

$$s_n(f;x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin(n+\frac{1}{2})u}{\sin(\frac{1}{2}u)} f(x+u)du.$$

The n-th partial sum of the Fourier series (1) and

$$K_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} \frac{\lambda_k}{(1+q)^{n-k}} \sum_{\nu=0}^{n-k} \binom{n-k}{\nu} q^{n-k-\nu} \frac{\sin(\nu+\frac{1}{2})t}{\sin(\frac{t}{2})}.$$

The motivation for this study stems from the need to refine approximation techniques for Fourier series, particularly for functions in the $Lip(\alpha, r)$ and $Lip(\zeta(t), r)$ classes. Traditional methods often struggle with capturing the nuanced smoothness properties of these functions. By employing the Natarajan-Euler product summability method, this research addresses these limitations, offering sharper bounds and enhanced approximation results. This approach bridges gaps in harmonic analysis and provides a robust framework for studying summability's impact on Fourier approximations in both theory and applications.

2. Known Results. Dealing with Degree of approximation of Fourier series of a function of class $Lip(\alpha, r)$, $r \geqslant 1$ and $Lip(\zeta(t), r)$, $r \geqslant 1$ by $(M, \lambda_n)(E, 1)$ transform, Das et al. [1] proved the following results.

Theorem 1. *let (M, λ_n) be a regular Natarajan method defined by a positive sequence $\{\lambda_n\}$, such that*

$$(n+1)\lambda_n = \mathcal{O}(1) \;\; \text{and} \;\; \sum_{k=0}^{n-1}(\Delta\lambda_k) = \mathcal{O}\Big(\frac{1}{n+1}\Big). \tag{2}$$

Let $f \in L^p[0, 2\pi]$ belong to the class $Lip(\alpha, r)$, $r \geqslant 1$; then the degree of approximation of f by $(M, \lambda_n)(E, 1)$ transform

$$t_n^{ME} = \sum_{k=0}^{n} \lambda_k \frac{1}{2^{n-k}} \sum_{\nu=0}^{n-k} \binom{n-k}{\nu} s_\nu$$

of its Fourier series satisfies

$$\|t_n^{ME} - f\|_L^r(\alpha) = \mathcal{O}\Big(\frac{1}{(n+1)^{\alpha-1}}\Big) + \begin{cases} \mathcal{O}\Big(\frac{1}{(n+1)^\alpha}\Big), 0 < \alpha < 1, \\ \mathcal{O}\Big(\frac{log(n+1)}{(n+1)}\Big), \alpha = 1, \end{cases} \quad n = 0, 1, \ldots$$

Theorem 2. *Let (M,λ_n) be a regular Natarajan method defined by a positive sequence $\{\lambda_n\}$ satisfying condition (2). Let $f \in L^p[0,2\pi]$ belong to the class $Lip(\zeta(t),r)$, $r \geqslant 1$; then the degree of approximation of f by t_n^{ME} of its Fourier series (1) is given by*

$$\|t_n^{ME}-f\|_L^r(\zeta)=\mathcal{O}\Big((n+1)^2\int_0^{\frac{\pi}{n+1}}\zeta(t)dt\Big)+\mathcal{O}\Big(\frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\zeta(t)}{t^2}dt\Big).$$

3. Main Result. In this paper, we have established Theorems on Degree of approximation of the function belonging to the class $Lip(\alpha,r)$ and $Lip(\zeta(t),r)$, $r \geqslant 1$ of the Fourier series (1) by the product Natarajan-Euler mean in the following form.

Theorem 3. *Let f be a 2π-periodic L-integrable and belong to the class $Lip(\alpha,r)$, $r \geqslant 1$. Let (M,λ_n) be a regular Natarajan method defined by a positive sequence $\{\lambda_n\}$ satisfying condition (2). Then the degree of approximation of the Fourier series (1) by the product $(M,\lambda_n)(E,q)$ summability mean is given by*

$$\|\tau_n-f\|_L^r(\alpha)=\mathcal{O}\Big(\frac{1}{(n+1)^{\alpha-1}}\Big),\ 0<\alpha\leqslant 1. \tag{3}$$

Theorem 4. *Let (M,λ_n) be a regular Natarajan method defined by a positive sequence $\{\lambda_n\}$ satisfying condition (2). Let f be a 2π-periodic, L-integrable, and belong to the class $Lip(\zeta(t),r)$, $r \geqslant 1$. Then the degree of approximation of f by the product $(M,\lambda_n)(E,q)$ method of Fourier series (1) is given by*

$$\|\tau_n-f\|_L^r(\zeta)\ =\ \mathcal{O}\Big((n+1)^2\int_0^{\frac{\pi}{n+1}}\zeta(t)dt\Big)+\mathcal{O}\Big((n+1)\int_{\frac{\pi}{n+1}}^{\pi}\frac{\zeta(t)}{t}dt\Big). \tag{4}$$

4. Required Lemmas. We need the following lemmas for the proof of the Theorem 3 and Theorem 4.

Lemma 1. *For $0<t\leqslant\dfrac{\pi}{n+1}$, $|K_n(t)|=\mathcal{O}\big((n+1)^2\big)$.*

Proof. We have, for $0\leqslant t\leqslant\dfrac{\pi}{n+1}$: $\sin nt\leqslant nt$ and $\sin\Big(\dfrac{t}{2}\Big)\geqslant\dfrac{t}{\pi}$. So,

$$
\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi}\Bigg|\sum_{k=0}^{n}\frac{\lambda_k}{(1+q)^{n-k}}\Bigg\{\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\frac{\sin(\nu+\frac{1}{2})t}{\sin(\frac{t}{2})}\Bigg\}\Bigg|\\
&\leqslant \frac{1}{2\pi}\sum_{k=0}^{n}\frac{|\lambda_k|}{(1+q)^{n-k}}\Bigg\{\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\frac{(2\nu+1)\frac{t}{2}}{t}\Bigg\}\\
&\leqslant \frac{1}{4}\sum_{k=0}^{n}|\lambda_k|(2(n-k)+1)\frac{1}{(1+q)^{n-k}}\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu} = \mathcal{O}(n+1)^2.
\end{aligned}
$$

This completes the proof. □

Lemma 2. *For* $\frac{\pi}{n+1} \leqslant t \leqslant \pi$, $|K_n(t)| = \mathcal{O}\Big(\frac{n+1}{t}\Big)$.

Proof. For $\frac{\pi}{n+1} \leqslant t \leqslant \pi$, we have $\sin\left(\frac{t}{2}\right) \geqslant \frac{t}{\pi}$, $\sin nt \leqslant 1$ and

$$
\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi}\Bigg|\sum_{k=0}^{n}\frac{\lambda_k}{(1+q)^{n-k}}\Bigg\{\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\frac{\sin(\nu+\frac{1}{2})t}{\sin(\frac{t}{2})}\Bigg\}\Bigg|\\
&\leqslant \frac{1}{2\pi}\sum_{k=0}^{n}\frac{|\lambda_k|}{(1+q)^{n-k}}\Bigg\{\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\left(\frac{\pi}{t}\right)\Bigg\}\\
&= \frac{1}{2t}\sum_{k=0}^{n}\frac{|\lambda_k|}{(1+q)^{n-k}}\Bigg\{\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\Bigg\} = \frac{1}{2t}\sum_{k=0}^{n}|\lambda_k| = \mathcal{O}\Big(\frac{n+1}{t}\Big).
\end{aligned}
$$

The proof is completed. □

Lemma 3. [6] *Let* $f \in Lip(\alpha, r)$, $0 < \alpha \leqslant 1$, $r \geqslant 1$; *then*

$$\left(\int_0^{2\pi}|f(x,t)|^r dx\right)^{\frac{1}{r}} = \mathcal{O}(|t|^{\alpha}).$$

Lemma 4. [1] *Let* $f \in Lip(\zeta(t), r)$, $r \geqslant 1$; *then*

$$\left(\int_0^{2\pi}|\phi(x,t)|^r dx\right)^{\frac{1}{r}} = \mathcal{O}(\zeta(t)).$$

5. Proof of the Main Theorems.

Proof of Theorem 3. Using the Riemann-Lebesgue theorem for the n-th partial sum $s_n(f,x)$ of the Fourier series (1) and following Titchmarsh [16], we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(x,t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt. \tag{5}$$

If T_n denotes the n-th (E,q) mean of $s_n(f;x)$, then we have

$$\begin{aligned} T_n - f(x) &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(s_k(f;x)-f(x)) \\ &= \frac{1}{2\pi(1+q)^n}\int_0^\pi \phi(x,t)\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{\sin(k+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}dt. \end{aligned}$$

If τ_n denotes the Natarajan (M,λ_n) mean of $\{T_n\}$, then we have

$$\begin{aligned} \tau_n - f(x) &= \sum_{k=0}^{n}\lambda_k\frac{1}{2\pi(1+q)^{n-k}}\int_0^\pi \phi(x,t)\left\{\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\frac{\sin(\nu+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}dt \\ &= \frac{1}{2\pi}\int_0^\pi \phi(x,t)\left\{\sum_{k=0}^{n}\frac{\lambda_k}{(1+q)^{n-k}}\left(\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\frac{\sin(\nu+\frac{1}{2})t}{\sin(\frac{t}{2})}\right)\right\}dt \\ &= \int_0 \pi\phi(x,t)K_n(t)dt, \end{aligned}$$

where

$$K_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n}\frac{\lambda_k}{(1+q)^{n-k}}\left\{\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}q^{n-k-\nu}\frac{\sin(\nu+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}.$$

Now,

$$\|\tau_n - f\|_L^r(\alpha) = \left(\int_0^{2\pi}|\tau_n - f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_0^{2\pi}\left|\int_0^\pi \phi(x,t)K_n(t)\,dt\right|^r dx\right)^{\frac{1}{r}}$$

$$\leqslant \int_0^\pi\left(\int_0^{2\pi}|\phi(x,t)|^r\,dx\right)^{\frac{1}{r}}|K_n(t)|dt \quad \text{(by generalized Minkowski inequality)}$$

$$= \int_0^{\pi} \mathcal{O}(t^{\alpha})|K_n(t)|\,dt \quad \text{(using Lemma 4.3)}$$

$$= \mathcal{O}\Big(\int_0^{\frac{\pi}{n+1}} t^{\alpha}|K_n(t)|dt\Big) + \mathcal{O}\Big(\int_{\frac{\pi}{n+1}}^{\pi} t^{\alpha}|K_n(t)|dt\Big)$$

$$= \mathcal{O}\Big((n+1)^2\int_0^{\frac{\pi}{n+1}} t^{\alpha}dt\Big) + \mathcal{O}\Big((n+1)\int_{\frac{\pi}{n+1}}^{\pi} \frac{t^{\alpha}}{t}dt\Big) \text{(using Lemmas 4.1 and 4.2)}$$

$$= \mathcal{O}\Big(\frac{(n+1)^2}{(n+1)^{\alpha+1}}\Big) + \mathcal{O}\Big(\frac{n+1}{(n+1)^{\alpha}}\Big) = \mathcal{O}\Big(\frac{1}{(n+1)^{\alpha-1}}\Big).$$

Thus

$$\|\tau_n - f\|_L^r(\alpha) = \mathcal{O}\Big(\frac{1}{(n+1)^{\alpha-1}}\Big),\ 0 < \alpha \leqslant 1.$$

The proof is completed.

Proof of Theorem 4. If f is a class of $Lip(\zeta(t), r)$, $r \geqslant 1$, then we have

$$\|\tau_n - f(x)\|_L^r(\zeta) = \Big(\int_0^{2\pi} |\tau_n - f(x)|^r dx\Big)^{\frac{1}{r}} = \Big(\int_0^{2\pi}\Big|\int_0^{\pi} \phi(x,t)K_n(t)dt\Big|^r dx\Big)^{\frac{1}{r}}$$

$$\leqslant \int_0^{\pi}\Big(\int_0^{2\pi} |\phi(x,t)|^r\,dx\Big)^{\frac{1}{r}}|K_n(t)|\,dt \text{ (by generalized Minkowski inequality)}$$

$$= \int_0^{\pi} \mathcal{O}(\zeta(t))|K_n(t)|\,dt \quad \text{(using Lemma 4.4)}$$

$$= \mathcal{O}\Big(\int_0^{\frac{\pi}{n+1}} (\zeta(t)|K_n(t)|dt\Big) + \mathcal{O}\Big(\int_{\frac{\pi}{n+1}}^{\pi} \zeta(t)|K_n(t)|dt\Big)$$

$$= \mathcal{O}\Big(\int_0^{\frac{\pi}{n+1}} \zeta(t)(n+1)^2dt\Big) + \mathcal{O}\Big(\int_{\frac{\pi}{n+1}}^{\pi} \zeta(t)(\frac{n+1}{t})dt\Big)$$

$$= \mathcal{O}\left((n+1)^2 \int_0^{\frac{\pi}{n+1}} \zeta(t)dt\right) + \mathcal{O}\left((n+1) \int_{\frac{\pi}{n+1}}^{\pi} \frac{\zeta(t)}{t}dt\right).$$

Thus, we have

$$\|\tau_n - f(x)\|_L^r(\zeta) = \mathcal{O}\left((n+1)^2 \int_0^{\frac{\pi}{n+1}} \zeta(t)dt\right) + \mathcal{O}\left((n+1) \int_{\frac{\pi}{n+1}}^{\pi} \frac{\zeta(t)}{t}dt\right).$$

This completes the proof.

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Priyadarsini Parida
Department of Mathematics, Kuntala Kumari Sabat Women's College,
Balasore 756003, Odisha, India
E-mail: priyadarsiniparida1@gmail.com

Bidu Bhusan Jena
Faculty of Science (Mathematics), Sri Sri University
Cuttack 754006, Odisha, India
E-mail: bidumath.05@gmail.com

Susanta Kumar Paikray
Department of Mathematics, Veer Surendra Sai University of Technology
Burla 768018, Odisha, India
E-mail: skpaikray_math@vssut.ac.in