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ON A THEOREM OF TUEN-WAI NG

Abstract. In 1981, Steve Smale conjectured that for any polynomial P of degree $n \geqslant 2$ normalized by $P(0) = 0$, $P'(0) = 1$, there exists a critical point ζ, such that $|P(\zeta)/\zeta| \leqslant 1$ (or even $\leqslant (n-1)/n$). The conjecture has remained open even though it has been proven to be true for many classes of polynomials. At present, there are various approaches to its solution, as well as numerous upper bounds for $|P(\zeta)/\zeta|$. Unfortunately, in general, these bounds are of the form $4 + o(1)$ as $n \to \infty$. An exception is the result of Tuen-Wai Ng, who established a bound of 2 for a large class of polynomials, which includes the odd polynomials. In proving the main result, he made substantial use of the univalent functions. Following a similar approach, in this note we extend Tuen-Wai Ng's result in various directions. In particular, we refine the upper bound for odd polynomials and consider an analog of Smale's conjecture for finite Blaschke products, proposed by Sheil-Small. It is shown that for odd Blaschke products, the upper bound can also be set to 2. Previously, only estimates close to 4 were known.

Key words: *polynomials, critical points, critical values, condensers*

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1. Introduction. Let P be any polynomial; then b is a critical point of P if and only if $P'(b) = 0$, and v is a critical value of P if and only if $v = P(b)$ for some critical point b of P. In his work [1], in connection with the study of the convergence of Newton's method for calculating the roots of polynomials, S. Smale established inequalities for critical values and raised the question of the accuracy of these estimates [1, pp. 9, 11, 31–33]. In particular, for a polynomial P normalized by the conditions $P(0) = 0$, $P'(0) \neq 0$, he showed that

$$\min\{|P(b)/b| \colon P'(b) = 0\} \leqslant 4|P'(0)|. \qquad (1)$$

In connection with this inequality, S. Smale posed the problem [1, Problem 1E]: is it possible to replace the number 4 in the right-hand side of (1) by 1 (or also by $(n-1)/n$, where n is the degree of the polynomial P)? To the best of the author's knowledge, S. Smale's problem has not yet been solved in the general case, although it has generated wide interest. At present, there are various approaches to its solution, as well as numerous upper bounds for the left-hand side of (1) (see, for example, [2]–[10]). Unfortunately, these bounds are not asymptotically sharp in the general case. Moreover, they have the form $4+o(1)$ as $n \to \infty$. An exception is the following result of Tuen-Wai Ng.

Theorem A. [5] *Let P be a polynomial of degree $n \geqslant 2$, such that $P(0)=0$ and $P'(0) \neq 0$. Let $b_1, \ldots, b_{n-1}$ be its critical points, such that $|b_1| \leqslant |b_2| \leqslant \ldots \leqslant |b_{n-1}|$. Suppose that $b_2 = -b_1$; then*

$$\min_k \left| \frac{P(b_k)}{b_k} \right| \leqslant 2|P'(0)|. \tag{2}$$

As a consequence, it follows that (2) holds for all nonlinear odd polynomials with nonzero linear term. In the proof of Theorem A in [5], the Lavrent'ev covering theorem for univalent functions is used.

In this note, we show that, following [5], we can refine inequality (2) for odd polynomials and also obtain (2) for finite Blaschke products instead of polynomials. In the case of Blaschke products, inequality (2) with the constant 1 instead of 2 in the right-hand side was conjectured by Sheil-Small [11, Sec. 10.4.8]. This conjecture has not yet been proven or refuted. There are only upper bounds in the left-hand side of (2) in the form $4+o(1)$ as $n \to \infty$ [11], [12]. The proof of the theorems uses the technique developed in [13].

2. Polynomials. Denote by $S(P)$ the maximal domain in the z-plane that contains the origin and that is univalently mapped by the polynomial $w = P(z)$, $P(0) = 0$, $P'(0) \neq 0$, onto a domain starlike with respect to the origin in the w-plane. We recall that a domain is starlike with respect to $w = 0$ if it contains a point together with the segment connecting this point with the origin. It is easily seen that the range

$$S^*(P) := P(S(P)),$$

is a domain bounded by some radial rays of the form $L = \{P(b)t \colon t \geqslant 1\}$, where b is a critical point of the polynomial P.

Theorem 1. *Let P be a polynomial of degree $n \geqslant 2$, $P(0) = 0$, $P'(0) \neq 0$, and let the boundary of the domain $S^*(P)$ consist of m rays, $m \leqslant n-1$. Then inequality*

$$\left|\frac{P(b)}{bP'(0)}\right| \leqslant 1 \tag{3}$$

holds for $m = 1$, and inequality

$$\left|\frac{P(b)}{bP'(0)}\right| \leqslant 4^{\frac{m-1}{m}} \left|\frac{b'}{b-b'}\right| \tag{4}$$

holds for $m > 1$, where b is a critical point for which

$$|P(b)| = \min\{|P(\zeta)| \colon P'(\zeta) = 0,\, \zeta \in \partial S(P)\},$$

and b' is any other critical point of P on $\partial S(P)$ other than b.

Proof. We need the properties of the inner radius $r(B, z_0)$ of the domain B with respect to the point $z_0 \in B$. In the case of a simply connected domain B, the inner radius coincides with the well-known definition of the conformal radius (for more details, see, e. g., [13, Ch. 2.1]).

Consider the case $m = 1$. The function

$$w(z) = \frac{z 4P(b)}{(1+z)^2}$$

maps the unit disk $|z| < 1$ conformally and univalently onto a plane with the radial cut

$$B := \mathbb{C} \backslash \{P(b)t \colon t \geqslant 1\}.$$

Therefore, the inner radius is

$$r(B, 0) = 4|P(b)|.$$

Applying the Koebe One-Quarter Theorem, we arrive at the inequality

$$r(B, 0) = r(P^{-1}(B), 0)|P'(0)| \leqslant 4|bP'(0)|,$$

which proves (3).

Now assume that $m > 1$. The function

$$w_m(z) = \frac{z\,|P(b)|\,\sqrt[m]{4}}{(1+z^m)^{2/m}}$$

maps the disk $|z| < 1$ conformally and univalently onto a simply connected domain B_m, which is a plane with cuts along the rays

$$\{t|P(b)|\exp(2\pi ik/m)\colon t \geqslant 1\}, \quad k = 1, \ldots, m.$$

Therefore,

$$r(B_m, 0) = |P(b)|\,\sqrt[m]{4}.$$

According to the dissymmetrization theorem [13, Theorem 4.16],

$$r(B_m, 0) \leqslant r(S^*(P), 0).$$

Superposition of functions

$$\varphi \circ P^{-1},$$

where P^{-1} is the branch of the inverse function, $P^{-1}(0) = 0$, and $\varphi(z) = z/(z - b')$ maps the domain $S^*(P)$ onto a simply connected domain G of an open plane that does not contain the point $b/(b - b')$. By the property of the inner radius [13, (2.2)], we obtain

$$r(S^*(P), 0) = \left|\frac{P'(0)}{\varphi'(0)}\right| r(G, 0) = |b'P'(0)|\, r(G, 0).$$

Again, the Koebe One-Quarter Theorem gives an estimate

$$r(G, 0) \leqslant \frac{4|b|}{|b - b'|}.$$

Combining the above relations, we conclude that

$$|P(b)|\,\sqrt[m]{4} \leqslant 4|P'(0)| \left|\frac{bb'}{b - b'}\right|.$$

Therefore, inequality (4) holds. □

A polynomial P is called odd if for any z, $P(-z) = -P(z)$.

Corollary 1. *For any odd polynomial P of degree $n \geqslant 2$, $P'(0) \neq 0$, there exists a critical point b at which*

$$\left|\frac{P(b)}{bP'(0)}\right| \leqslant 2^{\frac{n-3}{n-1}}.$$

Proof. In inequality (4), we set $b' = -b$. □

3. Blaschke products. By a *Blaschke product* of degree $n \geqslant 2$ we mean a rational function

$$B(z) = \alpha \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z}_k z},$$

where $|\alpha| = 1$ and the complex numbers $z_1, \dots, z_n$ lie in the unit disk $|z| < 1$. The functions $w = B(z)$ realize holomorphic n-sheeted coverings of the disk $|w| < 1$ by the disk $|z| < 1$ and, therefore, are rational analogs of polynomials of degree n. Denote by $S(B)$ the maximal domain in the z-plane that contains the origin and that is univalently mapped by the Blaschke product $w = B(z)$, $B(0) = 0$, $B'(0) \neq 0$, onto a domain starlike with respect to the origin in the w-plane, $S^*(B) := B(S(B))$.

Theorem 2. *Let B be a Blaschke product of degree $n \geqslant 2, B(0) = 0$, $B'(0) \neq 0$, the boundary of the domain $S^*(B)$ consists of m rays, $m \leqslant n-1$, and let b be a critical point of B for which*

$$|B(b)| = \min\{|B(\zeta)| \colon B'(\zeta) = 0,\ \zeta \in \partial S(B)\}.$$

In the case $m > 1$, we additionally assume that on the boundary of the domain $S(B)$ there are two critical points b_1 and b_2, such that $\arg b_1 = \pi + \arg b_2$.

Then, for $m = 1$, inequality

$$\left| \frac{B(b)}{bB'(0)} \right| \leqslant 1 \tag{5}$$

holds, and for $m > 1$, inequality

$$\frac{|B(b)|\, \sqrt[m]{4}}{(1 + |B(b)|^m)^{2/m}} \leqslant 2|B'(0)|\sqrt{|b_1 b_2|} \tag{6}$$

holds.

Proof. Following the idea of the proof of Theorem 1, we first consider the case $m = 1$. The function $u = u(z)$, defined by the relations

$$u(z) = \tilde{u}(z)\frac{B(b)}{|B(b)|}, \quad \frac{\tilde{u}(z)}{(1 + \tilde{u}(z))^2} = \frac{\tilde{u}'(0)z}{(1+z)^2}, \quad \tilde{u}'(0) = \frac{4|B(b)|}{(1 + |B(b)|)^2},$$

maps the disk $|z| < 1$ conformally and univalently onto the domain H, which is the disk $|u| < 1$ with a cut along a segment $[B(b), B(b)/|B(b)|]$. Therefore,

$$r(H, 0) = |u'(0)| = \tilde{u}'(0).$$

By Koebe's theorem for bounded univalent functions, we obtain

$$r(H,0) = r(B^{-1}(H),0)|B'(0)| \leqslant \frac{4|bB'(0)|}{(1+|b|)^2}.$$

Hence,

$$\frac{|B(b)|}{(1+|B(b)|)^2} \leqslant \frac{|bB'(0)|}{(1+|b|)^2}.$$

Taking into account Schwarz's lemma,

$$|B(b)| \leqslant |b|,$$

we arrive at inequality (5).

In the case $m > 1$, we introduce the function $u = u_m(z)$:

$$\frac{u}{(1+u^m)^{2/m}} = \frac{u_m'(0)\, z}{(1+z^m)^{2/m}},$$

where

$$u_m'(0) = \frac{|B(b)|\, \sqrt[m]{4}}{(1+|B(b)|^m)^{2/m}}.$$

This function maps the disk $|z| < 1$ conformally and univalently onto the domain H_m, which is the disk $|u| < 1$ with cuts

$$\{t|B(b)|\exp(2\pi ik/m)\colon 1 \leqslant t \leqslant 1/|B(b)|\}, \quad k = 1,\dots,m.$$

Hence,

$$r(H_m,0) = u_m'(0).$$

By the dissymmetrization theorem,

$$r(H_m,0) \leqslant r(S^*(B),0).$$

On the other hand,

$$r(S^*(B),0) = r(S(B),0)|B'(0)|.$$

Consider the function F, which maps the disk $|v| < 1$ conformally and univalently onto the domain $S(B)$, such that $F(0) = 0$. Applying Corollary 7.4 from [13] to the function $F(v)/F'(0)$ leads to the estimate

$$\frac{\sqrt{|b_1 b_2|}}{|F'(0)|} \geqslant \left(1 + \sqrt{1-|F'(0)|^2}\right)^{-1}.$$

Therefore,

$$r(S(B), 0) = |F'(0)| \leqslant \sqrt{|b_1 b_2|}\left(1 + \sqrt{1 - |F'(0)|^2}\right) \leqslant 2\sqrt{|b_1 b_2|}.$$

Combining the relations written out, we arrive at inequality (6). □

Corollary 2. *For any odd Blaschke product of degree $n \geqslant 2$, $B'(0) \neq 0$, there exists a critical point b at which the inequality*

$$\left|\frac{B(b)}{bB'(0)}\right| \leqslant 2$$

holds.

Proof. It suffices to take in inequality (6) $b_1 = -b_2 = b$ and notice that $|B(b)| \leqslant 1$. □

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