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ON THE TIME-OPTIMAL CONTROL PROBLEM FOR A HEAT EQUATION WITH A NONLOCAL BOUNDARY CONDITION

Abstract. This paper studies a time-optimal control problem for the one-dimensional heat equation with a nonlocal boundary condition of the Bitsadze–Samarskii type. The control is applied at the left endpoint, and the goal is to steer the system from the zero initial state to a prescribed temperature level in minimal time. Using spectral decomposition, we reduce the problem to a Volterra integral equation of the first kind. The kernel of this equation is shown to be positive, continuous, and integrable. Under natural assumptions on the weight function, we establish an explicit upper bound for the minimal time in terms of the first eigenvalue and the Fourier coefficients. The results demonstrate the influence of the nonlocal boundary condition on the system’s controllability.

Key words: *heat equation, Volterra integral equation, admissible control, weight function, minimal time, nonlocal condition*

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1. Introduction. In this paper, we consider the heat conduction equation

$$u_t(x, t) = u_{xx}(x, t), \quad x \in (0, 1), \quad t > 0, \quad (1)$$

with a nonlocal boundary condition of the Bitsadze–Samarskii type

$$u(0, t) = \nu(t), \quad u(1, t) = u(1/2, t), \quad t \geq 0, \quad (2)$$

and the homogeneous initial condition

$$u(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (3)$$

where $\nu(t)$ is a boundary control function applied at the left endpoint.

Physically, equation (1) describes the temperature $u(x, t)$ in a thin homogeneous rod of unit length with unit thermal diffusivity. The initial condition (3) means the rod is initially at zero temperature. The left boundary condition $u(0, t) = \nu(t)$ represents a time-varying thermal actuation, where $\nu(t)$ is the control input. The nonlocal condition $u(1, t) = u(1/2, t)$ couples the right endpoint with the midpoint of the rod, requiring these temperatures to coincide for all $t \geq 0$. Such conditions arise in feedback control systems where a sensor located at the midpoint regulates the temperature at the right endpoint, in temperature equalization problems in material processing, or in inverse problems of thermography. This nonlocal constraint fundamentally alters the heat transfer dynamics compared to classical local boundary value problems, introducing spectral anomalies and controllability challenges.

Control problems for parabolic equations have been extensively studied due to their importance in modeling heat conduction. Friedman [1] studied such problems from a variational perspective. Foundational results on controllability were obtained by Fattorini and Russell [2]. Egorov [3] extended Pontryagin's maximum principle to Banach spaces and proved a bang-bang principle. Comprehensive treatments can be found in the monographs by Lions and Fursikov [4], [5]. The bang-bang principle for time-optimal boundary control was further examined in [6].

A boundary control problem for parabolic equation in three-dimensional spatial domain was considered in [7], where admissibility of the control function was established using geometric estimates. The time-optimal heating of a non-homogeneous rod to a prescribed average temperature was studied in [8], where the control function was explicitly constructed via the Laplace transform method and its admissibility was proved. Boundary control problem for one-dimensional heat equation was further investigated in [9], while the time-optimal control of a fourth-order parabolic equation in a two-dimensional domain was analyzed in [10].

Initial studies of time-optimal control problem for heat conduction equation in bounded n -dimensional domain were presented in [11], where sharp estimates for the minimal time required to achieve a prescribed average temperature were derived. Various related problems for nonlinear and degenerate parabolic equations have also attracted considerable attention; see, for instance, [12], where local existence and uniqueness of weak solutions were established via the Galerkin method and the Banach fixed point theorem, and [13], where singular parabolic equations with

logarithmic nonlinearities were studied.

The boundary control problem for a pseudo-parabolic equation was studied in [14], where a necessary control function was constructed to achieve a prescribed temperature level in an inhomogeneous thin plate. Reaction-diffusion systems and semilinear parabolic equations with singular or nonlocal terms were investigated in [15], [16], focusing on global existence, finite-time blow-up, and long-time behavior of solutions.

In [17], spectral inequalities for the Laplace-Beltrami operator were derived, leading to observability and null-controllability results for the spherical heat equation. Boundary control problems for generalized heat equations involving Schrödinger-type operators were studied in [18]. Numerical aspects of boundary control and discretization schemes for heat equations were analyzed in [19], [20]. Applications to fluid dynamics and thermally coupled systems were considered in [21].

In this paper, we study a time-optimal control problem for the one-dimensional heat equation with a Bitsadze–Samarskii type nonlocal boundary condition. The control is applied at the left endpoint, and the goal is to steer the system from zero initial state to a prescribed temperature in minimal time. Using an auxiliary function, we reduce the problem to one with homogeneous boundary conditions and apply the spectral method. The eigenfunctions of the nonlocal spectral problem form a complete system. By introducing a class of weight functions and an integral condition, we derive a Volterra integral equation of the first kind for the control function. The kernel of this equation is expressed through the eigenvalues and Fourier coefficients. We establish necessary estimates for the kernel and obtain an explicit upper bound for the minimal time in terms of the first eigenvalue and Fourier coefficients. The result quantifies the influence of the nonlocal boundary condition on the controllability of the system.

2. Problem statement. In this section, we give a precise statement of the control problem and explain what it means from the physical perspective. The idea comes from thermal processes where we regulate the overall temperature of the medium by adjusting an internal heat source distributed throughout the domain.

If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0) = 0$ and $|\nu(t)| \leq 1$ on the half-line $t \geq 0$, then we call it an *admissible control*.

We define the class of weight functions as follows:

$$\mathcal{P} := \left\{ \rho \in W_2^1(0, 1) \left| \begin{array}{l} \rho(x) \geq 0 \quad \text{a.e. in } (0, 1), \\ \int_0^1 \rho(x) dx = 1, \\ \rho_{k,j} \geq 0 \quad \text{for all } k \in \mathbb{N}, j = 1, 2, \end{array} \right. \right\},$$

where $\rho_{k,j}$ are the Fourier coefficients of the function $\rho(x)$ defined later by formula (8).

The main objective of this work is to investigate a time-optimal control problem associated with the heat equation.

Time-optimal problem. *For a given constant $\theta > 0$, find the minimal time $T_{\min} \geq 0$, such that for $t \geq T_{\min}$ the solution $u(x, t)$ of the initial-boundary value problem (1)–(3) with some admissible control $\nu(t)$ exists, and satisfies the integral condition*

$$\int_0^1 \rho(x) u(x, t) dx = \theta, \quad t \in [T_{\min}, T_1], \quad (4)$$

for some $T_1 > T_{\min}$.

This formulation requires that the weighted temperature in the rod attains and maintains a prescribed level θ over $[T_{\min}, T_1]$. Unlike pointwise constraints, this averaged condition characterizes a global thermal performance criterion relevant in applications, such as thermal processing, manufacturing, and heating systems. From a control-theoretic viewpoint, the integral constraint offers greater flexibility while guaranteeing the required thermal effect.

We consider the spectral problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < 1, \quad (5)$$

with the nonlocal boundary conditions

$$X(0) = 0, \quad X(1) = X(1/2). \quad (6)$$

For $\lambda \leq 0$, problem (5)–(6) admits only the trivial solution. Hence, we consider $\lambda > 0$. In this case, the spectrum consists of two countable sequences of eigenvalues:

$$\lambda_{k,1} = \left(\frac{2(2k-1)\pi}{3} \right)^2, \quad \lambda_{k,2} = (4k\pi)^2, \quad k \in \mathbb{N}. \quad (7)$$

The corresponding eigenfunctions are

$$X_{k,1}(x) = \sin(\sqrt{\lambda_{k,1}}x), \quad X_{k,2}(x) = \sin(\sqrt{\lambda_{k,2}}x), \quad k \in \mathbb{N}.$$

The system $\{X_{k,j}\}$ is not orthogonal. Following [22], we introduce the adjoint problem:

$$Y''(x) + \lambda Y(x) = 0, \quad x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1),$$

with boundary conditions

$$Y(0) = 0, \quad Y(1) = 0,$$

and transmission conditions at $x = 1/2$:

$$Y(1/2 + 0) = Y(1/2 - 0), \quad Y'(1/2 + 0) - Y'(1/2 - 0) = Y'(1).$$

The eigenvalues of the adjoint problem are the same as in (7). The adjoint eigenfunctions are given explicitly by

$$Y_{k,1}(x) = \begin{cases} \frac{8}{3} \sin(\sqrt{\lambda_{k,1}}x), & 0 \leq x < \frac{1}{2}, \\ -\frac{4}{3 \cos \sqrt{\lambda_{k,1}}} \sin(\sqrt{\lambda_{k,1}}(1-x)), & \frac{1}{2} < x \leq 1, \end{cases}$$

$$Y_{k,2}(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ -\frac{4}{\cos \sqrt{\lambda_{k,2}}} \sin(\sqrt{\lambda_{k,2}}(1-x)), & \frac{1}{2} < x \leq 1. \end{cases}$$

The systems $\{X_{k,j}\}$ and $\{Y_{k,j}\}$ satisfy the biorthogonality condition

$$\int_0^1 X_{k,j}(x) Y_{\ell,m}(x) dx = \delta_{k\ell} \delta_{jm},$$

and are complete and biorthogonal in $L_2(0, 1)$ (see [22]).

For a weight function $\rho \in \mathcal{P}$, its expansion in the adjoint eigenfunctions $\{Y_{k,j}\}$ is

$$\rho(x) = \sum_{j=1}^2 \sum_{k=1}^{\infty} \rho_{k,j} Y_{k,j}(x),$$

where the Fourier coefficients are given by

$$\rho_{k,j} = \int_0^1 \rho(x) X_{k,j}(x) dx. \quad (8)$$

For the constant function 1, its Fourier coefficients with respect to $\{Y_{k,j}\}$ are

$$B_{k,j} = \int_0^1 Y_{k,j}(x) dx. \quad (9)$$

Now we set

$$\Phi_{k,j} := B_{k,j} \rho_{k,j} \lambda_{k,j}, \quad j = 1, 2, \quad k \in \mathbb{N}. \quad (10)$$

Theorem 1. *Assume that the average temperature θ satisfies*

$$0 < \theta < \frac{\Phi_{1,1}}{\lambda_{1,1}},$$

where $\lambda_{1,1}$ is the smallest eigenvalue of the spectral problem (5)–(6), and $\Phi_{1,1}$ is defined by (10) with $k = 1, j = 1$. Set

$$T_0 = -\frac{1}{\lambda_{1,1}} \ln \left(1 - \frac{\theta \lambda_{1,1}}{\Phi_{1,1}} \right).$$

Then a solution $T_{\min}$ of the time-optimal control problem exists, and the estimate $T_{\min} \leq T_0$ is valid.

Physically, the condition $0 < \theta < \Phi_{1,1}/\lambda_{1,1}$ guarantees that the desired temperature θ does not exceed the maximal achievable temperature under the unit control constraint $|\nu(t)| \leq 1$.

3. Volterra integral equation for the control function.

In this section, we reduce the control problem to an equivalent integral representation using spectral decomposition.

Let $T > 0$ be fixed. We denote by $C([0, T]; B)$ the space of continuous functions from $[0, T]$ into a Banach space B .

Due to the nonlocal boundary condition, we introduce the space

$$V := \{v \in W_2^1(0, 1) : v(0) = 0, v(1) = v(1/2)\}.$$

Definition 1. A function $u(x, t)$ is called a generalized solution of problem (1)–(3) if it can be represented in the form

$$u(x, t) = \nu(t) - w(x, t), \quad (11)$$

where $w \in C([0, T]; V)$ is a generalized solution of

$$\begin{cases} w_t - w_{xx} = \nu'(t), & x \in (0, 1), \quad t > 0, \\ w(0, t) = 0, \quad w(1, t) = w(1/2, t), & t \geq 0, \\ w(x, 0) = 0, \end{cases} \quad (12)$$

in the weak sense: for all $\varphi \in V$ and almost all $t \in (0, T)$,

$$\frac{d}{dt}(w(\cdot, t), \varphi) + (w_x(\cdot, t), \varphi_x) = \nu'(t)(1, \varphi), \quad (13)$$

with $w(x, 0) = 0$.

Let $\{X_{k,j}\}$ and $\{Y_{k,j}\}$ be the biorthogonal systems defined above. Using the Fourier method, we expand w in the eigenfunctions $\{X_{k,j}\}$:

$$w(x, t) = \sum_{j=1}^2 \sum_{k=1}^{\infty} w_{k,j}(t) X_{k,j}(x), \quad w_{k,j}(t) = \int_0^1 w(x, t) Y_{k,j}(x) dx.$$

Substituting into (12) and using biorthogonality gives

$$w'_{k,j}(t) + \lambda_{k,j} w_{k,j}(t) = \nu'(t) B_{k,j}, \quad w_{k,j}(0) = 0,$$

where $B_{k,j}$ is defined by (9).

Solving this ODE yields

$$w(x, t) = \sum_{j=1}^2 \sum_{k=1}^{\infty} B_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)} \nu'(s) ds \right) X_{k,j}(x). \quad (14)$$

Integrating by parts and using $\nu(0) = 0$, we obtain the representation for $u = \nu - w$:

$$u(x, t) = \sum_{j=1}^2 \sum_{k=1}^{\infty} B_{k,j} \lambda_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)} \nu(s) ds \right) X_{k,j}(x). \quad (15)$$

Lemma 1. *Let $\nu \in W_2^1(0, T)$ and $\nu(0) = 0$. Then the function $u(x, t)$ defined by (15) is a generalized solution of problem (1)–(3).*

Proof. We show that $w \in C([0, T]; V)$. Differentiating (14) with respect to x gives

$$w_x(x, t) = \sum_{j=1}^2 \sum_{k=1}^{\infty} B_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)} \nu'(s) ds \right) X'_{k,j}(x).$$

Using the completeness of the system $\{X_{k,j}\}$ and standard L_2 estimates, we obtain

$$\|w_x(\cdot, t)\|_{L_2}^2 \leq \sum_{j=1}^2 \sum_{k=1}^{\infty} B_{k,j}^2 \|X'_{k,j}\|_{L_2}^2 \left(\int_0^t e^{-\lambda_{k,j}(t-s)} \nu'(s) ds \right)^2.$$

Since $X''_{k,j} = -\lambda_{k,j} X_{k,j}$, we have

$$\|X'_{k,j}\|_{L_2}^2 = \lambda_{k,j} \|X_{k,j}\|_{L_2}^2 = \lambda_{k,j} A_{k,j},$$

where $A_{k,j} = \int_0^1 X_{k,j}^2(x) dx$ is uniformly bounded.

Applying the Cauchy–Schwarz inequality,

$$\left(\int_0^t e^{-\lambda_{k,j}(t-s)} \nu'(s) ds \right)^2 \leq \frac{1}{2\lambda_{k,j}} \|\nu'\|_{L_2(0,T)}^2.$$

Thus,

$$\|w_x(\cdot, t)\|_{L_2}^2 \leq \frac{1}{2} \|\nu'\|_{L_2(0,T)}^2 \sum_{j=1}^2 \sum_{k=1}^{\infty} B_{k,j}^2 A_{k,j}.$$

By the estimate $|B_{k,j}| \leq \frac{C}{k}$ and boundedness of $A_{k,j}$, the series converges. Hence $\|w_x(\cdot, t)\|_{L_2}$ is uniformly bounded on $[0, T]$. A similar argument shows that $\|w(\cdot, t)\|_{L_2}$ is bounded.

Moreover, the mapping $t \mapsto w(\cdot, t)$ is continuous in $L_2(0,1)$ by the dominated convergence theorem applied to the series representation. Therefore, $w \in C([0, T]; W_2^1(0, 1))$. Since each $X_{k,j}$ satisfies $X_{k,j}(0) = 0$ and $X_{k,j}(1) = X_{k,j}(1/2)$, it follows that $w(0, t) = 0$ and $w(1, t) = w(1/2, t)$ for all t , hence $w \in C([0, T]; V)$.

Substituting the series into (13) and using the biorthogonality of $\{X_{k,j}\}$ and $\{Y_{k,j}\}$ verifies the equation. The initial condition $w(x, 0) = 0$ follows from (14). Thus $u = \nu - w$ is a generalized solution. $\square$

By substituting the expression (15) into the integral condition with weight function $\rho \in \mathcal{P}$, we arrive at the following relation:

$$\begin{aligned} \phi(t) &= \int_0^1 \rho(x) u(x, t) dx \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} B_{k,j} \lambda_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)} \nu(s) ds \right) \int_0^1 \rho(x) X_{k,j}(x) dx, \end{aligned}$$

where $\phi(t) = \theta$ for $t \in [T_{\min}, T_1]$.

Using (8), we obtain

$$\phi(t) = \sum_{j=1}^2 \sum_{k=1}^{\infty} B_{k,j} \rho_{k,j} \lambda_{k,j} \int_0^t e^{-\lambda_{k,j}(t-s)} \nu(s) ds.$$

Define the kernel function

$$K(t) := \sum_{j=1}^2 \sum_{k=1}^{\infty} \Phi_{k,j} e^{-\lambda_{k,j}t}, \quad t \geq 0, \quad (16)$$

where $\Phi_{k,j}$ are defined by (10).

Consequently, we obtain the following Volterra integral equation of the first kind for the boundary control $\nu(t)$:

$$\int_0^t K(t-\tau) \nu(\tau) d\tau = \phi(t), \quad t > 0, \quad (17)$$

where $\phi(t)$ is a prescribed function determined by the desired thermal state. In particular, for the time-optimal problem considered in this paper, we have $\phi(t) = \theta$ for $t \in [T_{\min}, T_1]$.

The integral equation (17) plays a central role in determining the control function $\nu(t)$ that drives the system to the desired thermal state. The solvability of this equation depends on the properties of the kernel $K(t)$.

Therefore, we first establish the necessary estimates for the kernel, which are essential for the existence of a solution.

Lemma 2. *For any $\rho \in \mathcal{P}$, the Fourier coefficients*

$$\rho_{k,j} = \int_0^1 \rho(x) X_{k,j}(x) dx$$

satisfy the estimate

$$0 \leq \rho_{k,j} \leq \frac{C}{k}, \quad j = 1, 2, k \in \mathbb{N},$$

where $C > 0$ is a constant independent of k and j .

Proof. The nonnegativity $\rho_{k,j} \geq 0$ follows from the definition of the class $\mathcal{P}$. We now prove the upper bound.

From the definition of $\rho_{k,j}$, we have

$$\rho_{k,j} = \int_0^1 \rho(x) \sin(\sqrt{\lambda_{k,j}}x) dx.$$

Integrating by parts, we obtain

$$\rho_{k,j} = \frac{1}{\sqrt{\lambda_{k,j}}} \int_0^1 \rho'(x) \cos(\sqrt{\lambda_{k,j}}x) dx - \frac{\rho(1) \cos \sqrt{\lambda_{k,j}} - \rho(0)}{\sqrt{\lambda_{k,j}}}.$$

Taking absolute values and using $|\cos(\sqrt{\lambda_{k,j}}x)| \leq 1$, we get

$$|\rho_{k,j}| \leq \frac{1}{\sqrt{\lambda_{k,j}}} \int_0^1 |\rho'(x)| dx + \frac{|\rho(1)| + |\rho(0)|}{\sqrt{\lambda_{k,j}}}.$$

Since $\rho \in W_2^1(0, 1)$, by the Cauchy–Schwarz inequality we have

$$\int_0^1 |\rho'(x)| dx \leq \|\rho'\|_{L_2(0,1)}.$$

Moreover, by the Sobolev embedding in one dimension, ρ is continuous on $[0, 1]$, hence $\rho(0)$ and $\rho(1)$ are bounded. Therefore, there exists a constant $C_1 > 0$, such that

$$\int_0^1 |\rho'(x)| dx + |\rho(0)| + |\rho(1)| \leq C_1.$$

Thus,

$$|\rho_{k,j}| \leq \frac{C_1}{\sqrt{\lambda_{k,j}}}.$$

Finally, since $\lambda_{k,j} \geq ck^2$ for some constant $c > 0$ and all $k \in \mathbb{N}$, it follows that

$$\frac{1}{\sqrt{\lambda_{k,j}}} \leq \frac{C_2}{k}.$$

Hence,

$$\rho_{k,j} \leq \frac{C}{k},$$

where $C > 0$ is independent of k and j . $\square$

The estimate obtained in Lemma 2 allows us to establish the integrability properties of the kernel function $K(t)$, which is essential for the analysis of the Volterra integral equation (17). In particular, we need to show that $K(t)$ is positive and decays like $1/\sqrt{t}$ as $t \rightarrow 0^+$, which guarantees that the integral operator is well-defined and that the control problem is solvable.

Lemma 3. *The kernel function $K(t)$ defined by (16) is positive and satisfies the estimate*

$$0 < K(t) \leq \frac{C}{\sqrt{t}}, \quad 0 < t \leq 1,$$

where $C > 0$ is a constant independent of t .

Proof. First, we show that $K(t) > 0$ for all $t > 0$. From the definition,

$$K(t) = \sum_{j=1}^2 \sum_{k=1}^{\infty} \Phi_{k,j} e^{-\lambda_{k,j} t}.$$

Since $B_{k,1} > 0$ for all k , $B_{k,2} = 0$, $\rho_{k,j} \geq 0$ by Lemma 2, and $\lambda_{k,j} > 0$, we have $\Phi_{k,j} \geq 0$. Moreover, $\rho_{1,1} > 0$ and $B_{1,1} > 0$, hence $\Phi_{1,1} > 0$. Therefore,

$$K(t) \geq \Phi_{1,1} e^{-\lambda_{1,1} t} > 0 \quad \text{for all } t > 0.$$

Next, we estimate $K(t)$ from above. From Lemma 2, we have

$$\rho_{k,j} \leq \frac{C_1}{k}.$$

Moreover, by direct computation of $B_{k,j} = \int_0^1 Y_{k,j}(x) dx$, one obtains the estimate

$$|B_{k,j}| \leq \frac{C_2}{k}, \quad j = 1, 2.$$

Since $\lambda_{k,j} \leq C_3 k^2$, it follows that

$$\Phi_{k,j} = B_{k,j} \rho_{k,j} \lambda_{k,j} \leq C_4 \cdot \frac{1}{k} \cdot \frac{1}{k} \cdot k^2 \leq C_4,$$

where C_4 is independent of k and j . Thus, $\Phi_{k,j}$ is uniformly bounded.

Consequently,

$$K(t) \leq C_4 \sum_{j=1}^2 \sum_{k=1}^{\infty} e^{-\lambda_{k,j} t}.$$

Since the eigenvalues satisfy $\lambda_{k,j} \geq c_1 k^2$ for all $k \in \mathbb{N}$ and $j = 1, 2$, we obtain

$$K(t) \leq C_5 \sum_{k=1}^{\infty} e^{-c_1 k^2 t}.$$

Now we estimate the series $\sum_{k=1}^{\infty} e^{-c_1 k^2 t}$. For any $\zeta > 0$, we have

$$\sum_{k=1}^{\infty} e^{-\zeta k^2} \leq \int_0^{\infty} e^{-\zeta s^2} ds = \frac{1}{2} \sqrt{\frac{\pi}{\zeta}}. \quad (18)$$

Setting $\zeta = c_1 t$, we obtain

$$\sum_{k=1}^{\infty} e^{-c_1 k^2 t} \leq \frac{C}{\sqrt{t}}.$$

Therefore,

$$K(t) \leq \frac{C}{\sqrt{t}}, \quad 0 < t \leq 1.$$

This completes the proof. $\square$

The estimate obtained in Lemma 2 allows us to derive precise bounds for the kernel function $K(t)$, which are essential for the analysis of the Volterra integral equation (17). In particular, we show that $K(t)$ is positive and satisfies the estimate $K(t) \leq \frac{C}{\sqrt{t}}$ for $0 < t \leq 1$. This behavior ensures that $K \in L^1(0, T)$ for any $T > 0$, so that the corresponding integral operator is well defined. These properties play a crucial role in the subsequent analysis of the solvability of the control problem.

4. Existence of the control function. In this section, we prove that the Volterra integral equation derived above admits a solution. More precisely, we show that for a suitable class of right-hand side functions $\phi(t)$, there exists a control function $\nu(t)$ that satisfies the integral equation and is bounded by one. The proof is based on the Laplace transform method and on the properties of the kernel $K(t)$ established in the previous section.

For any constant $M > 0$, we denote by $W(M)$ the set of functions $\phi \in W_2^2(-\infty, +\infty)$, such that $\phi(t) = 0$ for all $t \leq 0$ and

$$\|\phi\|_{W_2^2(\mathbb{R}_+)} \leq M.$$

We present the following main theorem:

Theorem 2. *There exists $M > 0$, such that for any function $\phi \in W(M)$ the solution $\nu(t)$ of the Volterra integral equation (17) exists and satisfies $|\nu(t)| \leq 1$.*

We solve the Volterra integral equation (17) by means of the Laplace transform. Recall that the Laplace transform of the control function $\nu(t)$ is defined by

$$\tilde{\nu}(p) = \int_0^\infty e^{-pt} \nu(t) dt, \quad \operatorname{Re} p > 0.$$

Applying the Laplace transform to (17) and using the convolution theorem gives

$$\tilde{\nu}(p) = \frac{\tilde{\phi}(p)}{\tilde{K}(p)}, \quad p = \xi + i\tau, \quad \xi > 0, \quad \tau \in \mathbb{R}.$$

By the inverse Laplace transform, the control function $\nu(t)$ can be written as

$$\nu(t) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} \frac{\tilde{\phi}(p)}{\tilde{K}(p)} e^{pt} dp = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{\phi}(\xi + i\tau)}{\tilde{K}(\xi + i\tau)} e^{(\xi + i\tau)t} d\tau. \quad (19)$$

Lemma 4. For $\xi > 0$ and $\tau \in \mathbb{R}$ the estimate

$$|\tilde{K}(\xi + i\tau)| \geq \frac{C_\xi}{\sqrt{1 + \tau^2}}$$

holds, where $C_\xi > 0$ is a constant depending only on ξ .

Proof. Using the definition of $K(t)$, we write

$$\tilde{K}(p) = \int_0^\infty K(t)e^{-pt} dt = \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}}{p + \lambda_{k,j}}.$$

For $p = \xi + i\tau$, we have

$$\tilde{K}(\xi + i\tau) = \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}(\xi + \lambda_{k,j})}{(\xi + \lambda_{k,j})^2 + \tau^2} - i\tau \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}}{(\xi + \lambda_{k,j})^2 + \tau^2}.$$

Therefore,

$$\operatorname{Re} \tilde{K}(\xi + i\tau) = \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}(\xi + \lambda_{k,j})}{(\xi + \lambda_{k,j})^2 + \tau^2},$$

$$\operatorname{Im} \tilde{K}(\xi + i\tau) = -\tau \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}}{(\xi + \lambda_{k,j})^2 + \tau^2}.$$

It is clear that

$$\frac{1}{(\xi + \lambda_{k,j})^2 + \tau^2} \geq \frac{1}{1 + \tau^2} \cdot \frac{1}{(\xi + \lambda_{k,j})^2 + 1}.$$

Consequently,

$$\begin{aligned} |\operatorname{Re} \tilde{K}(\xi + i\tau)| &= \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}(\xi + \lambda_{k,j})}{(\xi + \lambda_{k,j})^2 + \tau^2} \\ &\geq \frac{1}{1 + \tau^2} \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}(\xi + \lambda_{k,j})}{(\xi + \lambda_{k,j})^2 + 1} = \frac{C_{1,\xi}}{1 + \tau^2}, \end{aligned}$$

and

$$|\operatorname{Im} \tilde{K}(\xi + i\tau)| = |\tau| \sum_{j=1}^2 \sum_{k=1}^\infty \frac{\Phi_{k,j}}{(\xi + \lambda_{k,j})^2 + \tau^2}$$

$$\geq \frac{|\tau|}{1 + \tau^2} \sum_{j=1}^2 \sum_{k=1}^{\infty} \frac{\Phi_{k,j}}{(\xi + \lambda_{k,j})^2 + 1} = \frac{C_{2,\xi} |\tau|}{1 + \tau^2},$$

where

$$C_{1,\xi} = \sum_{j=1}^2 \sum_{k=1}^{\infty} \frac{\Phi_{k,j}(\xi + \lambda_{k,j})}{(\xi + \lambda_{k,j})^2 + 1}, \quad C_{2,\xi} = \sum_{j=1}^2 \sum_{k=1}^{\infty} \frac{\Phi_{k,j}}{(\xi + \lambda_{k,j})^2 + 1}.$$

Hence,

$$|\tilde{K}(\xi + i\tau)|^2 = |\operatorname{Re} \tilde{K}|^2 + |\operatorname{Im} \tilde{K}|^2 \geq \frac{C_{1,\xi}^2 + C_{2,\xi}^2 \tau^2}{(1 + \tau^2)^2} \geq \frac{\min(C_{1,\xi}^2, C_{2,\xi}^2)}{1 + \tau^2},$$

which proves the lemma. $\square$

Now, proceeding to the limit as $\xi \rightarrow 0$ in (19), we obtain the equality

$$\nu(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{\phi}(i\tau)}{\tilde{K}(i\tau)} e^{i\tau t} d\tau. \quad (20)$$

Lemma 5. [10] Assume that $\phi \in W(M)$. Then for the imaginary part of the Laplace transform of $\phi(t)$ the following inequality holds:

$$\int_{-\infty}^{+\infty} |\tilde{\phi}(i\tau)| \sqrt{1 + \tau^2} d\tau \leq C \|\phi\|_{W_2^2(\mathbb{R}_+)},$$

where $C > 0$ is a constant.

Now we prove Theorem 2.

Proof of Theorem 2. First, we prove that $\nu \in W_2^1(\mathbb{R}_+)$. Using Lemma 4 with $\xi \rightarrow 0$ and Lemma 5, we obtain the estimate

$$\begin{aligned} \int_{-\infty}^{+\infty} |\tilde{\nu}(\tau)|^2 (1 + |\tau|^2) d\tau &= \int_{-\infty}^{+\infty} \left| \frac{\tilde{\phi}(i\tau)}{\tilde{K}(i\tau)} \right|^2 (1 + |\tau|^2) d\tau \\ &\leq C_0 \int_{-\infty}^{+\infty} |\tilde{\phi}(i\tau)|^2 (1 + |\tau|^2)^2 d\tau = C_0 \|\phi\|_{W_2^2(\mathbb{R})}^2, \end{aligned}$$

where $C_0 = \min(C_{1,0}, C_{2,0})$. The regularity $\nu \in W_2^1(\mathbb{R}_+)$ is obtained in the Fourier sense. Indeed, by the Plancherel theorem,

$$\|\nu\|_{W_2^1(\mathbb{R}_+)}^2 \asymp \int_{-\infty}^{+\infty} |\widehat{\nu}(\tau)|^2 (1 + |\tau|^2) d\tau.$$

The estimate derived above from Lemmas 4 and 5 shows that this integral is bounded by $C_0 \|\phi\|_{W_2^2(\mathbb{R}_+)}^2$, which is finite by the assumption $\phi \in W(M)$. Hence, $\nu \in W_2^1(\mathbb{R}_+)$, with the bound depending only on $\|\phi\|_{W_2^2(\mathbb{R}_+)}$.

From (20), together with Lemmas 4 and 5, we obtain

$$\begin{aligned} |\nu(t)| &\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\tilde{\phi}(i\tau)|}{|\tilde{K}(i\tau)|} d\tau \\ &\leq \frac{1}{2\pi C_0} \int_{-\infty}^{+\infty} |\tilde{\phi}(i\tau)| \sqrt{1 + \tau^2} d\tau \leq \frac{C}{2\pi C_0} \|\phi\|_{W_2^2(\mathbb{R}_+)} \leq \frac{CM}{2\pi C_0} = 1, \end{aligned}$$

where we choose

$$M = \frac{2\pi C_0}{C}.$$

Thus, for any $\phi \in W(M)$, the control function $\nu(t)$ exists, belongs to $W_2^1(\mathbb{R}_+)$, and satisfies $|\nu(t)| \leq 1$ for all $t \geq 0$. $\square$

Remark. Equation (17) is a Volterra integral equation of the first kind. Existence of the solution $\nu \in W_2^1(\mathbb{R}_+)$ follows from Lemmas 4 and 5 via the Laplace transform. Uniqueness follows from Titchmarsh's convolution theorem (see, e.g., [23, Chapter VI, Section 5]), since $K * \delta\nu = 0$ with $K \not\equiv 0$ implies $\delta\nu = 0$.

5. Estimate of the minimal time. In this section, we derive an explicit upper bound for the minimal time required to reach a prescribed averaged temperature level in the domain. The estimate is obtained by means of a lower bound for the kernel of the corresponding Volterra integral equation, which plays a key role in the time-optimal control analysis.

Lemma 6. *The following estimate holds:*

$$K(t) \geq \Phi_{1,1} e^{-\lambda_{1,1} t}, \quad t > 0,$$

where $\lambda_{1,1}$ is the smallest eigenvalue of the corresponding spectral problem, and $\Phi_{1,1}$ is defined by (10) with $k = 1$, $j = 1$.

This follows from the nonnegativity of all coefficients $\Phi_{k,j}$ and the fact that $\Phi_{1,1} > 0$.

We introduce the function

$$Q(t) := \int_0^t K(t - \tau) d\tau = \int_0^t K(\tau) d\tau, \quad t > 0. \quad (21)$$

The function $Q(t)$ has a clear physical interpretation: it characterizes the cumulative thermal effect at the right-hand side of the Volterra equation when a unit control input is applied. In other words, it represents the total response of the system up to time t generated by a unit impulse.

From the properties of $K(t)$ established above, we immediately have

$$Q(0) = 0, \quad Q'(t) = K(t) \geq 0,$$

so $Q(t)$ is nonnegative, continuous, and strictly increasing for $t > 0$.

Define

$$Q^* := \int_0^\infty K(\tau) d\tau. \quad (22)$$

From Lemma 3, $K(t) \leq C/\sqrt{t}$ for $0 < t \leq 1$, and $K(t) \leq Ce^{-ct}$ for $t \geq 1$ with some $c > 0$. Hence $Q^* < \infty$. Thus, for any admissible control $|\nu(t)| \leq 1$, the total accumulated response cannot exceed Q^* .

Lemma 7. Assume that θ satisfies $0 < \theta < Q^*$. Then one can find a time $T_{\min} > 0$ and an admissible control $\nu(t)$ with $|\nu(t)| \leq 1$, such that

$$\int_0^{T_{\min}} K(T_{\min} - \tau) \nu(\tau) d\tau = \theta. \quad (23)$$

Proof. Consider the simplest admissible control, namely the constant function $\nu(t) \equiv 1$. Substituting this into the left-hand side of (23) gives

$$\int_0^t K(t - \tau) d\tau = \int_0^t K(\tau) d\tau = Q(t),$$

where $Q(t)$ is defined by (21).

As established above, $Q(t)$ is continuous, strictly increasing, and satisfies $Q(0) = 0$ and $\lim_{t \rightarrow \infty} Q(t) = Q^*$. Therefore, the function $Q(t)$ increases continuously from 0 to Q^* .

Since $0 < \theta < Q^*$, by the intermediate value theorem, there exists a unique $T_{\min} > 0$, such that

$$Q(T_{\min}) = \theta. \quad (24)$$

For this $T_{\min}$ and the constant control $\nu(t) \equiv 1$, equation (23) holds. This control satisfies $|\nu(t)| \leq 1$. So, the required time and control exist. $\square$

The main goal of this work is to derive a quantitative estimate for the minimal time $T_{\min}$ needed to achieve the prescribed temperature level θ .

Lemma 8. *Assume that the target temperature θ satisfies*

$$0 < \theta < \frac{\Phi_{1,1}}{\lambda_{1,1}}, \quad (25)$$

where $\Phi_{1,1}$ is defined by (10) with $k = 1$, $j = 1$, and $\lambda_{1,1}$ is the first eigenvalue of the spectral problem (5)–(6). Then there exists a minimal time $T_{\min} > 0$, such that $T_{\min} \leq T_0$ and equation (23) holds, with T_0 given explicitly by

$$T_0 = -\frac{1}{\lambda_{1,1}} \ln \left(1 - \frac{\theta \lambda_{1,1}}{\Phi_{1,1}} \right).$$

Proof. We start from the lower bound for the kernel $K(t)$ obtained in Lemma 6:

$$K(t) \geq \Phi_{1,1} e^{-\lambda_{1,1} t}, \quad t > 0.$$

Integrating this estimate gives a corresponding lower bound for $Q(t)$:

$$Q(t) = \int_0^t K(\tau) d\tau \geq \int_0^t \Phi_{1,1} e^{-\lambda_{1,1} \tau} d\tau = \frac{\Phi_{1,1}}{\lambda_{1,1}} (1 - e^{-\lambda_{1,1} t}).$$

Moreover, since $Q'(t) = K(t) > 0$ for all $t > 0$ by Lemma 6, Q is strictly increasing on $(0, \infty)$. The minimal time $T_{\min}$ is determined by the condition $Q(T_{\min}) = \theta$ (for the constant control $\nu \equiv 1$). The right-hand side of the above inequality is increasing in t and tends to $\Phi_{1,1}/\lambda_{1,1}$ as $t \rightarrow \infty$. Therefore, under condition (25), the equation $Q(t) = \theta$ has a unique solution $T_{\min} > 0$.

Now let T_0 be the time at which the lower bound itself reaches θ , i.e.,

$$\frac{\Phi_{1,1}}{\lambda_{1,1}} (1 - e^{-\lambda_{1,1}t}) = \theta.$$

Solving this equation yields

$$T_0 = -\frac{1}{\lambda_{1,1}} \ln \left(1 - \frac{\theta \lambda_{1,1}}{\Phi_{1,1}} \right).$$

Since $Q(t)$ is increasing and satisfies $Q(t) \geq \frac{\Phi_{1,1}}{\lambda_{1,1}} (1 - e^{-\lambda_{1,1}t})$ for all t , at time T_0 we have

$$Q(T_0) \geq \theta = Q(T_{\min}).$$

Monotonicity of Q then implies $T_0 \geq T_{\min}$. Hence $T_{\min} \leq T_0$, which is the desired estimate. $\square$

Proof of Theorem 1. Lemma 7 gives the existence of $T_{\min}$ and an admissible control $\nu(t)$ with $|\nu(t)| \leq 1$, such that

$$\int_0^{T_{\min}} K(T_{\min} - \tau) \nu(\tau) d\tau = \theta$$

for any $0 < \theta < Q^*$. Lemma 8 provides the estimate

$$T_{\min} \leq -\frac{1}{\lambda_{1,1}} \ln \left(1 - \frac{\theta \lambda_{1,1}}{\Phi_{1,1}} \right) = T_0$$

under the condition $0 < \theta < \Phi_{1,1}/\lambda_{1,1}$. Thus the minimal time exists and satisfies $T_{\min} \leq T_0$. $\square$

Remark. For sufficiently small θ , Theorem 1 gives the linear approximation $T_0 \approx \theta/\Phi_{1,1}$.

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