DOI: 10.15393/j3.art.2026.20650

UDC 517.521.2

R. M. Gadzhimirzaev

APPROXIMATION PROPERTIES OF THE FOURIER SERIES IN HERMITE-SOBOLEV POLYNOMIALS

Abstract. For functions from a weighted Sobolev space, we consider Fourier series in a system of polynomials, orthonormal with respect to the Sobolev inner product and associated with the system of classical Hermite polynomials. Previously, the author investigated the uniform and absolute convergence of these series. However, the approximation properties of the partial sums of these series have not been studied. In the present paper, this problem is investigated for the particular case ($r = 1$). The main attention is paid to estimating the corresponding Lebesgue function.

Key words: *Hermite-Sobolev polynomials, Fourier series, Lebesgue function*

2020 Mathematical Subject Classification: *41A10*

1. Introduction. A Sobolev inner product can be expressed by the formula [9]

$$\langle f, g\rangle = \sum_{k=0}^{m} \int_{\mathbb{R}} f^{(k)}(x) g^{(k)}(x) d\mu_k, \tag{1}$$

where $d\mu_k$ are Borel measures. Polynomials orthogonal with respect to Sobolev inner products attract the interest of researchers, particularly in connection with problems of simultaneous approximation of functions and their derivatives. Fourier series in these polynomials are used in the spectral method for solving boundary value problems for differential equations (see [1], [18], [15], [4]). In solving such problems, the issues of convergence and the rate of convergence of Fourier series in various functional spaces play an important role. The difficulty of studying these issues for Sobolev orthogonal polynomials is related to the lack of the Christoffel–Darboux formula for them, which is a consequence of the absence of three-term recurrence relation. For special cases of the inner product (1), some results were obtained, for example, in the works [10], [11], [3], [5], [14], [6], [8].

Consider a special case of the inner product (1) of the following form:

$$\langle f,g\rangle=\sum_{\nu=0}^{r-1} f^{(\nu)}(0)g^{(\nu)}(0)+\int_{\mathbb{R}} f^{(r)}(x)g^{(r)}(x)\rho(x)dx,\quad \rho(x)=e^{-x^2}. \tag{2}$$

In [16], the Hermite – Sobolev polynomials orthonormal with respect to (2) were introduced:

$$\hat{H}_{r,n}(x)=\frac{x^n}{n!},\quad n=0,1,\dots,r-1,$$

$$\hat{H}_{r,n}(x)=\frac{1}{(r-1)!}\int_0^x (x-t)^{r-1}\hat{H}_{n-r}(t)dt,\quad n\geqslant r,$$

where $\hat{H}_n(t)$ is the orthonormal Hermite polynomial of degree n. In the following, we will need some notation. Let $r\in\mathbb{N}$, $1\leqslant p<\infty$, $L^p_\rho(\mathbb{R})$ be the space of measurable functions f defined on $\mathbb{R}$, such that

$$\|f\|_{L^p_\rho(\mathbb{R})}=\left(\int_{\mathbb{R}}|f(x)|^p\rho(x)dx\right)^{\frac{1}{p}}<\infty,$$

and $W^r_{L^p_\rho(\mathbb{R})}$ be the space of functions f that are continuously differentiable $r-1$ times, for which $f^{(r-1)}$ is absolutely continuous on any segment $[-A,A]\subset\mathbb{R}$, and $f^{(r)}\in L^p_\rho(\mathbb{R})$. Note that the system of polynomials $\hat{H}_{r,n}(x)$ is complete in the space $W^r_{L^2_\rho(\mathbb{R})}$ [16]. The Fourier series of a function $f\in W^r_{L^2_\rho(\mathbb{R})}$ in this system has the form

$$f(x)\sim\sum_{k=0}^{r-1} f^{(k)}(0)\frac{x^k}{k!}+\sum_{k=r}^{\infty} c_{r,k}(f)\hat{H}_{r,k}(x), \tag{3}$$

where

$$c_{r,k}(f)=\int_{\mathbb{R}} f^{(r)}(t)\hat{H}_{k-r}(t)\rho(t)dt,\quad k\geqslant r.$$

The convergence of the series (3) was investigated in [7]. Namely, the following statements were proved.

Theorem A. [7, Theorem 1] *Let $1\leqslant p<\infty$, $\rho(x)=e^{-x^2}$. Then, given $f\in W^r_{L^p_\rho(\mathbb{R})}$, for $p\geqslant 2$ the series (3) converges uniformly to f on every* *interval* $[-A, A]$. *If* $1 \leqslant p < 2$ *then there exists a function* $f \in W^r_{L^p_\rho(\mathbb{R})}$ *whose Fourier series diverges at the point* $x_0 = 1$.

Theorem B. [7, Theorem 2] *Let* $\rho(x) = e^{-x^2}$, $f \in W^r_{L^2_\rho(\mathbb{R})}$. *Then the series* (3) *converges absolutely on every interval* $[-A, A]$.

However, the approximation properties of the partial sums of the series (3) have not been studied. In the present paper, this problem is investigated for $r = 1$.

2. Some properties of Hermite and Hermite-Sobolev polynomials. The Hermite polynomials $H_n(x)$ can be defined by the formula

$$H_n(x) = \frac{(-1)^n}{\rho(x)} \frac{d^n}{dx^n} \rho(x).$$

These polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function $\rho(x)$:

$$\int\limits_{\mathbb{R}} H_n(x) H_m(x) \rho(x) dx = h_n \delta_{nm}, \quad h_n = \sqrt{\pi} 2^n n!.$$

By $\hat{H}_n(x)$ we denote the corresponding orthonormal Hermite polynomials:

$$\hat{H}_n(x) = \frac{H_n(x)}{\sqrt{h_n}}.$$

We also note the following properties, which can be found in [17, p. 106]:

$$(H_{n+r}(x))^{(r)} = 2^r (n+r)^{[r]} H_n(x), \quad n^{[r]} = n(n-1)\cdots(n-r+1); \tag{4}$$

$$H_n(-x) = (-1)^n H_n(x);$$

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \quad H_{2n+1}(0) = 0;$$

$$K_n(x,t) = \sum_{k=0}^{n} \hat{H}_k(x) \hat{H}_k(t) = \frac{1}{2h_n} \frac{H_{n+1}(x) H_n(t) - H_n(x) H_{n+1}(t)}{x - t}. \tag{5}$$

By means of elementary transformations, the formula (5) can be written in the following form [13, (2.15)]:

$$K_n(x,t) = a_n \hat{H}_n(x) \hat{H}_n(t) + b_n \sqrt{n} \hat{H}_n(t) \frac{\hat{H}_{n+1}(x) - \hat{H}_{n-1}(x)}{x - t} +$$

$$+ b_n\sqrt{n}\hat{H}_n(x)\frac{\hat{H}_{n+1}(t)-\hat{H}_{n-1}(t)}{t-x}, \quad (6)$$

where $\frac{1}{3} \leqslant |a_n|, |b_n| \leqslant 1$. For the polynomials $\hat{H}_n(x)$, the following estimates hold [2, p. 700]:

$$\sqrt{\rho(x)}|\hat{H}_n(x)| \leqslant c\begin{cases} \left(N^{\frac{1}{3}}+N^{\frac{1}{2}}|x-\sqrt{N}|\right)^{-\frac{1}{4}}, & x\in[0,\sqrt{N}+N^{-\frac{1}{6}}];\\ \dfrac{e^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}}}{N^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}}}, & x\in[\sqrt{N}+N^{-\frac{1}{6}},\sqrt{2N}];\\ e^{-\gamma x^2}, & x\geqslant\sqrt{2N}, \end{cases} \quad (7)$$

$$\sqrt{\rho(x)}\left|\hat{H}_{n+1}(x)-\hat{H}_{n-1}(x)\right| \leqslant$$
$$\leqslant c\begin{cases} \dfrac{\left(N^{\frac{1}{3}}+N^{\frac{1}{2}}|x-\sqrt{N}|\right)^{\frac{1}{4}}}{\sqrt{N}}, & x\in[0,\sqrt{N}+N^{-\frac{1}{6}}];\\ \dfrac{N^{-\frac{3}{8}}(x-\sqrt{N})^{\frac{1}{4}}}{e^{\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}}}, & x\in[\sqrt{N}+N^{-\frac{1}{6}},\sqrt{2N}];\\ e^{-\gamma x^2}, & x\geqslant\sqrt{2N}. \end{cases} \quad (8)$$

where $N = 2n+1$, c is a positive constant, $0 < \gamma < \frac{1}{2}$. Since $\hat{H}_n(-x) = (-1)^n\hat{H}_n(x)$, similar estimates also hold for $x < 0$. Consequently, for $|x| \leqslant \sqrt{2N}$ we can write one general estimate [13, (2.3), (2.4)]:

$$\sqrt{\rho(x)}|\hat{H}_n(x)| \leqslant c\left(N^{\frac{1}{3}}+|x^2-N|\right)^{-\frac{1}{4}}, \quad (9)$$

$$\sqrt{\rho(x)}\left|\hat{H}_{n+1}(x)-\hat{H}_{n-1}(x)\right| \leqslant c\frac{\left(N^{\frac{1}{3}}+|x^2-N|\right)^{\frac{1}{4}}}{\sqrt{N}}. \quad (10)$$

In what follows, we denote by c, $c(\gamma)$ positive constants depending only on the indicated parameters, which may vary from one occurrence to another.

Lemma 1. *Let $n \in \mathbb{N}$, $x \in [0,\infty)$. Then the following estimates hold:*

$$\rho(x)K_n(x,x) \leqslant c \begin{cases} \dfrac{n}{\left(N^{\frac{1}{3}} + N^{\frac{1}{2}}|x-\sqrt{N}|\right)^{\frac{1}{2}}}, & x \in [0, \sqrt{N} + N^{-\frac{1}{6}}]; \\ \dfrac{n^{\frac{3}{4}} e^{-2\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}}}{(x-\sqrt{N})^{\frac{1}{2}}}, & x \in [\sqrt{N} + N^{-\frac{1}{6}}, \sqrt{2N}]; \\ n e^{-2\gamma x^2}, & x \geqslant \sqrt{2N}. \end{cases} \tag{11}$$

Proof. Since

$$H_{n+1}(x)H_n(t) - H_n(x)H_{n+1}(t) = \\ = \left(H_{n+1}(x) - H_{n+1}(t)\right)H_n(t) - \left(H_n(x) - H_n(t)\right)H_{n+1}(t),$$

then, as $t \to x$, we obtain from (5) and (4):

$$K_n(x,x) = \frac{1}{\sqrt{\pi}2^{n+1}n!}\left(H'_{n+1}(x)H_n(x) - H'_n(x)H_{n+1}(x)\right) = \\ = \frac{1}{\sqrt{\pi}2^{n+1}n!}\left[2(n+1)H_n(x)H_n(x) - 2nH_{n-1}(x)H_{n+1}(x)\right] = \\ = (n+1)\left(\hat{H}_n(x)\hat{H}_n(x) - \frac{\sqrt{n}}{\sqrt{n+1}}\hat{H}_{n-1}(x)\hat{H}_{n+1}(x)\right) \leqslant \\ \leqslant (n+1)\left(|\hat{H}_n(x)\hat{H}_n(x)| + \frac{\sqrt{n}}{\sqrt{n+1}}|\hat{H}_{n-1}(x)\hat{H}_{n+1}(x)|\right).$$

From this and (7) we obtain estimates (11). □

Since $K_n(-x,-x) = K_n(x,x)$, the estimates for $x \in (-\infty, 0]$ can be obtained by replacing x with $-x$.

In what follows, let $L^\infty_u(\mathbb{R})$ denote the Lebesgue space with norm

$$\|f\|_{L^\infty_u(\mathbb{R})} = ess\sup_{x\in\mathbb{R}}|f(x)u(x)|, \ u(x) = \sqrt{\rho(x)}.$$

In [12], for $f \in L^\infty_u(\mathbb{R})$, the Cesáro means of order $\delta \geqslant 0$ are considered:

$$(C,\delta)_n(f,x) = \frac{1}{A^\delta_n}\int_{\mathbb{R}} f(t)\sum_{k=0}^{n} A^\delta_{n-k}\frac{H_k(x)H_k(t)}{h_k}\rho(t)dt, \quad A^\delta_n = \binom{n+\delta}{n}$$

and estimates for the operator norm $\|(C,\delta)_n\|_{L^\infty_u(\mathbb{R})\to L^\infty_u(\mathbb{R})}$ are obtained (see Theorem 1). Observe that for $\delta=0$, the means $(C,\delta)_n(f,x)$ coincide with the Fourier – Hermite partial sums:

$$(C,0)_n(f,x)=S_n(f,x)=\int_{\mathbb{R}} f(t)\sum_{k=0}^{n}\frac{H_k(x)H_k(t)}{h_k}\rho(t)dt.$$

From Theorem 1 of the paper [12] it follows that for $\|S_n\|_{L^\infty_u(\mathbb{R})\to L^\infty_u(\mathbb{R})}$ the estimate

$$\|S_n\|_{L^\infty_u(\mathbb{R})\to L^\infty_u(\mathbb{R})}\leqslant cn^{\frac{1}{6}}. \tag{12}$$

Moreover, in the same paper it is shown (see Theorem 2) that there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$, such that

$$\|S_{n_k}\|_{L^\infty_u(\mathbb{R})\to L^\infty_u(\mathbb{R})}\geqslant cn_k^{\frac{1}{6}}. \tag{13}$$

For $\hat{H}_{r,n+r}(x)$ the equality holds [7]:

$$\hat{H}_{r,n+r}(x)=\frac{1}{\sqrt{h_n}}\frac{1}{2^r(n+r)^{[r]}}\left(H_{n+r}(x)-\mathcal{A}_{r,n+r}(x)\right), \tag{14}$$

where

$$\mathcal{A}_{r,n+r}(x)=\sum_{\nu=0}^{r-1}a_\nu(n,r)\frac{x^\nu}{\nu!},$$

$$a_\nu(n,r)=\left(H_{n+r}(x)\right)^{(\nu)}\Big|_{x=0}=2^\nu(n+r)^{[\nu]}\begin{cases}(-1)^l\frac{(2l)!}{l!}, & n+r-\nu=2l,\\ 0, & n+r-\nu=2l+1.\end{cases}$$

3. Approximation properties of the series (3). Let $f\in W^1_{L^2_\rho(\mathbb{R})}$ and $\lim\limits_{|x|\to\infty}f(x)\sqrt{\rho(x)}=0$. The partial sum of the series (3) for $r=1$ has the form

$$S_{1,n}(f,x)=f(0)+\sum_{k=1}^{n}c_{1,k}(f)\hat{H}_{1,k}(x), \tag{15}$$

where

$$c_{1,k}(f)=\frac{1}{\sqrt{h_{k-1}}}\int_{\mathbb{R}}f'(t)H_{k-1}(t)\rho(t)dt.$$

Lemma 2. *Let* $\lim\limits_{|x|\to\infty}f(x)\sqrt{\rho(x)}=0$. *Then for* $S_{1,n}(f,x)$ *the following integral representation holds:*

$$S_{1,n}(f,x)=f(0)+\int_{\mathbb{R}}f(t)\rho(t)\left(K_n(x,t)-K_n(0,t)\right)dt. \tag{16}$$

Proof. Since $(\rho(x)H_k(x))' = -\rho(x)H_{k+1}(x)$ and $\lim\limits_{|x|\to\infty} f(x)\sqrt{\rho(x)} = 0$, applying integration by parts to $c_{1,k}(f)$, we obtain

$$c_{1,k}(f) = \frac{1}{\sqrt{h_{k-1}}} \int\limits_{\mathbb{R}} f(t)H_k(t)\rho(t)dt. \tag{17}$$

On the other hand, using (14), we have

$$\hat{H}_{1,k}(x) = \frac{1}{2k\sqrt{h_{k-1}}} \left(H_k(x) - H_k(0)\right). \tag{18}$$

Note that $2kh_{k-1} = 2k\sqrt{\pi}2^{k-1}(k-1)! = h_k$. Then from (15), (17) and (18) we get (16). □

Lemma 3. *Let $N = 2n+1$, $x \in [N + N^{-1/6}, \sqrt{2N}]$. Then*

$$n^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}} \int\limits_{-\sqrt{2N}}^{x-1/n} \frac{dt}{(x-t)\left(N^{\frac{1}{3}} + |t^2-N|\right)^{\frac{1}{4}}} \leqslant cN^{\frac{1}{8}},$$

$$\frac{1}{n^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}}} \int\limits_{-\sqrt{2N}}^{x-1/n} \frac{\left(N^{\frac{1}{3}} + |t^2-N|\right)^{\frac{1}{4}}}{x-t} dt \leqslant c\Big(\frac{N^{\frac{1}{8}}}{(x-\sqrt{N})^{\frac{1}{4}}} + \ln n\Big).$$

Proof. Both integrals are estimated in exactly the same way; therefore, we only provide the details for the first one. We split the integration interval $[-\sqrt{2N}, x-1/n]$ into three parts: $[-\sqrt{2N}, x/2]$, $(x/2, \sqrt{N}]$, $(\sqrt{N}, x-1/n)$. For $t \in [-\sqrt{2N}, x/2]$, we have $x - t \geqslant x/2 \geqslant c\sqrt{N}$. Therefore,

$$n^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}} \int\limits_{-\sqrt{2N}}^{x/2} \frac{dt}{(x-t)\left(N^{\frac{1}{3}} + |t^2-N|\right)^{\frac{1}{4}}} \leqslant$$

$$\leqslant cn^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}} \frac{1}{\sqrt{N}} \int\limits_{-\sqrt{2N}}^{x/2} \frac{dt}{|t-\sqrt{N}|^{\frac{1}{4}}|t+\sqrt{N}|^{\frac{1}{4}}} \leqslant$$

$$\leqslant c\frac{(x-\sqrt{N})^{\frac{1}{4}}}{N^{\frac{3}{8}}} \frac{1}{N^{\frac{1}{8}}} \int\limits_{-\sqrt{2N}}^{x/2} \frac{dt}{|t+\sqrt{N}|^{\frac{1}{4}}} =$$

$$= c\frac{(x-\sqrt{N})^{\frac{1}{4}}}{\sqrt{N}}\Bigg(\int\limits_{-\sqrt{2N}}^{-\sqrt{N}} + \int\limits_{-\sqrt{N}}^{x/2}\Bigg)\frac{dt}{|t+\sqrt{N}|^{\frac{1}{4}}} \leqslant$$

$$\leqslant c\frac{(x-\sqrt{N})^{\frac{1}{4}}}{\sqrt{N}}\left(N^{\frac{3}{8}} + (\sqrt{N}+x/2)^{\frac{3}{4}}\right) \leqslant c.$$

For the second interval, we obtain

$$n^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}}\int\limits_{x/2}^{\sqrt{N}}\frac{dt}{(x-t)\left(N^{\frac{1}{3}}+|t^2-N|\right)^{\frac{1}{4}}} \leqslant$$

$$\leqslant c(x-\sqrt{N})^{\frac{1}{4}}\int\limits_{x/2}^{\sqrt{N}}\frac{dt}{(x-t)(\sqrt{N}-t)^{\frac{1}{4}}}.$$

Set $s = x-\sqrt{N}$, where $N^{-1/6} \leqslant s \leqslant (\sqrt{2}-1)\sqrt{N}$, and let $y = \sqrt{N}-t$, so that $0 \leqslant y < \sqrt{N}-x/2$. Then $x-t = s+y$ and

$$(x-\sqrt{N})^{\frac{1}{4}}\int\limits_{x/2}^{\sqrt{N}}\frac{dt}{(x-t)(\sqrt{N}-t)^{\frac{1}{4}}} = s^{\frac{1}{4}}\int\limits_{0}^{\sqrt{N}-x/2}\frac{dy}{(s+y)y^{\frac{1}{4}}} \leqslant$$

$$\leqslant s^{\frac{1}{4}}\frac{1}{s}\int\limits_{0}^{1}\frac{dy}{y^{\frac{1}{4}}} + s^{\frac{1}{4}}\int\limits_{1}^{\sqrt{N}-x/2}\frac{dy}{y^{\frac{5}{4}}} \leqslant cN^{\frac{1}{8}}.$$

Similarly, we estimate the integral over the third interval:

$$n^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}}\int\limits_{\sqrt{N}}^{x-1/n}\frac{dt}{(x-t)\left(N^{\frac{1}{3}}+|t^2-N|\right)^{\frac{1}{4}}} \leqslant$$

$$\leqslant c(x-\sqrt{N})^{\frac{1}{4}}\int\limits_{\sqrt{N}}^{x-1/n}\frac{dt}{(x-t)(t-\sqrt{N})^{\frac{1}{4}}} = cs^{\frac{1}{4}}\int\limits_{0}^{s-1/n}\frac{d\tau}{(s-\tau)\tau^{\frac{1}{4}}} =$$

$$= cs^{\frac{1}{4}}\Bigg(\int\limits_{0}^{s/2} + \int\limits_{s/2}^{s-1/n}\Bigg)\frac{d\tau}{(s-\tau)\tau^{\frac{1}{4}}} \leqslant$$

$$\leqslant cs^{\frac{1}{4}}\left(\frac{2}{s}\int\limits_0^{s/2}\frac{d\tau}{\tau^{\frac{1}{4}}}+\frac{1}{(s/2)^{\frac{1}{4}}}\int\limits_{s/2}^{s-1/n}\frac{d\tau}{s-\tau}\right)\leqslant c\ln n.$$

Combining the estimates for the three intervals, we obtain the desired bound. □

Further, let $\mathcal{H}_n$ be the set of algebraic polynomials of degree at most n, and let $E_n^0(f)$ be the best approximation

$$E_n^0(f)=\inf_{\substack{p_n\in\mathcal{H}_n\\ p_n(0)=f(0)}}\sup_{x\in\mathbb{R}}|f(x)-p_n(x)|\sqrt{\rho(x)}.$$

By p_n^* we denote the polynomial, for which

$$E_n^0(f)=\sup_{x\in\mathbb{R}}|f(x)-p_n^*(x)|\sqrt{\rho(x)}.$$

Then

$$\sqrt{\rho(x)}|f(x)-S_{1,n}(f,x)|\leqslant\sqrt{\rho(x)}|f(x)-p_n^*(x)|+\sqrt{\rho(x)}|S_{1,n}(f,x)-p_n^*(x)|.$$

Since $S_{1,n}(p_n^*,x)=p_n^*(x)$, then from Lemma 2 we obtain

$$\sqrt{\rho(x)}|S_{1,n}(f,x)-p_n^*(x)|=$$
$$=\sqrt{\rho(x)}\left|\int\limits_{\mathbb{R}}(f(t)-p_n^*(t))\rho(t)\left(K_n(x,t)-K_n(0,t)\right)\right|dt\leqslant$$
$$E_n^0(f)\sqrt{\rho(x)}\int\limits_{\mathbb{R}}\sqrt{\rho(t)}\,|K_n(x,t)-K_n(0,t)|\,dt.$$

Hence,

$$\sqrt{\rho(x)}|f(x)-S_{1,n}(f,x)|\leqslant(1+\mathcal{L}_n(x))E_n^0(f),\tag{19}$$

where

$$\mathcal{L}_n(x)=\sqrt{\rho(x)}\int\limits_{\mathbb{R}}\sqrt{\rho(t)}\,|K_n(x,t)-K_n(0,t)|\,dt.$$

To estimate the Lebesgue function $\mathcal{L}_n(x)$, we set

$$\Lambda_n(x)=\sqrt{\rho(x)}\int\limits_{\mathbb{R}}\sqrt{\rho(t)}\,|K_n(x,t)|\,dt.$$

Thus,
$$\mathcal{L}_n(x) \leqslant \Lambda_n(x) + \sqrt{\rho(x)}\Lambda_n(0). \tag{20}$$
Note that $\Lambda_n(-x) = \Lambda_n(x)$, so we restrict ourselves to estimating $\Lambda_n(x)$ for $x \in [0, \infty)$.

Theorem 1. *Let $N = 2n + 1$, $0 < \gamma < \frac{1}{2}$. Then the following estimates hold:*

1) if $x \in [0, \sqrt{N} + N^{-1/6}]$, then
$$\Lambda_n(x) \leqslant cn^{\frac{1}{6}};$$
2) if $x \in (\sqrt{N} + N^{-1/6}, \sqrt{2N}]$, then
$$\Lambda_n(x) \leqslant c\, e^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} N^{\frac{1}{8}}\Big(\frac{1}{(x-\sqrt{N})^{\frac{1}{4}}} + 1\Big);$$
3) if $x > \sqrt{2N}$, then
$$\Lambda_n(x) \leqslant c\, n^{\frac{5}{4}} e^{-\gamma x^2}. \tag{21}$$

Proof. Let $x > \sqrt{2N}$. Then
$$\Lambda_n(x) = \sqrt{\rho(x)}\Big(\int\limits_{|t|\leqslant\sqrt{2N}} + \int\limits_{|t|>\sqrt{2N}}\Big)\sqrt{\rho(t)}\,|K_n(x,t)|\,dt = I_1 + I_2.$$

First we estimate I_2. From (7), we have
$$I_2 = \sqrt{\rho(x)} \int\limits_{|t|>\sqrt{2N}} \sqrt{\rho(t)}\Big|\sum_{k=0}^{n} \hat{H}_k(x)\hat{H}_k(t)\Big| \leqslant$$
$$\leqslant cne^{-\gamma x^2} \int\limits_{|t|>\sqrt{2N}} e^{-\gamma t^2} dt =$$
$$= cne^{-\gamma x^2} \int\limits_{2N}^{\infty} \frac{e^{-\gamma y}}{\sqrt{y}} dy \leqslant c(\gamma)\sqrt{n}e^{-\gamma(x^2+2N)}. \tag{22}$$

We estimate I_1 using the Cauchy–Bunyakovsky inequality and Lemma 1 (we also use the fact that $\rho(-t)K_n(-t,-t) = \rho(t)K_n(t,t)$):

$$I_1 \leqslant 2\sqrt{\rho(x)K_n(x,x)} \int\limits_0^{\sqrt{2N}} \sqrt{\rho(t)K_n(t,t)}dt \leqslant$$

$$\leqslant ce^{-\gamma x^2} n^{\frac{7}{8}} \int\limits_0^{\sqrt{2N}} \frac{dt}{|t-\sqrt{N}|^{\frac{1}{4}}} \leqslant ce^{-\gamma x^2} n^{\frac{5}{4}}.$$

From this and (22), we obtain estimate (21).

Now let $x \in [0, \sqrt{N} + N^{-1/6}]$. In this case, we use the estimate (12). Note that

$$\|S_n\|_{L_u^\infty(\mathbb{R}) \to L_u^\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{x \in \mathbb{R}} \sqrt{\rho(x)} \int\limits_{\mathbb{R}} \sqrt{\rho(t)}\, |K_n(x,t)|\,.$$

Since the function $F_n(x) = \sqrt{\rho(x)} \int\limits_{\mathbb{R}} \sqrt{\rho(t)}\, |K_n(x,t)|$ is continuous on $\mathbb{R}$, then, instead of ess sup, we can write sup, i.e.

$$\|S_n\|_{L_u^\infty(\mathbb{R}) \to L_u^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} \sqrt{\rho(x)} \int\limits_{\mathbb{R}} \sqrt{\rho(t)}\, |K_n(x,t)|\,. \tag{23}$$

Therefore, $\Lambda_n(x) \leqslant \|S_n\|_{L_u^\infty(\mathbb{R}) \to L_u^\infty(\mathbb{R})}$ and

$$\Lambda_n(x) \leqslant cn^{\frac{1}{6}}.$$

Now let $x \in (\sqrt{N} + N^{-1/6}, \sqrt{2N}]$. For $|t| > \sqrt{2N}$, from the Cauchy-Bunyakovsky inequality and Lemma 1, we obtain:

$$I_2 = \sqrt{\rho(x)} \int\limits_{|t| > \sqrt{2N}} \sqrt{\rho(t)}\, |K_n(x,t)|\, dt \leqslant$$

$$\leqslant c\frac{n^{\frac{3}{8}} e^{-\gamma N^{\frac{1}{8}}(x-\sqrt{N})^{\frac{3}{4}}}}{(x-\sqrt{N})^{\frac{1}{4}}} \int\limits_{\sqrt{2N}}^{\infty} \sqrt{n} e^{-\gamma t^2} dt \leqslant c(\gamma) n^{\frac{5}{12}} e^{-2\gamma N}. \tag{24}$$

We now turn to estimating the quantity

$$I_1 = \sqrt{\rho(x)} \int\limits_{|t| \leqslant \sqrt{2N}} \sqrt{\rho(t)}\, |K_n(x,t)|\, dt.$$

To this end, we introduce the notation:

$$\mathcal{B}_1=\Big[-\sqrt{2N},x-\frac{1}{n}\Big),\ \mathcal{B}_2=\Big[x-\frac{1}{n},x+\frac{1}{n}\Big],\ \mathcal{B}_3=\Big(x+\frac{1}{n},\sqrt{2N}\Big].$$

We consider $\mathcal{B}_3$ to be empty when $x\geqslant\sqrt{2N}-\frac{1}{n}$. With this notation, we can write

$$I_1=I_1^1+I_1^2+I_1^3, \tag{25}$$

where

$$I_1^i=\sqrt{\rho(x)}\int\limits_{t\in\mathcal{B}_i}\sqrt{\rho(t)}\,|K_n(x,t)|\,dt.$$

From the Cauchy–Bunyakovsky inequality and Lemma 1, we obtain the estimate:

$$\begin{aligned}I_1^2&=\sqrt{\rho(x)}\int\limits_{t\in\mathcal{B}_2}\sqrt{\rho(t)}\,|K_n(x,t)|\,dt\leqslant\\&\leqslant(\rho(x)K_n(x,x))^{\frac12}\int\limits_{t\in\mathcal{B}_2}(\rho(t)K_n(t,t))^{\frac12}\,dt\leqslant\\&\leqslant c\,\frac{n^{\frac34}e^{-2\gamma N^{\frac14}(x-\sqrt{N})^{\frac32}}}{(x-\sqrt{N})^{\frac12}}\int\limits_{\mathcal{B}_2}dt=c\,\frac{e^{-2\gamma N^{\frac14}(x-\sqrt{N})^{\frac32}}}{n^{\frac14}(x-\sqrt{N})^{\frac12}}.\end{aligned} \tag{26}$$

Using (6) we write the inequality

$$I_1^1\leqslant I_1^{11}+I_1^{12}+I_1^{13},$$

where

$$I_1^{11}=|a_n|\sqrt{\rho(x)}|\hat{H}_n(x)|\int\limits_{\mathcal{B}_1}\sqrt{\rho(t)}|\hat{H}_n(t)|dt,$$

$$I_1^{12}=|b_n|\sqrt{n}\sqrt{\rho(x)}|\hat{H}_{n+1}(x)-\hat{H}_{n-1}(x)|\int\limits_{\mathcal{B}_1}\sqrt{\rho(t)}\frac{|\hat{H}_n(t)|}{x-t}dt,$$

$$I_1^{13}=|b_n|\sqrt{n}\sqrt{\rho(x)}|\hat{H}_n(x)|\int\limits_{\mathcal{B}_1}\sqrt{\rho(t)}\frac{|\hat{H}_{n+1}(t)-\hat{H}_{n-1}(t)|}{x-t}dt.$$

From estimates (7)–(10) and Lemma 3, we have

$$I_1^{11} \leqslant c\frac{e^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}}}{N^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}}} 2\int\limits_0^{\sqrt{2N}} \frac{dt}{N^{\frac{1}{8}}|t-\sqrt{N}|^{\frac{1}{4}}} \leqslant c\frac{e^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}}}{(x-\sqrt{N})^{\frac{1}{4}}} N^{\frac{1}{8}},$$

$$I_1^{12} \leqslant c\sqrt{n}\frac{N^{-\frac{3}{8}}(x-\sqrt{N})^{\frac{1}{4}}}{e^{\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}}} \int\limits_{\mathcal{B}_1} \frac{dt}{(x-t)\left(N^{\frac{1}{3}}+|t^2-N|\right)^{\frac{1}{4}}} \leqslant$$
$$\leqslant ce^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} N^{\frac{1}{8}},$$

$$I_1^{13} \leqslant c\frac{e^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}}}{N^{\frac{1}{8}}(x-\sqrt{N})^{\frac{1}{4}}} \int\limits_{\mathcal{B}_1} \frac{\left(N^{\frac{1}{3}}+|t^2-N|\right)^{\frac{1}{4}}}{x-t} dt \leqslant$$
$$\leqslant ce^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} \left(\frac{N^{\frac{1}{8}}}{(x-\sqrt{N})^{\frac{1}{4}}} + \ln n\right).$$

Therefore,

$$I_1^1 \leqslant ce^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} N^{\frac{1}{8}} \left(\frac{1}{(x-\sqrt{N})^{\frac{1}{4}}} + 1\right). \tag{27}$$

Applying the same argument to I_1^3 yields the estimate

$$I_1^3 \leqslant ce^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} \left(\frac{N^{\frac{1}{8}}}{(x-\sqrt{N})^{\frac{1}{4}}} + \ln n\right). \tag{28}$$

From (25)–(28), we deduce

$$I_1 \leqslant ce^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} N^{\frac{1}{8}} \left(\frac{1}{(x-\sqrt{N})^{\frac{1}{4}}} + 1\right). \tag{29}$$

Since $\Lambda_n(x) = I_1 + I_2$, from estimates (24) and (29) we obtain

$$\Lambda_n(x) \leqslant ce^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} N^{\frac{1}{8}} \left(\frac{1}{(x-\sqrt{N})^{\frac{1}{4}}} + 1\right), \; x \in (\sqrt{N}+N^{-1/6}, \sqrt{2N}].$$

□

Remark. *The estimate for $\Lambda_n(x)$ on the interval $[0, \sqrt{N}+N^{-\frac{1}{6}}]$ is sharp in order. This follows from (12), (13), and (23), together with the exponential decay of $\Lambda_n(x)$ on the two remaining intervals. The question of sharpness of the estimates on $(\sqrt{N}+N^{-\frac{1}{6}}, \sqrt{2N}]$ and $(\sqrt{2N}, \infty)$ remains open and requires further investigation.*

From Theorem 1 and inequality (20), we obtain

Corollary 1. *Let $N = 2n+1$, $0 < \gamma < \frac{1}{2}$. Then*

$$\mathcal{L}_n(x) \leqslant c \begin{cases} n^{\frac{1}{6}}, & x \in [0, \sqrt{N}+N^{-\frac{1}{6}}]; \\ e^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} N^{\frac{1}{8}} \left(\frac{1}{(x-\sqrt{N})^{\frac{1}{4}}}+1\right), & x \in (\sqrt{N}+N^{-\frac{1}{6}}, \sqrt{2N}]; \\ n^{\frac{5}{4}} e^{-\gamma x^2}, & x > \sqrt{2N}. \end{cases}$$

Combining this Corollary with inequality (19) yields the assertion.

Corollary 2. *Let $N = 2n+1$, $0 < \gamma < \frac{1}{2}$. Then for any $f \in W^1_{L^2_\rho(\mathbb{R})}$ with $\lim\limits_{|x|\to\infty} f(x)\sqrt{\rho(x)} = 0$, the following estimates hold:*

$$\sqrt{\rho(x)}|f(x) - S_{1,n}(f,x)| \leqslant$$
$$cE_n^0(f) \begin{cases} n^{\frac{1}{6}}, & |x| \in [0,\sqrt{N}+N^{-\frac{1}{6}}]; \\ 1 + e^{-\gamma N^{\frac{1}{4}}(x-\sqrt{N})^{\frac{3}{2}}} N^{\frac{1}{8}} \left(\frac{1}{(x-\sqrt{N})^{\frac{1}{4}}}+1\right), & |x| \in (\sqrt{N}+N^{-\frac{1}{6}}, \sqrt{2N}]; \\ 1, & |x| > \sqrt{2N}. \end{cases}$$

Acknowledgment. This work was supported by the Russian Science Foundation and the Government of the Republic of Dagestan (Project 25–21–20043).

References

[1] Ai Q., Li H., Wang Z. *Diagonalized Legendre spectral methods using Sobolev orthogonal polynomials for elliptic boundary value problems.* Appl. Numer. Math., 2018, vol. 127, pp. 196 – 210.
DOI: https://doi.org/10.1016/j.apnum.2018.01.003

[2] Askey R., Wainger S. *Mean convergence of expansions in Laguerre and Hermite series.* Amer. J. Math., 1965, vol. 87, pp. 698 – 708.
DOI: https://doi.org/10.2307/2373069

[3] Ciaurri Ó., Mínguez J. *Fourier series of Jacobi–Sobolev polynomials.* Integral Transforms Spec. Funct., 2019, vol. 30, no. 4, pp. 334 – 346. DOI: https://doi.org/10.1080/10652469.2018.1560279

[4] Fernández L., Marcellán F., Pérez T. E., Piñar M. A. *Sobolev orthogonal polynomials and spectral methods in boundary value problems.* Appl. Numer. Math., 2024, vol. 200, pp. 254 – 272. DOI: https://doi.org/10.1016/j.apnum.2023.07.027

[5] Gadzhimirzaev R. M. *Sobolev-orthonormal system of functions generated by the system of Laguerre functions.* Probl. Anal. Issues Anal., 2019, vol. 8(26), no. 1, pp. 32 – 46. DOI: https://doi.org/10.15393/j3.art.2019.5150

[6] Gadzhimirzaev R. M. *Estimates for the convergence rate of a Fourier series in Laguerre-Sobolev polynomials.* Sib. Math. J., 2024, vol. 65, pp. 751–763. DOI: https://doi.org/10.1134/S0037446624040025

[7] Gadzhimirzaev R. M. *Uniform and absolute convergence of the Fourier series in Hermite-Sobolev polynomials.* Sib. Math. J., 2025, vol. 66, pp. 1149 – 1157. DOI: https://doi.org/10.1134/S0037446625050052

[8] Magomed-Kasumov M. G. *The uniform convergence of Fourier series in a system of the Sobolev orthogonal polynomials associated to ultraspherical Jacobi polynomials.* Sib. Math. J., 2024, vol. 65, pp. 1343 – 1358. DOI: https://doi.org/10.1134/S0037446624060090

[9] Marcellán F., Xu Y. *On Sobolev orthogonal polynomials.* Expo. Math., 2015, vol. 33, no. 3, pp. 308 – 352. DOI: https://doi.org/10.1016/j.exmath.2014.10.002

[10] Marcellán F., Osilenker B. P., Rocha I. A. *On Fourier series of a discrete Jacobi-Sobolev inner product.* J. Approx. Theory, 2002, vol. 117, no. 1, pp. 1 – 22. DOI: https://doi.org/10.1006/jath.2002.3681

[11] Marcellán F., Quintana Y., Urieles A. *On the Pollard decompositionmethod applied to some Jacobi-Sobolev expansions.* Turkish J. Math., 2013, vol. 37, no. 6, pp. 934 – 948. DOI: https://doi.org/10.3906/mat-1208-29

[12] Markett C. *Norm estimates for (C, δ) means of Hermite expansions and bounds for δ_{eff}.* Acta Math. Hung., 1984, vol. 43, pp. 187 – 198. DOI: https://doi.org/10.1007/BF01958018

[13] Muckenhoupt B. *Mean convergence of Hermite and Laguerre series. II* Trans. Amer. Math. Soc., 1970, vol. 147, no. 2, pp. 433 – 460. DOI: https://doi.org/10.2307/1995205

[14] Osilenker B. P. *On multipliers for Fourier series in Sobolev orthogonal polynomials.* Sb. Math., 2022, vol. 213, no. 8, pp. 1058 – 1095. DOI: https://doi.org/10.4213/sm9556e

[15] Sharapudinov I. I. *Sobolev-orthogonal systems of functions and some of their applications.* Russian Math. Surveys, 2019, vol. 74, no. 4, pp. 659 – 733. DOI: https://doi.org/10.1070/RM9846

[16] Sharapudinov I. I., Gadzhieva Z. D., Gadzhimirzaev R. M. *Systems of functions orthogonal with respect to scalar products of Sobolev type with discrete masses generated by classical orthogonal systems.* Daghestan Electronic Mathematical Reports, 2016, no. 6, pp. 31 – 60. (in Russian) DOI: https://doi.org/10.31029/demr.6.3

[17] Szegö G. *Orthogonal Polynomials.* Amer. Math. Soc. Colloq. Publ, 23, Amer. Math. Soc., Providence R.I. 1975. Fourth Edition.

[18] Yu X., Wang Z., Li H. *Jacobi-Sobolev orthogonal polynomials and spectral methods for elliptic boundary value problems.* Commun. Appl. Math. Comput., 2019, vol. 1, pp. 283 – 308. DOI: https://doi.org/10.1007/s42967-019-00016-x

Received April 22, 2026.
In revised form, September 11, 2026.
Accepted September 16, 2026.
Published online September 27, 2026.

Dagestan Federal Research Center of the RAS
45, M. Gadzhieva st., Makhachkala, 367025, Russia
E-mail: ramis3004@gmail.com