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EXPANSIVE AND DEGENERATE BEHAVIOUR OF MULTIPLICATIVE KANNAN-TYPE CONTRACTIONS FOR $\beta > 1$

Abstract. This paper analyzes multiplicative Kannan-type contractions (MKC) $\mathfrak{p}(\mathcal{T}r,\mathcal{T}s) \leqslant \mu\,[\mathfrak{p}(r,\mathcal{T}r)]^{\beta}[\mathfrak{p}(s,\mathcal{T}s)]^{1-\beta}$, $0 \leqslant \mu < 1$, outside the contractive range, with emphasis on the case $\beta > 1$. We show that when $\beta > 1$ the MKC condition induces an *expansive* mapping, so that distances between successive iterates increase, rendering classical Cauchy-based fixed point techniques ineffective. A classification is obtained according to the behaviour of the distances between points and their images under the mapping. If these distances are unbounded or admit arbitrarily small values, then any map satisfying (MKC) on subsets of $\mathbb{R}$ is necessarily *degenerate*, in the sense that the image of all non-fixed points collapses to a singleton. Non-degenerate maps arise only when these distances remain bounded and bounded away from zero. These results refine and correct existing assertions on (MKC) for $\beta > 1$.

Key words: *Kannan type contractions, interpolation, fixed point, expansive map*

2020 Mathematical Subject Classification: *47H10; 54H25*

1. Introduction and Preliminaries. A central and productive theme in metric fixed point theory is the generalization of foundational results, such as the Banach Contraction Principle. Researchers frequently seek to relax the conditions of classic theorems to cover a broader class of mappings, thereby extending their applicability.

A recent and ambitious attempt in this vein was the paper by Kessy and Wangwe [6], which sought to generalize Karapinar's interpolative Kannan-type contraction [4]. To be precise, let $(X, \mathfrak{p})$ be a metric space and let $\mathcal{T}\colon X \to X$ be a map. The authors proposed a theorem for what is known as a Multiplicative Kannan-type Contraction (MKC), defined by the inequality:

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta} \tag{MKC}$$

for all distinct $r, s \in X \setminus \mathrm{Fix}(\mathcal{T})$, where $0 \leqslant \mu < 1$ and β is a real parameter.

While the original result in [4] held for $0 < \beta < 1$, the restriction $r, s \in X \setminus \mathrm{Fix}(\mathcal{T})$ was noticed in [5], the generalization in [6] claimed that the theorem holds for all $\beta \in \mathbb{R}$. While such generalizations are valuable, they require rigorous verification. The assertion by Kessy and Wangwe contained a subtle but critical flaw in its proof methodology for the case where $\beta \geqslant 1$.

Concerning Example 1 they gave, the map is continuous. Since X is finite, every function $\mathcal{T}\colon X \to X$ is continuous with respect to the subspace topology of $\mathbb{R}$. However, as noted in [6], the MKC inequality remains trivially true on X because for all non-fixed r,s the images coincide. Thus, MKC may hold for maps where the image of all non-fixed points collapses to a single value.

The purpose of this paper is to move beyond merely identifying that flaw. We provide a conclusive mathematical proof that the claimed generalization of the multiplicative Kannan-type contraction cannot hold when $\beta > 1$. In this parameter range, the defining inequality no longer yields a contractive procedure; instead, it produces an expansive map, in which distances between successive iterates necessarily increase. Such expansivity is fundamentally incompatible with the iterative Cauchy sequence methods in fixed point theory. Furthermore, we show that any function satisfying the (MKC) condition for $\beta > 1$ must either be constant or *degenerate*; that is, all non-fixed points share a common image. This reveals that the case $\beta > 1$ does not extend the Kannan framework but collapses it into trivial or pathological behaviour. The analysis in this paper follows the principles of metric topology as presented by Engelking [2], ensuring topological treatment of continuity and convergence. The fixed point arguments rely on the classical theory of metric contractions and nonexpansive mappings developed by Kirk and Khamsi [7] and some other results in [1], [3], [8] (see also references therein).

2. The Expansive Nature of the Inequality for $\boldsymbol{\beta > 1}$.

The defining inequality (MKC) [6] is

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}$$

with $0 < \mu < 1$. While this condition leads to a fixed point for $\beta < 1$ Karapinar [4], its character fundamentally changes when β exceeds 1.

Proposition 1. *Let $(X, \mathfrak{p})$ be a metric space and let $\mathcal{T}\colon X \to X$ satisfy the (MKC) condition for all $r, s \in X \setminus \operatorname{Fix}(\mathcal{T})$, where $0 < \mu < 1$ and $\beta > 1$. Fix any initial point $s_0 \in X \setminus \operatorname{Fix}(\mathcal{T})$ and define $s_{n+1} = \mathcal{T} s_n$ $(n \geqslant 0)$. Put*

$$a_n = \mathfrak{p}(s_n, s_{n+1}), \quad n \geqslant 0.$$

If $a_n > 0$ for all n (i.e., the sequence never reaches a fixed point), then the sequence (a_n) is strictly increasing and there exists a constant

$$c = \mu^{-1/(\beta-1)} > 1,$$

such that

$$a_n \geqslant c\, a_{n-1} \quad \textit{for all } n \geqslant 1,$$

hence $a_n \geqslant c^n a_0$ and, in particular, $a_n \to \infty$ as $n \to \infty$.

Proof. Apply (MKC) with $r = s_n$ and $s = s_{n-1}$ for $n \geqslant 1$:

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) = \mathfrak{p}(s_{n+1}, s_n) \leqslant \mu\, [\mathfrak{p}(s_n, s_{n+1})]^{\beta}\, [\mathfrak{p}(s_{n-1}, s_n)]^{1-\beta}.$$

In the notation $a_n = \mathfrak{p}(s_n, s_{n+1})$ this reads:

$$a_n \leqslant \mu\, a_n^{\beta}\, a_{n-1}^{1-\beta}.$$

Since by assumption each $a_n > 0$, we may divide both sides by a_n^{β}:

$$a_n^{1-\beta} \leqslant \mu\, a_{n-1}^{1-\beta}.$$

Let $\gamma = \beta - 1 > 0$. The inequality is $a_n^{-\gamma} \leqslant \mu a_{n-1}^{-\gamma}$, or

$$\frac{1}{a_n^{\gamma}} \leqslant \mu \frac{1}{a_{n-1}^{\gamma}}.$$

Rearranging yields (since $\mu > 0$)

$$a_{n-1}^{\gamma} \leqslant \mu a_n^{\gamma} \quad \Longleftrightarrow \quad \left(\frac{a_n}{a_{n-1}}\right)^{\gamma} \geqslant \frac{1}{\mu}.$$

Since $\gamma = \beta - 1 > 0$, we take the $1/\gamma$ root:

$$\frac{a_n}{a_{n-1}} \geqslant \mu^{-1/(\beta-1)} =: c > 1.$$

Thus $a_n \geqslant c\,a_{n-1}$ for every $n \geqslant 1$. Iterating this inequality gives $a_n \geqslant c^n a_0$, whence $a_n \to \infty$ as $n \to \infty$ because $c > 1$ and $a_0 > 0$. $\square$

Corollary 1. *Under the assumption of Proposition 1, if $(X, \mathfrak{p})$ has finite diameter (in particular, when $X \subset \mathbb{R}$ is bounded), then the assumption $a_n > 0$ for all n cannot hold. Consequently, every (s_n) starting in $X \setminus \mathrm{Fix}(\mathcal{T})$ must eventually have $a_N = 0$ for some N; equivalently $s_N = s_{N+1}$, so $s_N \in \mathrm{Fix}(\mathcal{T})$ and the sequence reaches a fixed point in finitely many steps.*

Proof. If X has a finite diameter $\mathrm{diam}(X) < \infty$, then the distances $a_n \leqslant \mathrm{diam}(X)$ for all n. But Proposition 1 gives $a_n \geqslant c^n a_0 \to \infty$: a contradiction. Hence some a_N must be zero, so $s_N = s_{N+1}$, s_N is a fixed point of $\mathcal{T}$. $\square$

Definition 1. *A map $\mathcal{T} \colon X \to X$ satisfying (MKC) with $\beta > 1$ is called* degenerate *if the image of the set of non-fixed points is a singleton, i.e., if $\mathcal{T}(X \setminus \mathrm{Fix}(\mathcal{T})) = \{q\}$ for some $q \in X$. If $\mathcal{T}(X \setminus \mathrm{Fix}(\mathcal{T}))$ contains more than one point, $\mathcal{T}$ is* non-degenerate.

3. Characterization of (MKC) Maps and on Subsets of $\mathbb{R}$.

Theorem 1. *Let $(X, \mathfrak{p})$ be a metric space. Let $\mathcal{T} \colon X \to X$ satisfy (MKC) for all distinct $r, s \in X \setminus \mathrm{Fix}(\mathcal{T})$ with $0 \leqslant \mu < 1$ and $\beta > 1$.*

Let $a \in \mathrm{Fix}(\mathcal{T})$ be a fixed point of $\mathcal{T}$. Assume that

1) *$\mathcal{T}$ is continuous at a.*
2) *a is an accumulation point of $X \setminus \mathrm{Fix}(\mathcal{T})$.*

Then $\mathcal{T}$ is constant on $X \setminus \mathrm{Fix}(\mathcal{T})$.

Proof. Let $a \in \mathrm{Fix}(\mathcal{T})$ be the point of continuity, and let $s \in X \setminus \mathrm{Fix}(\mathcal{T})$ be arbitrary.

Step 1. We find a sequence (r_n) with $\lim\limits_{n\to\infty} \mathfrak{p}(r_n, \mathcal{T} r_n) = 0$. Since a is an accumulation point of $X \setminus \mathrm{Fix}(\mathcal{T})$, there exists a sequence $(r_n) \subset X \setminus \mathrm{Fix}(\mathcal{T})$ with $r_n \to a$. Since $a \in \mathrm{Fix}(\mathcal{T})$, we have $\mathcal{T} a = a$. Since $\mathcal{T}$ is continuous at a, $r_n \to a \implies \mathcal{T} r_n \to \mathcal{T} a = a$. By continuity of the metric $\mathfrak{p}$, we have

$$\lim_{n\to\infty} \mathfrak{p}(r_n, \mathcal{T} r_n) = \mathfrak{p}(\lim r_n, \lim \mathcal{T} r_n) = \mathfrak{p}(a, a) = 0.$$

Therefore, $\lim\limits_{n\to\infty} [\mathfrak{p}(r_n, \mathcal{T} r_n)]^{\beta} = 0$.

Step 2. We show that $\mathcal{T}s = a$. Apply MKC to the pair (r_n, s) for n large enough $(r_n \neq s)$ to get

$$0 \leqslant \mathfrak{p}(\mathcal{T}r_n, \mathcal{T}s) \leqslant \mu[\mathfrak{p}(r_n, \mathcal{T}r_n)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}.$$

Take the limit as $n \to \infty$. The limit of the right-hand side is

$$\lim_{n\to\infty} \left(\mu[\mathfrak{p}(r_n, \mathcal{T}r_n)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}\right) = \mu \cdot 0 \cdot [\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta} = 0.$$

By the Squeeze Theorem, $\lim\limits_{n\to\infty} \mathfrak{p}(\mathcal{T}r_n, \mathcal{T}s) = 0$.

We also know from Step 1 that $\mathcal{T}r_n \to a$. By the properties of limits in a metric space,

$$\mathfrak{p}(a, \mathcal{T}s) = \lim_{n\to\infty} \mathfrak{p}(\mathcal{T}r_n, \mathcal{T}s) = 0.$$

This implies $\mathcal{T}s = a$. Since s was arbitrary, $\mathcal{T}(x) = a$ for all $x \in X \setminus \mathrm{Fix}(\mathcal{T})$. $\square$

Example 1. Let $X = [-1, 1] \cup \{3\}$ be equipped with the usual metric $\mathfrak{p}(x, y) = |x - y|$. Define the mapping $\mathcal{T} \colon X \to X$ by

$$\mathcal{T}x = \begin{cases} 0, & \text{if } x \in [-1, 1], \\ 3, & \text{if } x = 3. \end{cases}$$

This mapping is non-constant on X since $\mathcal{T}(0) = 0$ while $\mathcal{T}(3) = 3$. We verify that it satisfies all assumptions of Theorem 1:

1) Fixed points: $\mathrm{Fix}(\mathcal{T}) = \{0, 3\}$.
2) Non-fixed points: $X \setminus \mathrm{Fix}(\mathcal{T}) = [-1, 0) \cup (0, 1]$.
3) Accumulation point: The fixed point $a = 0$ is an accumulation point of $X \setminus \mathrm{Fix}(\mathcal{T})$.
4) Continuity at a: The mapping $\mathcal{T}$ is continuous at $a = 0$ (it is constant on the interval [−1,1]).
5) MKC condition: For any distinct $r, s \in X \setminus \mathrm{Fix}(\mathcal{T})$, we have $\mathcal{T}r = \mathcal{T}s = 0$, so $\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) = 0$. Therefore, the inequality

   $$0 \leqslant \mu\, [\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\, [\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}$$

   holds trivially for every $0 \leqslant \mu < 1$ and $\beta > 1$.

This example illustrates that continuity at an accumulation fixed point a, together with MKC condition for $\beta > 1$, forces $\mathcal{T}$ to be constant on the entire set $X \setminus \mathrm{Fix}(\mathcal{T})$ (with value a), even when the space is disconnected and possesses multiple fixed points.

Corollary 2. *Let $X \subset \mathbb{R}$, $\mathfrak{p}(r,s) = |r-s|$. Let $\mathcal{T}\colon X \to X$ satisfy* (MKC)

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}$$

for all distinct $r, s \in X \setminus \mathrm{Fix}(\mathcal{T})$, where $0 \leqslant \mu < 1$ and $\beta > 1$. Let $I \subset X$ be a nonempty interval and suppose $X \setminus \mathrm{Fix}(\mathcal{T})$ is dense in I. If $\mathcal{T}$ is continuous at some $a \in I$ and $a \in \mathrm{Fix}(\mathcal{T})$, then $\mathcal{T}$ is constant on $X \setminus \mathrm{Fix}(\mathcal{T})$.

Remark 1. *The assumption of continuity at the fixed point a in Theorem 1 and Corollary 2 imposes a structural constraint when $\beta > 1$.*

In the contractive case, where $0 < \beta < 1$, the MKC condition typically forces the iterates toward a fixed point. By contrast, when $\beta > 1$, MKC condition becomes fundamentally expansive. Without the continuity assumption at a, points arbitrarily close to a may be mapped to points arbitrarily far from a.

Continuity at a, together with the assumption that a is an accumulation point of $X \setminus \mathrm{Fix}(\mathcal{T})$, forces the image of all non-fixed points sufficiently close to a to coincide with a. This collapse, combined with the MKC inequality for $\beta > 1$, extends and implies that $\mathcal{T}$ is constant on the entire set $X \setminus \mathrm{Fix}(\mathcal{T})$.

If the continuity assumption at a is removed, then the mapping may admit a non-degenerate image on $X \setminus \mathrm{Fix}(\mathcal{T})$, but it must be discontinuous at every fixed point that is an accumulation point of $X \setminus \mathrm{Fix}(\mathcal{T})$.

Theorem 2. *Let $(X, \mathfrak{p})$ be a metric space. Let $\mathcal{T}\colon X \to X$ satisfy (MKC) with $0 \leqslant \mu < 1$, $\beta > 1$. If there exists a sequence $(r_n) \subset X \setminus \mathrm{Fix}(\mathcal{T})$ with $\mathfrak{p}(r_n, \mathcal{T}r_n) \to 0$, then $\mathcal{T}$ is constant on $X \setminus \mathrm{Fix}(\mathcal{T})$.*

Proof. Let (r_n) be the sequence from the assumption, so $\mathfrak{p}(r_n, \mathcal{T}r_n) \to 0$. Let s_1 and s_2 be two arbitrary points in $X \setminus \mathrm{Fix}(\mathcal{T})$. Applying (MKC) with $r = r_n$ and $s = s_1$, we have

$$\mathfrak{p}(\mathcal{T}r_n, \mathcal{T}s_1) \leqslant \mu\,[\mathfrak{p}(r_n, \mathcal{T}r_n)]^{\beta}\,[\mathfrak{p}(s_1, \mathcal{T}s_1)]^{1-\beta}.$$

As $n \to \infty$, $\mu\,[\mathfrak{p}(r_n, \mathcal{T}r_n)]^{\beta}\,[\mathfrak{p}(s_1, \mathcal{T}s_1)]^{1-\beta} \to 0$ because $\mathfrak{p}(r_n, \mathcal{T}r_n) \to 0$ and $\beta > 0$.

By the Squeeze Theorem, $\lim_{n\to\infty} \mathfrak{p}(\mathcal{T}r_n, \mathcal{T}s_1) = 0$, which means $\lim_{n\to\infty} \mathcal{T}r_n = \mathcal{T}s_1$.

Now apply (MKC) with $r = r_n$ and $s = s_2$ to get

$$\mathfrak{p}(\mathcal{T}r_n, \mathcal{T}s_2) \leqslant \mu\,[\mathfrak{p}(r_n, \mathcal{T}r_n)]^{\beta}\,[\mathfrak{p}(s_2, \mathcal{T}s_2)]^{1-\beta}.$$

As $n \to \infty$, the RHS also $\to 0$.

By the Squeeze Theorem again, $\lim_{n\to\infty} \mathfrak{p}(\mathcal{T}r_n, \mathcal{T}s_2) = 0$, which means $\lim_{n\to\infty} \mathcal{T}r_n = \mathcal{T}s_2$.

By the uniqueness of limits in $\mathbb{R}$, the single sequence $(\mathcal{T}r_n)$ has a single limit. Therefore,

$$\mathcal{T}s_1 = \mathcal{T}s_2.$$

Since s_1 and s_2 were arbitrary points in $X \setminus \mathrm{Fix}(\mathcal{T})$, this proves that $\mathcal{T}$ is constant on $X \setminus \mathrm{Fix}(\mathcal{T})$. □

Example 2. Let $X = \{0, 1\} \cup \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$ be equipped with the usual metric $\mathfrak{p}(x, y) = |x - y|$. Define the mapping $\mathcal{T} \colon X \to X$ by

$$\mathcal{T}x = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \neq 0. \end{cases}$$

This mapping is non-constant on X since $\mathcal{T}(0) = 0$ while $\mathcal{T}x = 1$ for all $x \neq 0$.

1) Fixed points: $\mathrm{Fix}(\mathcal{T}) = \{0, 1\}$.
2) Non-fixed points: $X \setminus \mathrm{Fix}(\mathcal{T}) = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$.
3) Vanishing defects: Consider the sequence (r_n) in $X \setminus \mathrm{Fix}(\mathcal{T})$ given by $r_n = 1 + \frac{1}{n}$. Then

$$\mathfrak{p}(r_n, \mathcal{T}r_n) = |1 + \frac{1}{n} - 1| = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

4) MKC condition: For any distinct $r, s \in X \setminus \mathrm{Fix}(\mathcal{T})$, we have $\mathcal{T}r = \mathcal{T}s = 1$, so $\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) = 0$. Thus the inequality

$$0 \leqslant \mu\,[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\,[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}$$

holds trivially for every $0 \leqslant \mu < 1$ and $\beta > 1$.

This example shows that the existence of even a single sequence of non-fixed points with vanishing defects is sufficient to force the MKC condition with $\beta > 1$ to collapse the image of the entire set $X\setminus \mathrm{Fix}(\mathcal{T})$ to a singleton, as predicted by Theorem 2.

Proposition 2. *Let $(X, \mathfrak{p})$ be a metric space. Let $\mathcal{T}\colon X \to X$ satisfy (MKC) with $0 \leqslant \mu < 1$, $\beta > 1$. Suppose*

$$m = \inf_{x\in X\setminus \mathrm{Fix}(\mathcal{T})} \mathfrak{p}(x, \mathcal{T}x) > 0.$$

Then, for all distinct $r, s \in X\setminus \mathrm{Fix}(\mathcal{T})$,

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\, m^{1-\beta}\, [\mathfrak{p}(r, \mathcal{T}r)]^{\beta},$$

and hence

$$\operatorname{diam}\big(\mathcal{T}(X\setminus \mathrm{Fix}(\mathcal{T}))\big) \leqslant \mu\, m^{1-\beta} \sup_{r\in X\setminus \mathrm{Fix}(\mathcal{T})} [\mathfrak{p}(r, \mathcal{T}r)]^{\beta}.$$

Proof. Let $r, s \in X\setminus \mathrm{Fix}(\mathcal{T})$ be distinct. The (MKC) inequality is:

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\, [\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\, [\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}.$$

By the assumption, $\mathfrak{p}(s, \mathcal{T}s) \geqslant m$. Since $\beta > 1$, the exponent $1 - \beta$ is negative. Therefore, the function $f(x) = x^{1-\beta}$ is a decreasing function for $x > 0$. Applying this decreasing function to the inequality $\mathfrak{p}(s, \mathcal{T}s) \geqslant m$ reverses the sign, so we get

$$[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta} \leqslant m^{1-\beta}.$$

Substitute this into (MKC) inequality to get

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\, [\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\, m^{1-\beta}.$$

This proves the first inequality.

To find the diameter of the image set $\mathcal{T}(X\setminus \mathrm{Fix}(\mathcal{T}))$, we take the supremum of the left-hand side over all distinct pairs r, s. This is less than or equal to the supremum of the right-hand side over all r. Thus

$$\begin{aligned}\operatorname{diam}\big(\mathcal{T}(X\setminus \mathrm{Fix}(\mathcal{T}))\big) &= \sup_{r,s\in X\setminus \mathrm{Fix}(\mathcal{T}), r\neq s} \mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \\ &\leqslant \sup_{r\in X\setminus \mathrm{Fix}(\mathcal{T})} \left(\mu\, m^{1-\beta}\, [\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\right).\end{aligned}$$

Since $\mu\, m^{1-\beta}$ is a positive constant, we can pull it out of the supremum. Therefore,

$$\operatorname{diam}\big(\mathcal{T}(X\backslash\operatorname{Fix}(\mathcal{T}))\big) \leqslant \mu\, m^{1-\beta} \sup_{r\in X\backslash\operatorname{Fix}(\mathcal{T})} [\mathfrak{p}(r, \mathcal{T}r)]^{\beta}.$$

This proves the second inequality. □

Proposition 3. *There exist $X \subset \mathbb{R}$ and $\mathcal{T}\colon X \to X$ discontinuous (relative to $\mathbb{R}$), such that (MKC) holds for some $\beta > 1$ and some $0 \leqslant \mu < 1$, and $\mathcal{T}(X\backslash\operatorname{Fix}(\mathcal{T}))$ is a singleton.*

Proof. Fix $\beta > 1$ and any $0 \leqslant \mu < 1$. Choose $\alpha \in \mathbb{R}$ and a discrete sequence $x_n = 1/n$ $(n \geqslant 2)$; assume $\alpha \notin \{x_n \colon n \geqslant 2\}$. Put

$$X = \{\alpha\} \cup \{x_n \colon n \geqslant 2\} \subset \mathbb{R},$$

and define $\mathcal{T}\colon X \to X$ by

$$\mathcal{T}(x_n) = \alpha \quad (n \geqslant 2), \quad \mathcal{T}(\alpha) = \alpha.$$

Then $\operatorname{Fix}(\mathcal{T}) = \{\alpha\}$ and $X\backslash\operatorname{Fix}(\mathcal{T}) = \{x_n \colon n \geqslant 2\}$. Moreover

$$\mathcal{T}(X\backslash\operatorname{Fix}(\mathcal{T})) = \{\alpha\},$$

a singleton. Let $r, s \in X\backslash\operatorname{Fix}(\mathcal{T})$. Then $r = x_n$, $s = x_m$ for some $n, m \geqslant 2$, and

$$\mathfrak{p}(\mathcal{T}r,\mathcal{T}s) = \mathfrak{p}(\alpha, \alpha) = 0.$$

Hence, for every choice of $\beta \in \mathbb{R}$ and every $0 \leqslant \mu < 1$, the inequality

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\, [\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\, [\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}$$

holds (left-hand side 0). Thus (MKC) is satisfied.

Discontinuity. As $x_n \to 0$ in $\mathbb{R}$ while $\mathcal{T}(x_n) = \alpha$ does not approach a value $\mathcal{T}(0)$ (indeed $0 \notin X$), the map $\mathcal{T}$ is discontinuous when X is viewed as a subset of $\mathbb{R}$. This proves the claim. □

Remark 2. *Let $X = [0, 1]$. Define $\mathcal{T}\colon X \to X$ by*

$$\mathcal{T}(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x = 1, \\ 1/2, & \text{if } x \in (0, 1). \end{cases}$$

We check the conditions for any $0 < \mu < 1$ and $\beta > 1$.

1) *Discontinuity. The map $\mathcal{T}$ is discontinuous. At $x = 0$, $\mathcal{T}(0) = 0$, but $\lim_{x\to 0^+} \mathcal{T}(x) = 1/2$. Thus, $\mathcal{T}$ is discontinuous at $x = 0$ (and also at $x = 1$).*
2) *Degeneracy. The fixed points are $\mathrm{Fix}(\mathcal{T}) = \{0, 1\}$. The set of non-fixed points is $X\setminus \mathrm{Fix}(\mathcal{T}) = (0, 1)$. The image of this set is $\mathcal{T}((0, 1)) = \{1/2\}$. This is a singleton.*
3) *(MKC) inequality. The inequality holds trivially. We must check (MKC) for distinct $r, s \in X\setminus \mathrm{Fix}(\mathcal{T})$. For any such $r, s \in (0, 1)$:*

   LHS $\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) = |1/2 - 1/2| = 0.$

   RHS $\mu[\mathfrak{p}(r, 1/2)]^{\beta}[\mathfrak{p}(s, 1/2)]^{1-\beta}.$

   Since $r, s \in (0, 1)$, $\mathfrak{p}(r, 1/2) > 0$ and $\mathfrak{p}(s, 1/2) > 0$. The RHS is a well-defined positive number. The (MKC) inequality $0 \leqslant$ (RHS) is therefore true.

This shows that a discontinuous, degenerate (MKC) map exists.

Lemma 1. *Let $(X, \mathfrak{p})$ be a metric space and let $\mathcal{T}\colon X \to X$ satisfy (MKC)*

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\,[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\,[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta} \quad \text{for all distinct } r, s \in X\setminus \mathrm{Fix}(\mathcal{T}),$$

with constants $0 < \mu < 1$ and $\beta > 1$.

Assume there exists a sequence $\{x_n\} \subset X\setminus \mathrm{Fix}(\mathcal{T})$ with $x_n \to a \in \overline{X}\setminus X$ and a partition of a subsequence of (x_n) into finitely many disjoint subsequences $\{x_n^{(j)}\}$ $(j = 1, \ldots, m)$, such that $\mathcal{T}x_n^{(j)} \to L_j$ and $\inf_n \mathfrak{p}(x_n^{(j)}, \mathcal{T}x_n^{(j)}) \geqslant d_j > 0$ for each j.

Assume

$$\text{(H)} \quad \sup\{\mathfrak{p}(x,\mathcal{T}x)\colon x \in X\setminus \mathrm{Fix}(\mathcal{T})\} = +\infty.$$

Then $m = 1$.

Proof. Assume for contradiction that $m \geqslant 2$. This implies there exist at least two distinct cluster limits, $L_i \neq L_j$.

By assumption (H), $\sup\{\mathfrak{p}(x,\mathcal{T}x)\colon x \in X\setminus \mathrm{Fix}(\mathcal{T})\} = +\infty$. We can therefore choose a sequence $\{s_k\} \subset X\setminus \mathrm{Fix}(\mathcal{T})$, such that its distances $\delta_k = \mathfrak{p}(s_k, \mathcal{T}s_k)$ go to infinity, i.e., $\lim_{k\to\infty} \delta_k = +\infty$.

Apply the (MKC) inequality to the pair $(r, s) = (x_n^{(i)}, s_k)$ for any fixed k, thus

$$\mathfrak{p}(\mathcal{T}x_n^{(i)}, \mathcal{T}s_k) \leqslant \mu[\mathfrak{p}(x_n^{(i)}, \mathcal{T}x_n^{(i)})]^\beta [\mathfrak{p}(s_k, \mathcal{T}s_k)]^{1-\beta}.$$

Now, take the limit of this inequality as $n \to \infty$. The left-hand side converges to $\mathfrak{p}(L_i, \mathcal{T}s_k)$. The term $\mathfrak{p}(x_n^{(i)}, \mathcal{T}x_n^{(i)})$ converges to $\mathfrak{p}(a, L_i)$.

$$\lim_{n\to\infty} \mathfrak{p}(\mathcal{T}x_n^{(i)}, \mathcal{T}s_k) \leqslant \lim_{n\to\infty} \left(\mu[\mathfrak{p}(x_n^{(i)}, \mathcal{T}x_n^{(i)})]^\beta \delta_k^{1-\beta}\right),$$

$$\mathfrak{p}(L_i, \mathcal{T}s_k) \leqslant \mu \left(\lim_{n\to\infty} [\mathfrak{p}(x_n^{(i)}, \mathcal{T}x_n^{(i)})]^\beta\right) \delta_k^{1-\beta},$$

$$\mathfrak{p}(L_i, \mathcal{T}s_k) \leqslant \mu \mathfrak{p}(a, L_i)^\beta \delta_k^{1-\beta}.$$

Let $C_i = \mathfrak{p}(a,L_i)^\beta$. The assumption $\inf\limits_n \mathfrak{p}(x_n^{(i)},\mathcal{T}x_n^{(i)}) \geqslant d_i > 0$ ensures that $\mathfrak{p}(a, L_i) \geqslant d_i > 0$, so C_i is a finite positive constant. The inequality is

$$\mathfrak{p}(L_i, \mathcal{T}s_k) \leqslant \mu \frac{C_i}{\delta_k^{\beta-1}}.$$

By a similar argument for the cluster L_j, there exists a finite positive constant $C_j = \mathfrak{p}(a, L_j)^\beta$, such that

$$\mathfrak{p}(L_j, \mathcal{T}s_k) \leqslant \mu \frac{C_j}{\delta_k^{\beta-1}}.$$

We now take the limit of these two inequalities as $k \to \infty$. Since $\beta > 1$, the exponent $\beta - 1$ is positive. As $k \to \infty$, $\delta_k \to \infty$, which implies $\delta_k^{\beta-1} \to \infty$. Therefore, the right-hand side of both inequalities tends to 0:

$$\lim_{k\to\infty} \mu \frac{C_i}{\delta_k^{\beta-1}} = 0 \quad \text{and} \quad \lim_{k\to\infty} \mu \frac{C_j}{\delta_k^{\beta-1}} = 0.$$

By the Squeeze Theorem, this gives

$$\lim_{k\to\infty} \mathfrak{p}(L_i, \mathcal{T}s_k) = 0 \implies \lim_{k\to\infty} \mathcal{T}s_k = L_i$$

and

$$\lim_{k\to\infty} \mathfrak{p}(L_j, \mathcal{T}s_k) = 0 \implies \lim_{k\to\infty} \mathcal{T}s_k = L_j.$$

The sequence $\{\mathcal{T}s_k\}$ converges to both L_i and L_j. By the uniqueness of limits in a metric space, this implies $L_i = L_j$.

This contradicts our initial assumption that L_i and L_j were distinct. Thus, the assumption $m \geqslant 2$ must be false. We conclude that $m = 1$. □

Proposition 4. *Let $(X, \mathfrak{p})$ be a metric space and $\mathcal{T}\colon X \to X$ satisfy (MKC) with $0 < \mu < 1$ and $\beta > 1$. Let $\{x_n\} \subset X \setminus \operatorname{Fix}(\mathcal{T})$ satisfy $x_n \to a \in \overline{X} \setminus X$. Assume also that*

$$\text{(H)} \quad \sup\{\mathfrak{p}(x, \mathcal{T}x) \colon x \in X \setminus \operatorname{Fix}(\mathcal{T})\} = +\infty.$$

If (MKC) holds for all distinct ordered pairs (r, s) from $X \setminus \operatorname{Fix}(\mathcal{T})$, then the sequence $(\mathcal{T}x_n)$ converges (i.e., has exactly one cluster point).

Proof. Set $\delta_n = \mathfrak{p}(x_n, \mathcal{T}x_n)$. Assume, for contradiction, that (MKC) holds for all ordered pairs and that $(\mathcal{T}x_n)$ has at least two distinct cluster points. Extract two subsequences (x_{n_k}) and (x_{m_k}) with

$$\mathcal{T}x_{n_k} \to L_1, \quad \mathcal{T}x_{m_k} \to L_2, \quad L_1 \neq L_2.$$

As $k \to \infty$, the delta sequences converge to finite limits

$$\delta_{n_k} = \mathfrak{p}(x_{n_k}, \mathcal{T}x_{n_k}) \to \mathfrak{p}(a, L_1) = \Delta_1$$

$$\delta_{m_k} = \mathfrak{p}(x_{m_k}, \mathcal{T}x_{m_k}) \to \mathfrak{p}(a, L_2) = \Delta_2$$

Case 1. At least one cluster point equals a. Assume $L_1 = a$. This implies $\Delta_1 = 0$, so $\delta_{n_k} \to 0$. Since $L_1 \neq L_2$, we must have $L_2 \neq a$, which implies $\Delta_2 = \mathfrak{p}(a, L_2) > 0$. Apply (MKC) to the ordered pair $(r, s) = (x_{n_k}, x_{m_k})$:

$$\mathfrak{p}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k}) \leqslant \mu\,[\mathfrak{p}(x_{n_k}, \mathcal{T}x_{n_k})]^{\beta}\,[\mathfrak{p}(x_{m_k}, \mathcal{T}x_{m_k})]^{1-\beta} = \mu \frac{\delta_{n_k}^{\beta}}{\delta_{m_k}^{\beta-1}}.$$

Now, take the limit as $k \to \infty$.

1) LHS $\lim\limits_{k\to\infty} \mathfrak{p}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k}) = \mathfrak{p}(L_1, L_2) = \mathfrak{p}(a, L_2) = \Delta_2 > 0$.

2) RHS $\lim\limits_{k\to\infty} \mu \dfrac{\delta_{n_k}^{\beta}}{\delta_{m_k}^{\beta-1}} = \mu \dfrac{0^{\beta}}{\Delta_2^{\beta-1}} = 0.$

The inequality becomes $\Delta_2 \leqslant 0$, which contradicts $\Delta_2 > 0$. Thus, this case is impossible.

Case 2. No cluster point equals a. This means that for any two cluster points $L_1 \neq L_2$, we have $\Delta_1 > 0$ and $\Delta_2 > 0$. The subsequences (x_{n_k}) (for L_1) and (x_{m_k}) (for L_2) satisfy the conditions

1) $\mathcal{T}x_n^{(1)} \to L_1$, $\mathcal{T}x_n^{(2)} \to L_2$;
2) $\inf_k \delta_{n_k} > 0$ and $\inf_k \delta_{m_k} > 0$ (since their limits Δ_1, Δ_2 are positive).

By Lemma 1, it is impossible to have $m \geqslant 2$ such distinct cluster points. This is a contradiction.

Since both Case 1 and Case 2 lead to contradictions, our initial assumption that $(\mathcal{T}x_n)$ has at least two distinct cluster points is false. Therefore, $(\mathcal{T}x_n)$ must have at most one cluster point. Since $(\mathcal{T}x_n)$ is a sequence in a metric space, if it has at least one cluster point (which it must, if X is bounded, or by assumption) and at most one cluster point, it must converge. □

Theorem 3. *Let $(X, \mathfrak{p})$ be a metric space. Let $\mathcal{T}\colon X \to X$ satisfy*

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\,[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}, \quad 0 \leqslant \mu < 1,\ \beta > 1,$$

for all distinct $r, s \in X \setminus \operatorname{Fix}(\mathcal{T})$. Assume

$$\textit{(H)} \quad \sup\{\mathfrak{p}(x,\mathcal{T}x)\colon x \in X \setminus \operatorname{Fix}(\mathcal{T})\} = +\infty.$$

Let $a \in \overline{X} \setminus X$. If there exists a sequence $\{x_n\} \subset X \setminus \operatorname{Fix}(\mathcal{T})$ with $x_n \to a$, such that $(\mathcal{T}x_n)$ does not converge, then

$$\mathcal{T}(X \setminus \operatorname{Fix}(\mathcal{T})) = \{q\}$$

for some $q \in X$.

Proof. Let $\{x_n\} \subset X \setminus \operatorname{Fix}(\mathcal{T})$, $x_n \to a$, and suppose $(\mathcal{T}x_n)$ has at least two distinct cluster points ($m \geqslant 2$). Put $\delta_n = \mathfrak{p}(x_n, \mathcal{T}x_n) \in (0, \infty)$.

Case 1. $\inf_n \delta_n = 0$.
There exists a subsequence (x_{n_k}) with $\delta_{n_k} \to 0$. Fix arbitrary $s_1, s_2 \in X \setminus \operatorname{Fix}(\mathcal{T})$. From (MKC),

$$\mathfrak{p}(\mathcal{T}x_{n_k},\mathcal{T}s_1) \leqslant \mu\,\delta_{n_k}^{\beta}\,[\mathfrak{p}(s_1,\mathcal{T}s_1)]^{1-\beta} \to 0.$$

Hence, $\lim\limits_{k\to\infty} \mathcal{T}x_{n_k} = \mathcal{T}s_1$. Similarly,

$$\mathfrak{p}(\mathcal{T}x_{n_k},\mathcal{T}s_2) \leqslant \mu\,\delta_{n_k}^{\beta}\,[\mathfrak{p}(s_2,\mathcal{T}s_2)]^{1-\beta} \to 0.$$

Hence, $\lim_{k\to\infty} \mathcal{T}x_{n_k} = \mathcal{T}s_2$. By uniqueness of limits, $\mathcal{T}s_1 = \mathcal{T}s_2$. Let $q = \mathcal{T}s_1$. Since s_1, s_2 were arbitrary, $\mathcal{T}(X \setminus \operatorname{Fix}(\mathcal{T})) = \{q\}$. This case proves the theorem.

Case 2. $\inf_n \delta_n > 0$.
The sequence (x_n) and its subsequences satisfy all conditions of Lemma 1:

1) $x_n \to a \in \overline{X} \setminus X$.
2) The images have $m \geqslant 2$ cluster points.
3) The δ_n sequences are bounded away from zero ($\inf_n \delta_n > 0$).

Lemma 1 concludes that these conditions force $m = 1$. This contradicts our claim that $m \geqslant 2$. Therefore, Case 2 is impossible.

The $m \geqslant 2$ condition requires Case 1 to hold, as Case 2 is impossible. Case 1 implies $\mathcal{T}(X \setminus \operatorname{Fix}(\mathcal{T})) = \{q\}$. $\square$

Example 3. Let $X = [0, \infty)$ be equipped with the usual metric $\mathfrak{p}(x,y) = |x - y|$. Define the mapping $\mathcal{T} \colon X \to X$ by

$$\mathcal{T}x = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\operatorname{Fix}(\mathcal{T}) = \{0,1\}$ and $X \setminus \operatorname{Fix}(\mathcal{T}) = (0, 1) \cup (1, \infty)$.

For $x \in X \setminus \operatorname{Fix}(\mathcal{T})$, the defect is $\mathfrak{p}(x, \mathcal{T}x) = |x - 1|$. Clearly,

$$\sup\{\mathfrak{p}(x, \mathcal{T}x) \colon x \in X \setminus \operatorname{Fix}(\mathcal{T})\} = +\infty,$$

so Assumption (H) holds.

The MKC condition holds trivially: for any distinct $r, s \in X \setminus \operatorname{Fix}(\mathcal{T})$, we have $\mathcal{T}r = \mathcal{T}s = 1$; hence $\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) = 0 \leqslant \mu[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}$ for any $0 \leqslant \mu < 1$ and $\beta > 1$.

As predicted by Theorem 3, the image collapses to a singleton:

$$\mathcal{T}(X \setminus \operatorname{Fix}(\mathcal{T})) = \{1\}.$$

Theorem 4. *Let $(X, \mathfrak{p})$ be a metric space and let $\mathcal{T} \colon X \to X$ satisfy the Multiplicative Kannan Contraction*

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\,[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{\,1-\beta} \quad \text{for all } r, s \in X \setminus \operatorname{Fix}(\mathcal{T}),$$

where $0 \leqslant \mu < 1$ and $\beta > 1$. Assume also that

$$\text{(H)} \quad \sup\{\mathfrak{p}(x,\mathcal{T}x) \colon x \in X \setminus \operatorname{Fix}(\mathcal{T})\} = +\infty.$$

Then $\mathcal{T}$ is degenerate; there exists $q \in X$, such that

$$\mathcal{T}(x) = q \quad \text{for all } x \in X \setminus \operatorname{Fix}(\mathcal{T})\,.$$

Now assume additionally that $X \subset \mathbb{R}$, $\mathfrak{p}(x,y) = |x-y|$, and that $I \subset X$ is a nonempty interval for which $X \setminus \operatorname{Fix}(\mathcal{T})$ is dense in I. Then one of the following holds:

1) *$\mathcal{T}$ is constant on I (and hence continuous on I);*
2) *$\mathcal{T}$ is not constant on I (and hence discontinuous at some point of I).*

Proof. Part 1: First, we show that $\mathcal{T}$ is degenerate. Assume for contradiction that $\mathcal{T}$ is non-degenerate. This means there exist at least two points, $s_1, s_2 \in X \setminus \mathrm{Fix}(\mathcal{T})$, such that $\mathcal{T}s_1 \neq \mathcal{T}s_2$.

By Assumption (H), there exists a sequence $\{r_k\} \subset X \setminus \mathrm{Fix}(\mathcal{T})$, such that the sequence $\delta_k = \mathfrak{p}(r_k, \mathcal{T}r_k) \to \infty$.

We apply (MKC) to the pair (s_1, r_k) by setting $r = s_1$ and $s = r_k$.

1) $r = s_1$, so $\mathfrak{p}(r, \mathcal{T}r) = \mathfrak{p}(s_1, \mathcal{T}s_1) = C_1$ (a finite, non-zero constant).
2) $s = r_k$, so $\mathfrak{p}(s, \mathcal{T}s) = \mathfrak{p}(r_k, \mathcal{T}r_k) = \delta_k$.

The (MKC) inequality is

$$\mathfrak{p}(\mathcal{T}s_1, \mathcal{T}r_k) \leqslant \mu[\mathfrak{p}(s_1, \mathcal{T}s_1)]^{\beta}[\mathfrak{p}(r_k, \mathcal{T}r_k)]^{1-\beta},$$

$$\mathfrak{p}(\mathcal{T}s_1, \mathcal{T}r_k) \leqslant \mu \frac{[C_1]^{\beta}}{[\delta_k]^{\beta-1}}.$$

Since $\beta > 1$, the exponent $\beta - 1$ is positive. As $k \to \infty$, $\delta_k \to \infty$, so the denominator $\delta_k^{\beta-1} \to \infty$. The entire RHS (which has a constant numerator) must go to 0. By the Squeeze Test, $\lim\limits_{k\to\infty} \mathfrak{p}(\mathcal{T}s_1, \mathcal{T}r_k) = 0$, which implies $\lim\limits_{k\to\infty} \mathcal{T}r_k = \mathcal{T}s_1$.

Now, we apply (MKC) to the pair (s_2, r_k) (setting $r = s_2$ and $s = r_k$), and we get

$$\mathfrak{p}(\mathcal{T}s_2, \mathcal{T}r_k) \leqslant \mu[\mathfrak{p}(s_2, \mathcal{T}s_2)]^{\beta}[\mathfrak{p}(r_k, \mathcal{T}r_k)]^{1-\beta}.$$

As $k \to \infty$, this RHS also $\to 0$. This implies $\lim\limits_{k\to\infty} \mathcal{T}r_k = \mathcal{T}s_2$.

By the uniqueness of limits in $\mathbb{R}$, the single sequence $(\mathcal{T}r_k)$ must have a single limit. Therefore, $\mathcal{T}s_1 = \mathcal{T}s_2$. This contradicts our assumption that $\mathcal{T}s_1 \neq \mathcal{T}s_2$.

The assumption that $\mathcal{T}$ is non-degenerate is false. Thus, $\mathcal{T}$ must be degenerate; $\mathcal{T}(X \setminus \mathrm{Fix}(\mathcal{T})) = \{q\}$ for some $q \in X$.

Part 2: Part 1 shows that any map satisfying (MKC) + (H) must be degenerate ($\mathcal{T}(X \setminus \mathrm{Fix}(\mathcal{T})) = \{q\}$). Now we check continuity.

1) If $\mathcal{T}$ is constant on I. This means $\mathcal{T}(x) = q$ for all $x \in I$. This map is clearly continuous on I.

2) If $\mathcal{T}$ is not constant on I. This means there must be some point $y \in I$, such that $\mathcal{T}(y) \neq q$. Since $\mathcal{T}(X \backslash \operatorname{Fix}(\mathcal{T})) = \{q\}$, this point y must be a fixed point $y \in I \cap \operatorname{Fix}(\mathcal{T})$ and $\mathcal{T}(y) = y \neq q$. This map is discontinuous at y. The reasons:
   (a) Since $X \backslash \operatorname{Fix}(\mathcal{T})$ is dense in I, there is a sequence $s_k \to y$ with $s_k \in X \backslash \operatorname{Fix}(\mathcal{T})$.
   (b) $\lim_{k\to\infty} \mathcal{T}(s_k) = \lim_{k\to\infty} q = q$.
   (c) But $\mathcal{T}(y) = y$, and $y \neq q$.
   (d) Since the limit does not equal the value, $\mathcal{T}$ is discontinuous at y.

These two cases are mutually exclusive by definition (constant vs. not constant). The proof is complete. □

Proposition 5. *Let $X \subset \mathbb{R}$ be equipped with the usual metric $\mathfrak{p}(x, y) = |x - y|$. Suppose that $\mathcal{T} : X \to X$ satisfies the (MKC) condition*

$$|\mathcal{T}r - \mathcal{T}s| \leqslant \mu\, |r - \mathcal{T}r|^{\beta}\, |s - \mathcal{T}s|^{1-\beta} \quad \textit{for all distinct } r, s \in X \backslash \operatorname{Fix}(\mathcal{T}),$$

where $0 < \mu < 1$ and $\beta > 1$. Assume further that the defect distances are bounded and bounded away from zero, i.e.,

$$0 < d = \inf_{x \in X \backslash \operatorname{Fix}(\mathcal{T})} |x - \mathcal{T}x| \quad \textit{and} \quad D = \sup_{x \in X \backslash \operatorname{Fix}(\mathcal{T})} |x - \mathcal{T}x| < +\infty.$$

Then $\mathcal{T}$ need not be degenerate.

Proof. Choose $\beta = 2$ and $\mu = 0.5$. Let

$$X = \{0, 1, 10, 11\} \subset \mathbb{R}, \quad \mathfrak{p}(x, y) = |x - y|.$$

Define the map $\mathcal{T} : X \to X$ by

$$\mathcal{T}(10) = 0, \quad \mathcal{T}(11) = 1, \quad \mathcal{T}(0) = 0, \quad \mathcal{T}(1) = 1.$$

Then $\operatorname{Fix}(\mathcal{T}) = \{0, 1\}$ and $X \backslash \operatorname{Fix}(\mathcal{T}) = \{10, 11\}$.
Step 1.

$$|10 - \mathcal{T}(10)| = |10 - 0| = 10, \quad |11 - \mathcal{T}(11)| = |11 - 1| = 10.$$

Hence $d = D = 10$, so the distances are bounded and bounded away from zero.

Step 2. Non-degeneracy.

$$\mathcal{T}(X \setminus \mathrm{Fix}(\mathcal{T})) = \{0,1\},$$

which contains two distinct points; thus $\mathcal{T}$ is non-degenerate.

Step 3. Verification of (MKC). The only distinct non-fixed pair is $(r, s) = (10, 11)$ (and its reverse). For $\beta = 2$ and $\mu = 0.5$,

$$|\mathcal{T}(10) - \mathcal{T}(11)| = |0 - 1| = 1.$$

Meanwhile,

$$\mu\,|r - \mathcal{T}r|^{\beta}\,|s - \mathcal{T}s|^{1-\beta} = 0.5\,|10|^{2}\,|10|^{1-2} = 0.5 \times 100 \times 10^{-1} = 5.$$

Hence $|\mathcal{T}(10) - \mathcal{T}(11)| = 1 \leqslant 5$, and the (MKC) condition is satisfied. The same calculation holds for $(r, s) = (11, 10)$.

Therefore, $\mathcal{T}$ satisfies (MKC) with $\beta > 1$ and $0 < \mu < 1$, has $\mathfrak{p}(x, \mathcal{T}x)$ distance both bounded and positive, and is non-degenerate.

This shows that in the real line with the usual metric, the conditions

$$\beta > 1, \quad 0 < \mu < 1, \quad 0 < \inf |x - \mathcal{T}x| \leqslant \sup |x - \mathcal{T}x| < \infty$$

does *not* imply that $\mathcal{T}$ must be degenerate. □

Example 4. Let $X = \{a, b, c, d\}$ be equipped with the ultrametric $\mathfrak{p}$ defined by the following distances (corresponding to two clusters $\{a, b\}$ and $\{c, d\}$):

$\mathfrak{p}$	a	b	c	d
a	0	1	2	2
b	1	0	2	2
c	2	2	0	1
d	2	2	1	0

One easily verifies that $\mathfrak{p}$ satisfies the strong triangle inequality $\mathfrak{p}(x, z) \leqslant \max\{\mathfrak{p}(x, y), \mathfrak{p}(y, z)\}$ for all $x, y, z \in X$, so $(X, \mathfrak{p})$ is ultrametric space.

Define the mapping $\mathcal{T}\colon X \to X$ by

$$\mathcal{T}(a) = a, \quad \mathcal{T}(b) = b, \quad \mathcal{T}(c) = a, \quad \mathcal{T}(d) = b.$$

1) Fixed points: $\mathrm{Fix}(\mathcal{T}) = \{a, b\}$.
2) Non-fixed points: $X \setminus \mathrm{Fix}(\mathcal{T}) = \{c, d\}$.
3) Non-degeneracy: $\mathcal{T}(\{c, d\}) = \{a, b\}$, which contains two distinct points. Thus $\mathcal{T}$ is non-degenerate on the non-fixed points.
4) Defect distances:

$$\mathfrak{p}(c, \mathcal{T}c) = \mathfrak{p}(c, a) = 2, \quad \mathfrak{p}(d, \mathcal{T}d) = \mathfrak{p}(d, b) = 2.$$

The defects are bounded and bounded away from zero ($\inf = \sup = 2 > 0$).
5) MKC verification: For the only pair of distinct non-fixed points $r = c$, $s = d$,

$$\mathfrak{p}(\mathcal{T}c, \mathcal{T}d) = \mathfrak{p}(a,b) = 1.$$

The right-hand side of the MKC inequality is

$$\mu\, [2]^{\beta}\, [2]^{1-\beta} = 2\mu.$$

Since $\beta > 1$, the inequality $1 \leqslant 2\mu$ holds whenever $\mu \geqslant 1/2$. Thus it is satisfied for any $\mu \in [1/2, 1)$, e.g., $\mu = 0.6$.

This example demonstrates that, even in a non-Euclidean ultrametric space, bounded and positive defects allow non-degenerate behavior under the MKC condition with $\beta > 1$. This supports the broader applicability of the classification beyond subsets of $\mathbb{R}$.

Remark 3. *The multiplicative Kannan-type condition (MKC)*

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leqslant \mu\, [\mathfrak{p}(r, \mathcal{T}r)]^{\beta} [\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}, \quad 0 < \mu < 1,$$

behaves fundamentally differently according to the sign of $\beta - 1$. For $0 < \beta < 1$ it generates interpolative contractive behaviour as in Karapinar [4], while for $\beta > 1$ it becomes expansive *(see Proposition 1).*

Our analysis of $\beta > 1$ case reveals a complete classification based on the nature of the distances $\mathfrak{p}(x, \mathcal{T}x)$. We have proven that $\mathcal{T}$ must be degenerate (i.e., $\mathcal{T}(X \setminus \mathrm{Fix}(\mathcal{T})) = \{q\}$) in two cases:

1) *If the distances $\mathfrak{p}(x, \mathcal{T}x)$ are unbounded (Assumption (H)), as proven in Theorem 3.*
2) *If the distances $\mathfrak{p}(x, \mathcal{T}x)$ have an infimum of zero ($\inf \mathfrak{p}(x, \mathcal{T}x) = 0$), as proven in Proposition 2.*

Conversely, the case where the distances $\mathfrak{p}(x, \mathcal{T}x)$ are bounded and bounded away from zero, $0 < \inf \mathfrak{p} \leqslant \sup \mathfrak{p} < \infty$ is fundamentally different. Our counterexample (Proposition 5) demonstrates that non-degenerate maps are permitted in this case on the real line $\mathbb{R}$.

Consequently, the universal claim that (MKC) admits a fixed point for all $\beta \in \mathbb{R}$ [6] is incorrect, as is the claim that the $\beta > 1$ case universally forces degeneracy. This analysis clarifies the logical structure of (MKC), revealing that for $\beta > 1$, degeneracy is forced by unbounded or vanishing distances, but not by bounded, positive-infimum distances.

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